2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

# Dynamic Smooth Pursuit Gain Estimation from Eye Tracking Data

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*Abstract*— **The smooth pursuit gain (SPG) is defined as the ratio of the angular velocity of the eye to that of the moving target. Being evaluated at a certain frequency of harmonic visual stimuli, it has been widely used in medicine as a measure of oculomotor system performance. In this study, the smooth pursuit system (SPS) is modeled as a dynamical system whose output signal is the angular velocity of the eye and the input is the angular velocity of a moving stimulus. Then, by means of system identification, the entire dynamics of SPS can be estimated, provided the visual stimuli are properly designed. This technique is referred to as the dynamic SPG (DSPG). Systems appearing equivalent in terms of SPG, can therefore be distinguished between using DSPG. Modern eye tracking techniques register gaze direction over time, but do not measure gaze velocity. Hence, to estimate the SPG/DSPG, differentiation must be applied to the output of the eye tracker. Four approaches to differentiation of eye-tracking data are evaluated in this paper with respect to the estimation of DSPG, out of which the method based on Laguerre functions stands out as the most reliable technique for this particular application.**

## I. INTRODUCTION

In studies of the human smooth pursuit system (SPS), the smooth pursuit gain (SPG) is a quantity which is often used for the characterization of system performance, see e.g. [1], [2], [5] and [7]. The SPG is then given as a single value constituting the ratio of the angular velocity of the eye to that of the target. Hence, the SPG can be seen as the static gain of a dynamic system relating gaze velocity to target velocity. For a mono-harmonic visual stimuli, SPG is still just a point on the frequency characteristics of SPS and thus constitutes an uncertain measure of system performance. In this study, the dynamical nature of SPS is stressed and its entire frequency response is modeled to gain more insight into the system properties. The frequency response of SPS is therefore referred to as dynamic SPG (DSPG), indicating that the system gain is evaluated over all frequencies available in the visual stimuli spectrum.

Modern eye tracking techniques, both video (see e.g. [8]) and electrooculography-based (see e.g. [3]) ones, only allow for the registration of gaze direction and not gaze velocity. Hence, methods for differentiation of measured (i.e. noisy) signals must be utilized in order to obtain the output data to identify a DSPG model. Estimates of gaze velocity are also sought by neurologists and neurophysiologists for studies of various diseases such as Huntington's disease ([4]) and Parkinson's disease ([7]).

In this paper, four signal differentiation techniques are investigated and evaluated for the identification of DSPG from eye tracking data. An observer-based method, a method involving the projection of the signal onto Laguerre functions, analytical differentiation of the model output and, finally, simple three-point numerical differentiation are considered. The methods are evaluated on both simulated and experimental data.

The paper is composed as follows: In Section II, a mathematical model of SPS is proposed. Signal differentiation techniques are then summarized in Section III. This is followed by the identification results on both simulated and real data in Section IV. Finally, the results and methods are discussed in Section V.

## II. MATHEMATICAL MODEL

A block diagram of the SPS model adopted in this paper is presented in Fig. 1. It is a linear feedback model. The eye plant is modeled by a first order dynamics with the transfer function $G(s) = \frac{K_e}{s+a}$. The transfer function of the controller is $F(s) = K_p + \frac{K_i}{s+b}$, feeding back a signal comprising a proportional and a low-pass filtered angular velocity error terms. More sophisticated nonlinear models of SPS are available ([9], [10]), but this highly simplified one will be shown to suffice for the purpose of DSPG estimation.

Let $y(t)$ be the output of the model at time $t$ with $r(t)$ as input. In physical terms, $y(t)$ is the angular velocity of the eye and $r(t)$ is the angular velocity of the visual stimulus. Notice that $y(t)$ is not accessible for measurement and has to be estimated from its integral, the gaze direction, in order to identify the parameters of this model. The observability canonical form of the system is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -(a+b+K_eK_p) & 1 \\ -(ab+K_eK_pb+K_eK_i) & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} K_eK_p \\ (K_eK_i+K_eK_pb) \end{bmatrix} r$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{1}$$

or more compactly

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + Br \\ y &= Cx \end{aligned}.$$

System (1) has five unknown parameters. However, since the static gain of $G(s)$ and that of $F(s)$ are indistinguishable in an input-output experiment, the parameter vector is chosen

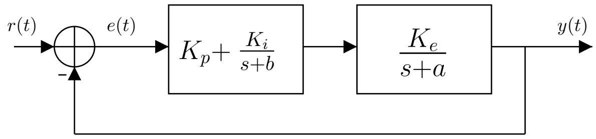


Fig. 1. Block diagram of the SPS model. $r(t)$ is the target angular velocity, $e(t)$ is the angular velocity error and $y(t)$ is the modeled angular velocity of the eye.

as

$$\theta = \left[ \begin{array}{cccc} a & K_eK_i & K_eK_p & b \end{array} \right]$$

to uniquely parameterize system (1).

## III. NUMERICAL DIFFERENTIATION

Differentiation of measured signals has been the topic of countless publications and many methods have been proposed. In this paper, four differentiation techniques are selected and evaluated for the purpose of DSPG estimation by applying them to both simulated and real data.

### A. The observer method

The first state variable of (1) is the angular velocity of the eye and has to be estimated from eye tracking gaze direction data. Therefore, a new model where gaze direction is the measured output is needed. Such a model can be obtained by augmenting (1) with the equation $\dot{x}_3 = x_1$ and letting the output of the system be the gaze direction $y_p = x_3$. The state vector in the augmented model is then $\mathbf{x}_p^T = [x_1 \quad x_2 \quad x_3]$. For the augmented plant

$$\begin{aligned} \dot{\mathbf{x}}_p &= A_p\mathbf{x}_p + B_pr \\ y_p &= C_p\mathbf{x}_p \end{aligned} \tag{2}$$

introduce a linear observer

$$\begin{aligned} \dot{\hat{\mathbf{x}}}_p &= A_p\hat{\mathbf{x}}_p + B_pr + K(y_p - C_p\hat{\mathbf{x}}_p) \\ \hat{y}_p &= C_p\hat{\mathbf{x}}_p \end{aligned} \tag{3}$$

where

$$\begin{aligned} K^T &= \left[ \begin{array}{ccc} K_1 & K_2 & K_3 \end{array} \right], \\ C_p &= \left[ \begin{array}{ccc} 0 & 0 & 1 \end{array} \right]. \end{aligned}$$

and $A_p$ and $B_p$ are the augmented versions of $A$ and $B$. The gain $K$ should be chosen so that the observer state estimation error dynamics are significantly faster than those of the actual system, but slower than the noise dynamics.

Provided the vector $\theta$ is known, observer (3) can be used to estimate the gaze velocity. However, parameter estimates are the purpose of identification and the latter requires the gaze velocity. This hints to the use of an iterative estimation approach. An initial guess for the model parameters in (1) is made and utilized in the observer to estimate gaze velocity. The obtained velocity estimate is used to identify the model parameters anew and re-design the observer. This is repeated until the parameter estimations converge. A rigorous examination of the convergence properties of this iterative process will not be dealt with here. The process converged for all eye tracking data sets used in this study.

### B. The Laguerre functions method

The Laguerre functions form an orthonormal basis which is complete in the $L_2$ function space. In Laplace domain, the Laguerre functions are rational in the Laplace operator, with all poles at the same point on the negative real axis. In time domain, the functions are polynomials multiplied by decaying exponentials. Consequently, Laguerre functions can be used to approximate transfer functions or the output of linear dynamic systems, see for example [11].

**Laguerre functions** The $k$:th time domain Laguerre function is given by

$$L_k(t) = pe^{-pt/2} \sum_{n=1}^{k+1} \frac{k!(-pt)^{n-1}}{((n-1)!)^2(k-n+1)!}, \tag{4}$$

where $p$ is the Laguerre (user-defined) parameter determining the decay rate of the function. The set of Laguerre functions is orthonormal so that

$$\langle L_m(t), L_n(t) \rangle = \delta_{mn}, \tag{5}$$

where $\delta_{mn}$ is the Kronecker delta and the inner product is given by

$$\langle f(t), g(t) \rangle = \int_0^\infty f(t)g(t)dt. \tag{6}$$

A function $f(t) \in L_2$ can be expressed in terms of Laguerre functions by

$$f(t) = \sum_{k=0}^{\infty} a_k L_k, \tag{7}$$

where

$$a_k = \langle f(t), L_k(t) \rangle, \tag{8}$$

are the Laguerre coefficients.

One feasible way of approximating the derivative of a noisy signal is by projecting the signal onto a finite set of Laguerre functions and then using analytically differentiated Laguerre functions to give an estimation of the signal derivative. However, if the signal variance is large, high order Laguerre functions are required in the approximation which may cause numerical problems. A way of bypassing this is by performing Laguerre approximation within a sliding window.

Assume the signal to be differentiated is sampled and stored in a vector

$$\mathbf{y} = \left[ \begin{array}{cccc} y_1 & y_2 & \dots & y_N \end{array} \right], \tag{9}$$

with $N$ samples. Choose a suitable window-length $L$ and create $M = N - L + 1$ new vectors $\mathbf{y}_m$ of length $L$ such that

$$\mathbf{y}_m = \left[ \begin{array}{cccc} y_m & y_{m+1} & \dots & y_{m+L-1} \end{array} \right]. \tag{10}$$

An approximation of the derivative of $\mathbf{y}_m$ can then be found by

$$\hat{\dot{\mathbf{y}}}_m = \sum_{k=0}^{\ell} \langle \mathbf{y}_m, \mathbf{L}_k \rangle \dot{\mathbf{L}}_k, \tag{11}$$

where $\mathbf{L}_k$ are sampled versions of the Laguerre functions, $\dot{\mathbf{L}}_k$ are the sampled versions of the Laguerre function derivatives and $\ell$ is the highest order of the Laguerre functions used. The integral in (11) is evaluated numerically.

Now form the $M$-by-$N$-matrix

$$\hat{\dot{\mathbf{Y}}} = \mathrm{diag}\ \left[\ \hat{\dot{\mathbf{y}}}_1 \quad \hat{\dot{\mathbf{y}}}_2 \quad \ldots \quad \hat{\dot{\mathbf{y}}}\ \right] \tag{12}$$

and let $V_m$ be the number of non-zero elements in column $m$ of $\hat{\dot{\mathbf{Y}}}$. The estimated derivative of the signal, $\hat{\dot{\mathbf{y}}}$, is finally given by

$$\hat{\dot{\mathbf{y}}} = \left[\ \hat{\dot{y}}_1 \quad \hat{\dot{y}}_2 \quad \ldots \quad \hat{\dot{y}}_N\ \right], \tag{13}$$

where

$$\hat{\dot{y}}_i = \frac{1}{V_i}\sum_{j=1}^{M} \hat{\dot{\mathbf{Y}}}_{ij}. \tag{14}$$

Hence, the approximation of the signal derivative is the average of the approximations obtained within each window.

### C. The analytical model output differentiation method

The solution to (2) is given by

$$y(t) = C_p e^{A_p t} B_p \mathbf{x}_{p_0} + \int_0^t C_p e^{A_p (t-\tau)} B_p u(\tau) d\tau \tag{15}$$

where $\mathbf{x}_{p_0}$ is the initial state vector. Differentiating this expression with respect to t gives

$$\dot{y}(t) = C_p \left[ A_p e^{A_p t} \mathbf{x}_{p_0} + \int_0^t A_p e^{A_p (t-\tau)} B_p u(\tau) d\tau + B_p u(t) \right] \tag{16}$$

Since this method relies on the model, the model parameters must be known beforehand. The model parameters for eye tracking data are un known and, therefore, have to be estimated using recorded gaze direction data. The angular velocity is a state variable in (1) and could be evaluated without derivation, directly from the solution of the state equation. However, for SPS models, where the angular velocity is not a state variable, (16) can be used.

### D. Three-point method

Assume the signal to be differentiated is sampled with sampling time $T_s$ and stored in a vector

$$\mathbf{y} = \left[\ y_1 \quad y_2 \quad \ldots \quad y_N\ \right], \tag{17}$$

with $N$ samples. The approximation of the signal derivative is a vector

$$\dot{\mathbf{y}} = \left[\ \dot{y}_1 \quad \dot{y}_2 \quad \ldots \quad \dot{y}_N\ \right], \tag{18}$$

of length $N$ where

$$\dot{y}_i = \begin{cases} \dfrac{y_{i+1} - y_i}{T_s} & i = 1 \\ \dfrac{y_i - y_{i-1}}{T_s} & i = N \\ \dfrac{y_{i+1} - y_{i-1}}{2T_s} & \text{otherwise} \end{cases} \tag{19}$$

## IV. Experiments and results

Experimental gaze direction data of test subjects attempting to track a moving dot on a computer monitor was gathered using a video-based eye tracker, described in [8]. Data was sampled with a sampling frequency of 60 Hz. Simulated data was generated by adding simulated zero mean white Gaussian noise to the output of the model in Fig. 1 for a given set of model parameters and input signals. The true derivatives of the input signals were known both in experiment and simulation.

The differentiation methods presented in Section III were evaluated for simulated and experimental data. For simulated data, the estimated derivative could simply be compared to the true derivative to obtain a performance measure. In the case of experimental data, the derivative of the signal was not known, so other means of evaluation had to be applied.

The purpose of this study is to identify model (1) and it is hence reasonable to compare the differentiation methods by their ability to give consistent estimates of the model parameters. The analytical model output differentiation method is implemented by using the estimates of the model parameters in (2) identified with gaze direction as output data. An approximation of the signal derivative is then readily given by (16). The method of analytical model output differentiation is only used to give estimates of gaze velocity. Its output cannot be used to further improve the model.

In each simulated case, the user-defined parameters of the observer method (the feedback gain $K$) and the Laguerre method (the parameter $p$, window-length and highest function order) were tuned to give the best result. For experimental data, the method parameters were chosen to give the most consistent estimates of the model parameters.

System identification was performed using the System Identification Toolbox in MathWorks MATLAB with the `pem`-function (prediction error method) for linear systems.

### A. Results with simulated data

Fig. 2 shows the first 12 seconds of a typical data set obtained from model simulation, representing horizontal eye movements. Simulated zero mean white Gaussian noise with unit variance was added to the measurements. The differentiation methods outlined in Section III were applied to the data set to give estimates of the derivative, shown in Fig. 3 and Fig. 4. The model parameters in the observer-based method and the analytical model output differentiation method were set equal to the simulation parameters, i.e. the underlying system was assumed to be fully known.

Since the observer-based and the analytical model output differentiation methods depend on the model, uncertainties in the model parameters may result in poor performance for these two techniques. To study the effects of imperfect models, the methods were once again applied to the signal of Fig. 2, but this time with the model parameters perturbed by $10\%$. The results are presented in Fig. 5. The output of the three-point method is omitted since the latter is not model-based. The output of the Laguerre functions method is provided for reference.

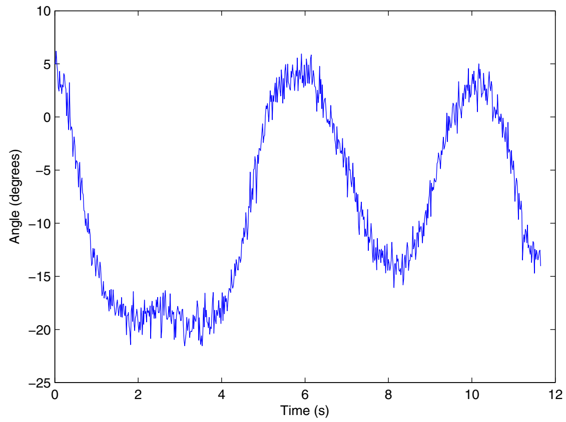


Fig. 2. The first 12 seconds of a typical data set obtained from model simulation. Simulated zero mean white Gaussian noise with unit variance was added.

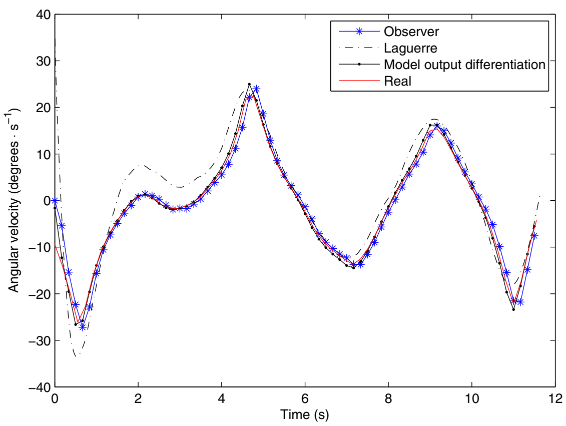


Fig. 3. Output of the differentiation methods applied to the signal in Fig. 2. The underlying system was known and the signal-to-noise ratio was 5 dB, simulated zero mean white Gaussian noise.

Fig. 6 shows the parameter estimates obtained from identification of the model for the considered differentiation methods. The method of analytical model output differentiation gives the same parameter estimates as identification of model (2) with eye position output data does and is thus marked as *Position* in the figure legend. Each method was run on several simulated data sets. The actual values of $\theta(2)$ and $\theta(3)$ were both equal to one and were used in the observer-based method to obtain the estimated derivative. Note that the iterative approach described in Section III-A was not used here.

### *B. Results with experimental data*

Fig. 7 shows the first 12 seconds of the horizontal part of a typical data set obtained from the eye tracker. Fig. 8 and Fig. 9 depict the results produced by the considered differentiation methods applied to the data in Fig. 7. Since the data set in Fig. 7 is a real gaze direction measurement obtained from an eye tracker, its true derivative was not known.

Identification was performed for eight sets of eye tracker data obtained from the same test subject to estimate $\theta(2)$ and $\theta(3)$. Fig. 10 summarizes the obtained parameter values. The frequency characteristics of the models estimated from a typical data set are provided in Fig. 11.

## V. Conclusions and Discussion

Four approaches to differentiation of measured signals were evaluated for the purpose of estimating eye velocity from eye tracking gaze direction data. In this study, the velocities were used to identify DSPG models, but accurate eye velocity estimates are also required in other types of medical and biomedical research.

The methods considered in this study use different ways of approaching the numerical differentiation problem. The analytical model output differentiation method completely relies on the estimated model and thus accurate modeling is required to obtain useful results. The signal to be differentiated only has an implicit effect on the result through the identified model parameters. The observer-based method also requires a model, but takes as well the signal to be differentiated into account. The feedback gain $K$ in the observer can be chosen to trust the data or the model to different degrees and the choice naturally depends on the noise variance. The steady-state Kalman filter can be used for gain optimization. The Laguerre functions method requires no prior knowledge and is preferable if the model quality is poor. The Laguerre functions method relies on appropriate values of the window length, function order and the parameter $p$. Choosing $p$ too small may result in noise amplification while a large $p$ combined with low function order may fail to give accurate approximations of high frequency content. A large window length will require high order of the approximation which may result in numerical problems. Short windows will increase computation time.

Fig. 3 shows that when the underlying system is known, both the observer-based method and the analytical model output differentiation method will give nearly exact results.

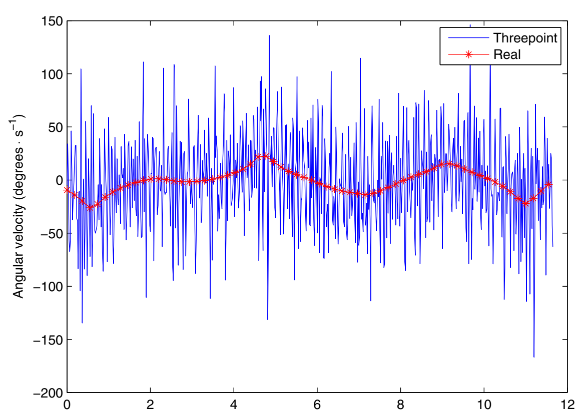


Fig. 4. Output of the three-point method applied to the signal in Fig. 2. The signal-to-noise ratio was 5 dB, simulated zero mean white Gaussian noise.

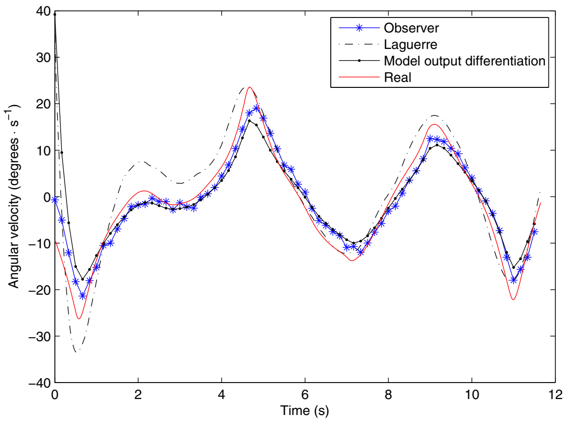


Fig. 5. Output of the differentiation methods applied to the signal in fig. 2. The assumed parameters of the underlying system were perturbed by 10% and the signal-to-noise ratio was 5 dB, with simulated zero mean white Gaussian noise as the noise sequence.

The Laguerre method gives a smooth and fair result compared to the three-point method, but neither can compete with the model-based methods when the system is perfectly known. However, under uncertainty in the parameters of the underlying system, the performance of the observer-based method and the analytical model output differentiation method degrade, as can be seen in Fig. 5. The former performs slightly better than the latter since it relies on the model knowledge to a lesser extent.

Fig. 6 confirms that, for a perfectly known model, the obtained parameter estimates are accurate for the observer-based method and for the model identified from gaze direction data. This is expected since the added noise was Gaussian and white. In the noise-free case the parameter estimates would have been exact. The observer method-based estimates are more spread than the estimates from gaze direction data since the observer partly relies on the

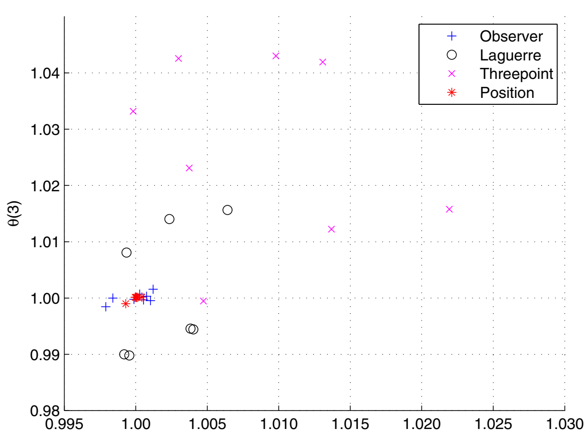


Fig. 6. Estimates of $\theta(2)$ and $\theta(3)$ ($K_eK_i$ and $K_eK_p$) for eight data sets.

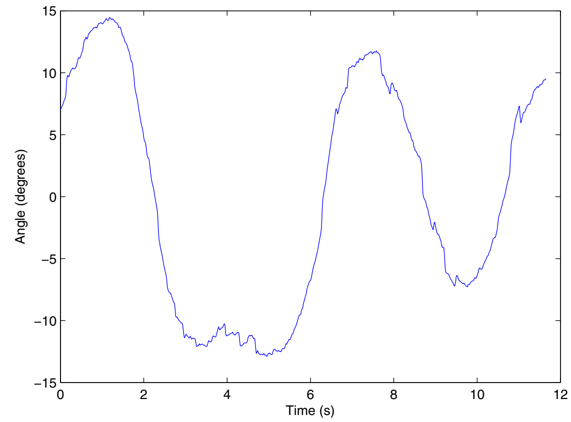


Fig. 7. The first 12 seconds of a typical data set from the eye tracker.

noisy data. The Laguerre functions method and the three-point method show higher parameter variance.

In Fig. 10, a somewhat larger variance is observed in the parameter estimates of the observer-based method and the estimates from gaze direction data, than in those obtained with the Laguerre functions method. This indicates that model (1) is not exact. A more accurate model of the system would give more consistent parameter estimates for these methods. However, accurate models are often complicated and cumbersome to work with. The Laguerre method provides an apparently robust way of estimating the derivatives of noisy gaze data without the need for an explicit model. It is worth noting that both the observer-based method and the Laguerre functions method give more consistent parameter estimates than identification from gaze direction data does, thus implying that efforts to find appropriate differentiation techniques are worthwhile.

Fig. 11 shows that the three-point method fails to give a feasible model. The static gain of the other models is close to unity. As was mentioned in Section I, the static gain of the SPS is the SPG at constant visual stimuli velocity and known

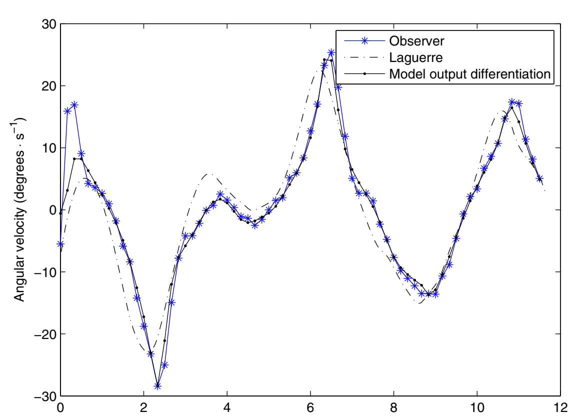


Fig. 8. The estimated derivatives of the signal in Fig. 7.

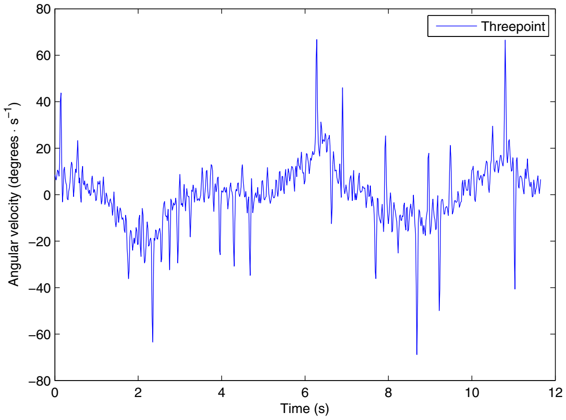


Fig. 9. Output of the three-point method when applied to the signal in Fig. 7.

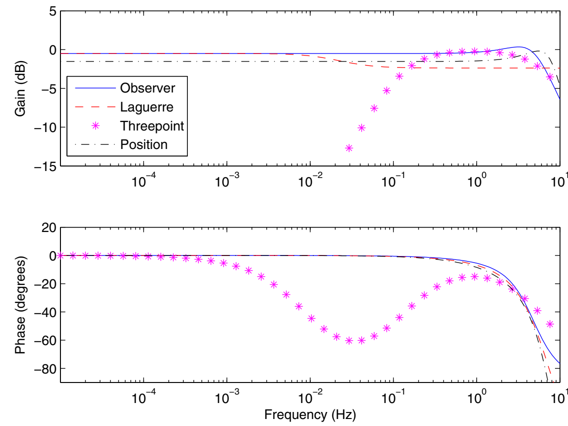


Fig. 11. Frequency characteristics of three models estimated with data obtained from applying the different differentiation methods to eye tracker data.

to be about 0.9 in healthy persons, [6]. In previous studies on SPS's, the SPG is given as a system performance measure. It is evident from Fig. 11 that this value does not fully characterize the system. The models obtained by using the observer-based method and the Laguerre functions method have the same static gain, but the corresponding systems are far from equivalent.

In conclusion, the SPG is a single point in the system frequency characteristics and performance measures based on it do not capture much of the system dynamics. The full frequency response, the DSPG, should therefore be considered when characterizing SPS's.

## VI. Acknowledgments

This study is in part financed by Advanced Grant from European Research Council entitled "*Systems and Signals Tools for Estimation and Analysis of Mathematical Models in Endocrinology and Neurology, SysTEAM*".

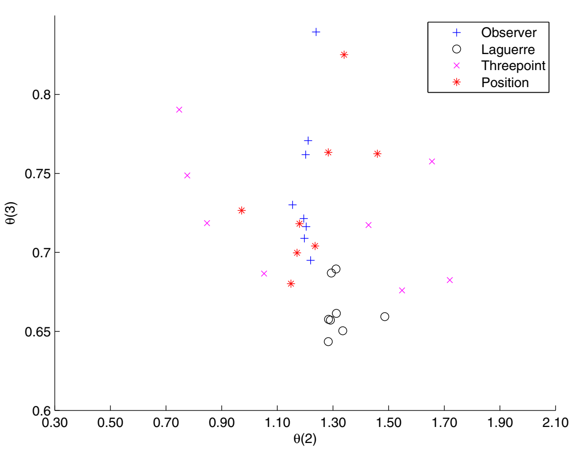


Fig. 10. Estimates of $\theta(2)$ and $\theta(3)$ ($K_eK_i$ and $K_eK_p$) for eight data sets of eye tracker data. All data sets pertain to the same test subject.

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