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Tube-based model predictive control for nonlinear systems with unstructured uncertainty

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Abstract— **This paper extends tube-based model predictive control methodology to the control of nonlinear systems with unmodelled dynamics. The problem of obtaining robustness against unstructured uncertainty is converted into the easier problem of achieving robustness against an (additional) bounded disturbance while satisfying an (additional) output constraint. The bound on the disturbance and the output constraint depend on the magnitude of the uncertainty. Robustness against the uncertainty is achieved by using tube-based model predictive control. Simulation experiments concerning the control of a solar collector plant illustrate the effectiveness of the proposed control strategy.**

I. INTRODUCTION

Robustness against dynamic uncertainty has received less attention in the model predictive control literature (see [1] and references therein) than robustness against parametric uncertainty or static time varying uncertainty that depends on the system variables. Motivated by the theory of linear robust control (linear H_∞), it is usual to require that the operator Δ corresponding to the unmodelled dynamics lies in the class of systems with bounded ℓ_2 gain. In this paper we assume that Δ lies in the class of systems with bounded ℓ_∞ gain; specifically, we assume that the operator Δ lies in the class of practical input/output stable (p-IOS) systems [2]. The motivation for this choice is that practical input-output stability generalizes the concept of finite gain with respect to supremum norms [3]. The problem of obtaining robustness against unstructured uncertainty can be converted into an easier problem of achieving robustness against an added bounded disturbance subject to the usual constraints and an added output constraint. It is well known that, if the controlled systems is ISS with respect to bounded uncertainties, then it is also robustly stable with respect to unmodelled dynamics (small gain theorem) but the amount of admissible uncertainty is unknown. Here we propose a design procedure, based on tube-based model predictive control [4], [5], that provides robustness against specified uncertainty provided the resultant optimal control problem is feasible. Tube-based MPC is well suited for this task because it is an implementable form of feedback MPC and possesses two degrees of freedom, the extra degree of freedom being employed to improve disturbance attenuation. The controller consists of a model predictive controller in the outer loop that steers trajectories of the uncertain system towards a reference trajectory that steers the initial state to the target state. In §2 the class of unmodelled dynamics is introduced. In §3 and §4 we specify the nominal trajectory and the ancillary controller. In §5 the properties of tube-based model predictive when a generic disturbance affects the system are examined. In §6 a discussion on the choice of the constraints for the Tube-based MPC is carried out and possible extensions are illustrated. In §7 we illustrate the performance of the method on outlet temperature control of a solar collector plant. Finally, in §8, we draw some conclusions.

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Notation: $\mathbb{I}$ is the set of integers. $|\cdot|$ denotes the usual Euclidean norm. ℓ_∞ is the set of all sequences $\phi(\cdot) = \{\phi(i)\}_{i=-\infty}^{\infty} \in \mathbb{R}^{n_\phi}$ such that $\|\phi\|_\infty \triangleq \sup\{|\phi(t)|,\ t \in \mathbb{I}\} < \infty$. $\phi_{[t_0,t]}(\cdot)$ denotes the sequence $\{\phi(i)\}_{i=t_0}^{t}$ defined on the interval $[t_0, t]$.

II. PROBLEM FORMULATION

The system to be controlled is described by

$$\begin{cases} x(t+1) = f(x(t), u(t), w(t)) \\ w(t) = \Delta(\phi_{(-\infty,t]}(\cdot))(t) \\ \phi(t) = h(x(t)) \end{cases} \quad (1)$$

where, $f(\cdot)$ is twice continuously differentiable, h is continuous and Δ represents unmodelled dynamics; $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $\phi \in \mathbb{R}^p$ and $w \in \mathbb{R}^q$. The controller is required to steer the initial state x to a neighborhood of a desired equilibrium point x_e while satisfying input and state constraints $u \in \mathbb{U}$ and $x \in \mathbb{X}$ for all admissible values of the uncertainty; x_e satisfies $x_e = f(x_e, u_e, 0)$ for some $u_e \in \mathbb{U}$. It is assumed that $\mathbb{U}$ and $\mathbb{X}$ are compact subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. The uncertainty is defined in a purely input/output form

Definition 1: The map $\Delta : \ell_\infty(\mathbb{R}^p) \to \ell_\infty(\mathbb{R}^q)$ is an input/output operator if it is causal. ■

We require Δ to lie in the class of i/o operators satisfying the practical input/output stability (p-IOS) property [2].

Definition 2: A causal operator $\Delta : \ell_\infty(\mathbb{R}^p) \to \ell_\infty(\mathbb{R}^q)$ is an admissible uncertainty if there exist $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ and a nonnegative constant c^* such that

$$|\Delta(\phi(\cdot))(t)| \le \max\{\beta(\|\phi_{(-\infty,0]}(\cdot)\|_\infty, t), \gamma(\|\phi_{[0,t]}(\cdot)\|_\infty), c^*\} \quad (2)$$

for all $t \ge 0$, i.e. such that Δ is p-IOS.

It follows from Definition 2 that, if the output $\phi(\cdot)$ of the system satisfies $|\phi_{(-\infty,0]}(\cdot)\|_\infty \le \beta^{-1}(c, 0)$[1] where $c \ge c^*$ and $\phi(t) \le \gamma^{-1}(c)$ for all $t \in \mathbb{I}_{\ge 0}$, then the input w to the

[1] $\beta^{-1}(c, 0) \triangleq z$ satisfying $\beta(z, 0) = c$

system satisfies $|w(t)| = |\Delta(\phi(\cdot))(t)| \leq c$ for all $t \in \mathbb{I}_{\geq 0}$. Hence robustness against this type of unmodelled uncertainty can be achieved by using a control u that ensures that the controlled system

$$x^+ = f(x, u, w), \tag{3}$$

where w is now regarded as a bounded disturbance taking values in the compact set $\mathbb{W} \triangleq \{w \in \mathbb{R}^q \mid |w| \leq c\}$, is such that an additional output constraint

$$\phi = h(x) \in \Phi_c \triangleq \{\phi \in \mathbb{R}^p \mid |\phi| \leq \gamma^{-1}(c)\} \tag{4}$$

is always satisfied (in addition to the original state and control constraints $x \in \mathbb{X}$, $u \in \mathbb{U}$). The output constraint restricts the input to, and, hence, the output of, the uncertainty Δ. We have converted the problem of obtaining robustness against unstructured uncertainty to the easier problem of achieving robustness against a bounded disturbance w while satisfying an additional output constraint. Both the output constraint set Φ_c and the set $\mathbb{W}$ in which the disturbance w lies depend on the design parameter c; the smaller c, the smaller $\mathbb{W}$ is and the tighter the output constraint becomes. The control problem is well posed if the following assumption is satisfied

Assumption 1:

$$h(x_e) \in \operatorname{int}(\Phi_c)$$

The control objectives can be achieved by using tube-based model predictive control [4].

III. The nominal (reference) trajectory

The *nominal* system is

$$z^+ = \tilde{f}(z, v) \tag{5}$$

where $\tilde{f}(z, v) \triangleq f(x, u, 0)$. The reference trajectory steers the nominal system from the initial state x_0 to the target state x_e and is obtained by minimizing

$$\bar{V}_N(x_0, \mathbf{v}; x_e) \triangleq \sum_{i=0}^{N-1} \ell(z(i) - x_e, v(i) - v_e) \tag{6}$$

subject to the state and control constraints $z \in \mathbb{Z}$ and $v \in \mathbb{V}$, and the terminal constraint $z(N) = x_e$; here $\mathbf{v} \triangleq \{v(0), v(1), \ldots, v(N-1)\}$ and $z(i)$ is the associated solution of (5) with initial state z. The function $\ell(\cdot)$ is defined by $\ell(z, v) \triangleq |z|_Q^2 + |v|_R^2$ where Q and R are positive definite, $|z|_Q^2 \triangleq (z^T Q z)$, and $|v|_R^2 \triangleq (v^T R v)$. The tightened constraint sets $\mathbb{Z}$ and $\mathbb{V}$ are chosen to ensure satisfaction of the original constraints $x(t) \in \mathbb{X}$, $h(x(t)) \in \Phi_c$ and $u(t) \in \mathbb{U}$ by the controlled uncertain system so that $\mathbb{Z}$ lies in the interior of $\mathbb{X}$, $h(\mathbb{Z})$ lies in the interior of Φ_c and $\mathbb{V}$ lies in the interior of $\mathbb{U}$. Let $\mathbf{v}^0(x_0; x_e)$ denote the solution of the nominal control problem with initial state x_0 and let $\mathbf{z}^0(x_0; x_e)$ denote the associated state trajectory. Clearly, $z^0(0; x_0, x_e) = x_0$ and $z^0(N; x_0, x_e) = x_e$; in the absence of the disturbance w, the control sequence $\mathbf{v}^0(x_0; x_e)$ steers x_0 to x_e. The reference state and control trajectories are the infinite sequences $\mathbf{z}^*(x_0, x_e)$ and $\mathbf{v}^*(x_0, x_e)$, respectively, defined by

$$\begin{aligned} \mathbf{z}^*(x_0, x_e) &= \{z^*(0; z, x_e), z^*(1; z, x_e), \ldots\} \triangleq \\ &\quad \{\mathbf{z}^0(x_0, x_e), x_e, x_e, \ldots\} \\ \mathbf{v}^*(x_0, x_e) &= \{v^*(0; z, x_e), v^*(1; z, x_e), \ldots\} \triangleq \\ &\quad \{\mathbf{v}^0(x_0, x_e), v_e, v_e, \ldots\}. \end{aligned} \tag{7}$$

IV. Model Predictive Controller

The task of the ancillary model predictive controller is to keep the state of the uncertain system close to the reference trajectory previously defined. The ancillary model predictive controller solves, at each state x of the uncertain system that is encountered, an open-loop optimal control problem in which the system is the nominal system and the cost is a measure of the deviation from the reference trajectories $z^*(x_0, x_e)$ and $v^*(x_0, x_e)$. The ancillary optimal control problem is therefore time-varying so we use t to denote current time. The optimal control problem $\mathbb{P}_N(x, t; x_0, x_e)$ solved by the ancillary controller at (x, t) (i.e. at state x, time t) is minimization of the cost $V_N(x, t, \mathbf{u}; x_0, x_e)$ defined by:

$$\begin{aligned} V_N(x, t, \mathbf{u}; x_0, x_e) \triangleq \sum_{i=0}^{N-1} \ell(x(i) - z^*(t+i; x_0, x_e), \\ u(i) - v^*(t+i; x_0, x_e)) + V_f(x(N); x_e) \end{aligned} \tag{8}$$

where $x(t) = x$ and, for each i, $x(i) \triangleq \bar{\phi}(i; x, \mathbf{u})$ the solution of $x^+ = \tilde{f}(x, u)$ if the initial state is x and the control sequence is $\mathbf{u} \triangleq \{u(0), u(1), \ldots, u(N-1)\}$. Problem $\mathbb{P}_N(x, t; x_0, x_e)$ is defined by

$$V_N^0(x, t; x_0, x_e) = \min_{\mathbf{u}} \{V_N(x, t, \mathbf{u}; x_0, x_e) \mid \mathbf{u} \in \mathbb{U}^N\} \tag{9}$$

The minimiser of $\mathbb{P}_N(x, t; x_0, x_e)$ is $\mathbf{u}^0(x, t; x_0, x_e)$ and the control $\kappa_N(x, t; x_0, x_e)$ applied to the system at (x, t) is the first element of this N-sequence:

$$\kappa_N(x, t; x_0, x_e) = u^0(0; x, t, x_0, x_e). \tag{10}$$

The corresponding state sequence is $\mathbf{x}^0(x, t; x_0, x_e)$. An important feature of $\mathbb{P}_N(x, t; x_0, x_e)$ is that it has no state constraints. If $x = z^*(t; x_0, x_e)$, then $V_N^0(x, t; x_0, x_e) = 0$ and the optimal trajectory coincides with the reference trajectory. The sampling period and stage cost $\ell(\cdot)$ may differ from that for the reference trajectory.

V. Analysis

Since $f(\cdot)$ is twice continuously differentiable, the terminal cost function $V_f(\cdot)$ may be chosen as follows. Using the linearization of $f(\cdot)$ at (x_e, u_e), a Control Lyapunov Function $\hat{V}_f(\cdot)$ of the form $\hat{V}_f(x; x_e) = |x - x_e|^2_{Q_f(x_e)}$ (where $Q_f(x_e)$ is positive definite) and an associated Control Invariant set $\mathbb{X}_f(x_e) = \{x \mid \hat{V}_f(x; x_e) \leq \alpha\}$ for some $\alpha > 0$ can be chosen similarly as shown in [6] (page 136) to satisfy the stability condition

$$J \leq \hat{V}_f(x - x_e; x_e) \quad \forall x \in \mathbb{X}_f(x_e) \tag{11}$$

where

$$J = \begin{array}{l} \min_u \hat{V}_f(\tilde{f}(x,u) - x_e; x_e) + \ell(x - x_e, u - u_e) \\ \text{subject to} \\ u \in \mathbb{U},\ x \in \mathbb{X}_f(x_e) \subset \mathbb{X},\ h(\mathbb{X}_f(x_e)) \subset \Phi_c \end{array} \tag{12}$$

Clearly x_e lies in the interior of $\mathbb{X}_f(x_e)$. Then $V_f(\cdot) \triangleq \lambda \hat{V}_f(\cdot)$ for some $\lambda \geq 1$. Let $\mathbb{M}$ denote the compact set of initial and target states (x_0, x_e) such that $\bar{\mathbb{P}}_N(x_0; x_e)$ is feasible. The parameter λ can be chosen to justify the omission of the terminal constraint in $\mathbb{P}_N$ as shown in the following extension of Proposition 6 in [7]

Proposition 1: For all $\zeta > 0$ there exists a $\lambda_\zeta \triangleq \zeta/\alpha$ such that, for all $(x_0, x_e) \in \mathbb{M}$ and all $t \in \mathbb{I}_{\geq 0}$, $x \in S_\zeta(t; x_0, x_e) \triangleq \{x \mid V_N^0(x, t; x_0, x_e) \leq \zeta\}$ implies that $\bar{\phi}(N; x, \mathbf{u}^0(x, t; x_0, x_e))$ lies in $\mathbb{X}_f(x_e)$ if $\lambda \geq \lambda_\zeta$. ■

Since $(\hat{V}_f(\cdot; x_e), \mathbb{X}_f(x_e))$ satisfies the stability condition, so does $(V_f(\cdot; x_e), \mathbb{X}_f(x_e))$ for all $\lambda \geq 1$. Our assumptions on $\ell(\cdot)$ and $V_f(\cdot)$ ensure that the stability condition is satisfied if $\lambda \geq \max\{1, \lambda_\zeta\}$; we assume this in the sequel.

Proposition 2: [5] For each $(x_0, x_e) \in \mathbb{M}$ there exist constants $c_2 > c_1 > 0$ such that

$$\begin{array}{ll} (i) & V_N^0(x, t; x_0, x_e) \geq c_1 |x - z^*(t; x_0, x_e)|^2 \\ & \qquad \forall (x, t) \in \mathbb{R}^n \times \mathbb{I}_{\geq 0} \\ (ii) & V_N^0(x, t; x_0, x_e) \leq c_2 |x - z^*(t; x_0, x_e)|^2 \\ & \qquad \forall (x, t) \in (\mathbb{X}_\zeta(x_0, x_e) + c_f \mathbb{W}) \times \mathbb{I}_{\geq 0} \\ (iii) & V_N^0(x^+, t^+; x_0, x_e) \leq V_N^0(x, t; x_0, x_e) - \\ & \ell(x - z^*(t; x_0, x_e), \kappa_N(x, t; x_0, x_e) - v^*(t; x_0, x_e)) \\ & \qquad \forall x \in S_\zeta(t; x_0, x_e),\ t \in \mathbb{I}_{\geq 0} \\ (iv) & V_N^0(x^+, t^+; x_0, x_e) \leq \rho V_N^0(x, t; x_0, x_e) \end{array}$$

where $\mathbb{X}_\zeta(x_0, x_e) \triangleq \bigcup_{t \in \mathbb{I}_{[0,N]}} S_\zeta(t; x_0, x_e)$, $\rho \triangleq 1 - c_1/c_2$, $x^+ = \tilde{f}(x, \kappa_N(x, t; x_0, x_e))$, $t^+ \triangleq t + 1$ and c_f is the Lipschitz constant of $f(\cdot)$.

Since the optimal control problem $\mathbb{P}_N$ has no state constraints, $V_N^0(\cdot)$ is Lipschitz continuous in x as stated in the following proposition

Proposition 3: For all $(x_0, x_e) \in \mathbb{M}$ there exists a constant $c_V > 0$ such that $|V_N^0(x, t; x_0, x_e) - V_N^0(z, t; x_0, x_e)| \leq c_V |x - z|$ for all x, z in $\mathbb{X}_\zeta(x_0, x_e) + c_f \mathbb{W}$, all $t > \mathbb{I}_{\geq 0}$. ■

Since $x \in S_\zeta(t; x_0, x_e) \subset \mathbb{X}_\zeta(x_0, x_e)$ implies $\tilde{f}(x, \kappa_N(x, t; x_0, x_e)) \in S_\zeta(t+1; x_0, x_e) \subset \mathbb{X}_\zeta(x_0, x_e)$ and since $f(x, \kappa_N(x, t; x_0, x_e), w) \in S_\zeta(t+1; x_0, x_e) + c_f \mathbb{W} \subset \mathbb{X}_\zeta(x_0, x_e) + c_f \mathbb{W}$, we have:

Corollary 1: For each $(x_0, x_e) \in \mathbb{M}$,

$$\begin{array}{l} V_N^0(x^+, t^+; x_0, x_e) \leq V_N^0(x, t; x_0, x_e) \\ \quad -c_1 |x - z^*(t; x_0, x_e)|^2 + c_V |w| \\ V_N^0(x^+, t^+; x_0, x_e) \leq \rho V_N^0(x, t; x_0, x_e) + c_V |w| \end{array}$$

for all $t \in \mathbb{I}_{\geq 0}$, all $x \in \mathbb{X}_\zeta(x_0, x_e)$ and all $x^+ \in f(x, \kappa_N(x, t; x_0, x_e), \mathbb{W})$. ■

The ancillary controller guarantees, as shown in the following proposition, that the state of the uncertain system belongs to a neighborhood of the reference trajectory.

Proposition 4: Let $S_d(t; x_0, x_e) \triangleq \{x | V_N^0(x, t; x_0, x_e) \leq d\}$ for each $d \in (0, \zeta)$ and suppose $(x_0, x_e) \in \mathbb{M}$. (i) For all $\varepsilon > 0$ there exists a $d(\varepsilon) \triangleq [c_V |\hat{\mathbb{W}}| + \varepsilon]/(1 - \rho)$ with $|\hat{\mathbb{W}}| \triangleq \max_w \{|w| \mid w \in \hat{\mathbb{W}} \subseteq \mathbb{W}\}$ such that, for all $d \geq d(\varepsilon)$, all $t \in \mathbb{I}_{\geq 0}$, $x \in S_\zeta(t; x_0, x_e) \setminus S_d(t; x_0, x_e)$ implies $V_N^0(x^+, t^+; x_0, x_e) \leq V_N^0(x, t; x_0, x_e) - \varepsilon$ and all $x^+ \in f(x, \kappa_N(x, t; x_0, x_e), \hat{\mathbb{W}})$. (ii) There exists a $d^* \triangleq c_V |\hat{\mathbb{W}}|/(1 - \rho)$ such that, for all $d \geq d^*$, all $t \in \mathbb{I}_{\geq 0}$, $x \in S_d(t; x_0, x_e)$ implies $x^+ \in S_d(t^+; x_0, x_e)$ for all $x^+ \in f(x, \kappa_N(x, t; x_0, x_e), \hat{\mathbb{W}})$. ■

Hence any state trajectory starting at (x, t) lying in the state tube $\mathbb{T}_x(x_0, x_e) \triangleq \{S_d(t; x_0, x_e) \mid t \in \mathbb{I}_{\geq 0}\}$ remains in the state tube thereafter and the corresponding control remains in the control tube $\mathbb{T}_u(x_0, x_e) \triangleq \{\kappa_N(S_d(t; x_0, x_e), t; x_0, x_e) \mid t \in \mathbb{I}_{\geq 0}\}$. Moreover, since $d^* \to 0$ as $\hat{\mathbb{W}} \to \{0\}$ in the Hausdorff metric, the state and control tubes shrink to the reference trajectories as $\hat{\mathbb{W}} \to \{0\}$. Then, the state constraints are fulfilled if the following assumption is satisfied for all $(x_0, x_e) \in \mathbb{M}$

Assumption 2:
1) $f : \mathbb{X} \times \mathbb{U} \times \mathbb{W} \to \mathbb{R}^n$ is twice continuously differentiable.
2) $\mathbb{X}$ and $\mathbb{U}$ are compact.
3) $\mathbb{W} = \{w \in \mathbb{R}^q \mid |w| \leq c\}$ for $c^* \leq c < \infty$.
4) $h(\mathbb{X}) \subseteq \Phi_c$
5) $\ell : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$ and $V_f(\cdot)$ are quadratic and positive definite.
6) $V_f(\cdot)$ satisfies the stability condition (11).
7) $d^* \leq d \leq \zeta$ and $\lambda \geq \max\{1, \zeta/\alpha\}$.
8) There exists $\mathbb{Z} \subset \mathbb{X}$ and $\mathbb{V} \subset \mathbb{U}$ such that for all $(x_0, x_e) \in \mathbb{M}$ and all $t \in \mathbb{I}_{[0:N]}$, $S_{d^*}(t; x_0, x_e) \subset \mathbb{X}$ and $h(S_{d^*}(t; x_0, x_e)) \subset \Phi_c$.

Assumption (8) ensures that constraint satisfaction is possible despite the disturbances and it is a necessary assumption when disturbances are present and constraints have to be satisfied. Since the state and control tubes tend, respectively, to the reference state and control trajectories as $\mathbb{W}$ tends to zero, Assumption (8) is satisfied if $\mathbb{W}$ is sufficiently small. Stability analysis is simplified by the fact that the reference trajectory remains constant at x_e for all $t \geq N$. Consequently, the value function $V_N^0(\cdot)$, the control law $\kappa_N(\cdot)$ and the level set $S_d(\cdot)$ are all independent of both t and x_0 for $t \geq N$ so that, for all $t \geq N$,

$$\begin{array}{l} V_N^0(x, t; x_0, x_e) = V_N^{x_e}(x) \triangleq V_N^0(x, N; x_0, x_e) \\ S_d(t; x_0, x_e) = S_d^{x_e} \triangleq \{x | V_N^{x_e}(x) \leq d\} \\ \kappa_N(x, t; x_0, x_e) = \kappa_N^{x_e}(x) \triangleq \kappa_N(x, N; x_0, x_e) \end{array}$$

Theorem 1: Suppose Assumption 2 is satisfied. Then, for each $(x_0, x_e) \in \mathbb{M}$, every state trajectory $\{x(t) \mid t \in \mathbb{I}_{\geq 0}\}$ and control trajectory $\{u(t) \mid t \in \mathbb{I}_{\geq 0}\}$ of the controlled uncertain system $x^+ = f(x, \kappa_N(x, t; x_0, x_e), w)$ with initial state $x(0) = x_0$ lie, respectively, in the state tube $\mathbb{T}_x(x_0, x_e)$ and the corresponding control tube $\mathbb{T}_u(x_0, x_e)$ for all $t \in \mathbb{I}_{\geq 0}$ thereby satisfying the state and control constraints. The state $x(t)$ converges in time N to the set $S_d^{x_e}$ and remains in this set thereafter. Moreover, if the Lipschitz constant $c_V^{x_e}$ for $V_N^{x_e}(x)$ in $S_d^{x_e}$ is such that $c_V^{x_e} < c_V$ and/or $\max\{\gamma(\|h(S_d^{x_e})\|_\infty), c^*\} < c$ there exists a $d^e < d$ such that every solution $x^+ = f(x, \kappa_N^{x_e}(x), w)$ with initial state in $S_d^{x_e}$ converges to $S_{d^e}^{x_e} \subset S_d^{x_e}$ in finite time and remains in this set thereafter.

VI. CHOICE OF CONSTRAINTS

A. Choice of $\mathbb{Z}$ and $\mathbb{V}$

Explicit determination of the tubes $\mathbb{T}_x(x_0, x_e)$ and $\mathbb{T}_u(x_0, x_e)$ is virtually impossible. Instead we propose, as described more fully in [8], that attention is restricted to a finite set $\mathcal{S}$ of pairs $(\mathbb{Z};\mathbb{V})$. If a pair $(\mathbb{Z};\mathbb{V})$ is such that some constraints are not satisfied for a suitably large sample of initial and target states and disturbance sequences, then $(\mathbb{Z};\mathbb{V})$ is replaced by a 'tighter' pair of constraint sets in $\mathcal{S}$; if all constraints are easily satisfied, then $(\mathbb{Z};\mathbb{V})$ is replaced by a 'looser' pair from $\mathcal{S}$. The procedure is continued until an acceptable pair is obtained. Although exact constraint satisfaction cannot be ensured with this procedure, it should be recognised that most design strategies for nonlinear robust control can be too conservative or implicitly require sophisticated global optimization for which exact solutions are computationally demanding; examples are the estimation and exact determination of a Lipschitz constant. This procedure is considerably simpler than the design of a non-linear control law in which the decision variable is also infinite dimensional; the fact that the state and control tubes are bounded makes the procedure possible but tuning and validation are required. Moreover it is worth to point out that the proposed controller is inherently robust because any $x \in S_\zeta(t; x_0, x_e) \setminus S_d(t; x_0, x_e)$, i.e belonging to the controllability set but not to the tube (the state constraints can be violated), is steered to a suitable neighborhood of the reference trajectory and finally of the equilibrium pair where likely the constraints satisfaction is fulfilled.

B. Extension for mixed constraints

The procedure may be adapted to handle systems whose the output ϕ is of the form $\phi = h(x, u)$ assuming that $h(x_e, u_e) \in \mathrm{int}(\Phi_c)$. The difficulty that arises when ϕ is of this form is due to the fact that theorem 1 holds if the ancillary controller is subject only to constraints on the control variables. To achieve this when $\phi = h(x, u)$ requires some modification.

(i) In some cases it is possible to ensure satisfaction of the mixed constraints $x \in \mathbb{X}$, $u \in \mathbb{U}$ and $h(x, u) \in \Phi_c$ by requiring x and u to satisfy

$$x \in \hat{\mathbb{X}} = \{x \in \mathbb{X} \mid h(x, \hat{\mathbb{U}}) \in \Phi_c\} \subset \mathbb{X}$$

and $u \in \hat{\mathbb{U}}$ if $\hat{\mathbb{U}} \subset \mathbb{U}$ is chosen appropriately. In effect we are replacing the mixed constraints $h(x, u) \in \Phi_c$ by the decoupled constraints $x \in \hat{\mathbb{X}}$, $u \in \hat{\mathbb{U}}$; this is possible in some cases albeit conservatively.

(ii) An alternative approach is to replace the constraints $h(x, u) \leq \hat{c} \stackrel{\Delta}{=} \gamma^{-1}(c)$ by $x_a \leq \hat{c}$ where the additional state x_a satisfies

$$\begin{aligned} x_a^+ &= \nu x_a + h(x, u) \\ \phi &= (1 - \nu) x_a \end{aligned}$$

where $|\nu| \ll 1$. Since $h(\cdot)$ is chosen to represent model error, the change modifies slightly the class of permissible uncertainties.

(iii) When the uncertainty class is the set of input-output-to-state stable (IOSS) finite dimensional systems, it is possible to exploit its Lyapunov characterization similarly to the one proposed in [9], [10]. If $\Delta(\phi(\cdot))(t)$ is IOSS there exist α_1, α_2, $\tilde{\gamma}$, $\tilde{\delta}$ of class $\mathcal{K}_\infty$ and a Lyapunov function V_Δ satisfying

$$\alpha_1(|y|) \leq V_\Delta \leq \alpha_2(|y|)$$

where y denotes the state variables of the unmodelled dynamic such that the uncertain system can be modeled in the following way

$$\begin{aligned} V_\Delta^+ &\leq \xi V_\Delta + \tilde{\gamma}(h(x, u)) \\ |w| &\leq \tilde{\delta}(V_\Delta) \end{aligned} \tag{13}$$

where $|\xi| < 1$. The worst case situation is given by

$$V_\Delta^+ = \xi V_\Delta + \tilde{\gamma}(h(x, u)) \tag{14}$$

Then it is possible to add the equation (14) to the control optimization problem and to impose the additional state constraint $0 \leq V_\Delta \leq \overline{V}_\Delta$ where $\overline{V}_\Delta$ is such that $|w(t)| \leq \tilde{\delta}(\overline{V}_\Delta) \leq c$ for all $t \geq 0$. Note that explicit knowledge of a Lyapunov function is not required. V_Δ is regarded as an additional variable with an initial condition satisfying the constraints.

VII. ILLUSTRATIVE EXAMPLE

The effectiveness of the proposed strategy is illustrated with the problem of controlling the outlet temperature of the 30 MWe SEGS VI Parabolic Trough Plant using a simplified model [11], [12]. This solar collector plant is described by three main parts: a collector to absorb the solar energy, a heat exchanger to discharge the heat energy to generate electricity in a power plant, a recycle tube to store the cooled fluid for recycling. The process is controlled by pumping the required flow rate of fluid from recycle tube to the collector. For system modeling, the collector and the recycle tube are discretized spatially at 6 points each. The states of the model represent the temperature at these locations. The first 6 states represent the temperature along the collector and the other 6 states represent the temperature along the recycle tube. The temperature at the outlet of heat exchanger has been approximated with a static gain since the heat exchange is described by a fast dynamic with respect to outlet temperature of the solar collector plant. The neglected dynamics comprising the effect of the power plant has been modeled with a high frequency uncertainty. The system is described by

$$
\begin{array}{ll}
\dot{x}_i = a_1 u(x_{i-1} - x_i) + \theta_1 (T_a - x_i) + s_r & 1 \leq i \leq 6 \\
\dot{x}_i = a_1 u((x_{i-1} + a_2)/(1+\theta_2) - x_i) + w & i = 7 \\
\dot{x}_i = a_1 u(x_{i-1} - x_i) & 7 < i \leq 12 \\
\phi = u - u_e; &
\end{array}
\tag{15}
$$

where the state x_i represents the temperature at the i-th location with $x_0 = x_{12}$, u is the input mass flow rate and the additive w represents (high frequency) uncertainty. The parameters are $a_1 = 2.5 \times 10^{-3}$, $\theta_2 = 5$, $s_r = 5.41 \times 10^{-1}$, $a_2 = 1875.75$, $\theta_1 = 1.19 \times 10^{-3}$, $T_a = 303.15$. The controller is required to steer the system state to the equilibrium state x_e corresponding to desired equilibrium value of 543°K for the variable x_6. The operator $\Delta(\phi(\cdot))$ belongs to the class of finite dimensional asymptotically stable linear systems with input-output state representation

$$
\begin{array}{l}
\dot{y} = A_\Delta y + B_\Delta (u - u_e) \\
w = C_\Delta y + D_\Delta (u - u_e)
\end{array}
\tag{16}
$$

Since $\Delta(\phi(\cdot))$ is a finite dimensional uncertainty we replace Definition 2 of admissible uncertainties by

$$
|\Delta(\phi(\cdot))(t)| \leq \max\{\beta(|y_0|, t), \gamma(\|\phi_{[0,t]}(\cdot)\|_\infty)\} \tag{17}
$$

for all $t \geq 0$. Each admissible uncertainty Δ satisfies (17) with $\beta(s,t) \triangleq \beta_0 e^{-\sigma t} s \leq c$ and $\gamma(s) \triangleq \gamma_0 s \leq c$ where $\sigma > 0$, $\beta_0 = 11$, $c = 0.48$, and $\gamma_0 = 0.19$. Hence the condition $\beta(|y_0|, 0) = \beta_0 |y_0| \leq c$ is satisfied if $|y_0| \leq c/\beta_0 = 0.48/11$. The state, control and disturbance constraints to be fulfilled by the system are

$$
\begin{array}{l}
\hat{\mathbb{X}} = \mathbb{X} = \{x \in \mathbb{R}^n \mid x_i \in [290, 620], i = 1, \ldots, 12\} \\
\mathbb{U} = [0.8, 8], \quad \mathbb{W} = [-0.48, 0.48],
\end{array}
$$

Note that, since ϕ depends only on u, tightening of the input constraints is straightforward; since $u_e = 5.53$, $\hat{\mathbb{U}} = [3, 8]$ implies $|\phi| \leq 2.5$ and $\gamma(|\phi|) \leq c$. Simulations were carried out with the following admissible realization of $\Delta(\cdot)$

$$
\begin{array}{l}
\dot{y} = -by - b(u - u_e) \\
w = K(y + (u - u_e))
\end{array}
\tag{18}
$$

where $b = 0.1$ and $K = 0.086$. The discrete-time model is implicitly obtained via the optimization process using the nonlinear optimization code IPOPT [13] together with the toolbox ICLOCS [8]. The sampling time used for the controller is $T_s = 20$ seconds. The horizon is $N = 250$. The length of the horizon is due to fact that the system is weakly controllable in a neighborhood of (x_e, u_e) and the desired domain of attraction is large. The nominal trajectory is generated by solving the optimal control problem with stage cost $\ell(z, v) = |z|^2 + |v|^2$ and the ancillary control uses the same stage cost $\ell(x, u) = |x|^2 + |u|^2$. The constraint sets for the central path optimal control problem are $\mathbb{Z} = \{z \in \mathbb{R}^n \mid z_i \in [295, 615], i = 1, \ldots, 12\}$ and $\mathbb{V} = \{v \in \mathbb{R}^m \mid v \in [3.4, 7]\}$ and were obtained using the procedure described above. For comparison a standard model predictive controller was determined using the same stage cost $\ell(\cdot)$, the same terminal constraint, the same sampling period and the same horizon as used for the central path controller. Simulations show that tube-based MPC restricts the spread of trajectories compared with standard MPC during the transient phase. Figures 1, 2, 3 illustrate the behavior of the controlled system starting from the initial conditions $x(0) = x_e - (100)\mathbf{1}$ and $y(0) = 0$ where $\mathbf{1}$ denotes a vector of ones having the same dimension as x_e. The disturbance dynamics are fast when compared with respect the system dynamics and the magnitude of the disturbance is significant (see figure 4). The state and control responses are shown in 1, 2, 3.

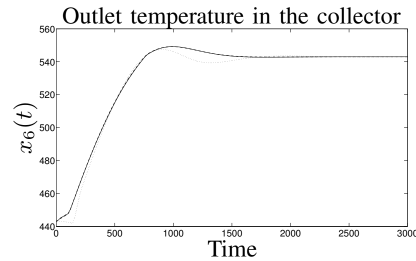

Fig. 1. State variable x_6 vs time: reference trajectory (dashed), perturbed trajectory (solid) and Standard MPC (dotted)

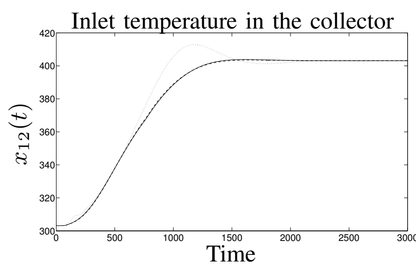

Fig. 2. State variable x_{12} vs time: reference trajectory (dashed), perturbed trajectory (solid) and Standard MPC (dotted)

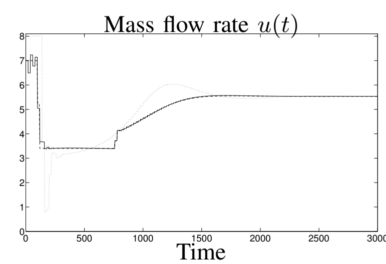

Fig. 3. Input variable u vs time: reference input (dashed), applied input (solid) and Standard MPC (dotted)

The intrinsic robustness of the Tube-MPC is illustrated in figures 5, 6, 7 when the uncertainty is described by (18) with $K = 0.45$. The selected uncertainty does not belong to the admissible class but Tube-MPC counteracts its effect whereas Standard-MPC does not.

VIII. CONCLUSIONS

We have described a method for achieving robust feedback MPC of nonlinear systems with unstructured uncertainty. The

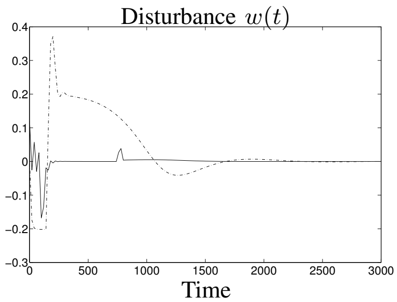

Fig. 4. Behavior of the disturbance w vs time for the Tube-based MPC (solid) and the Standard MPC (dot-dashed)

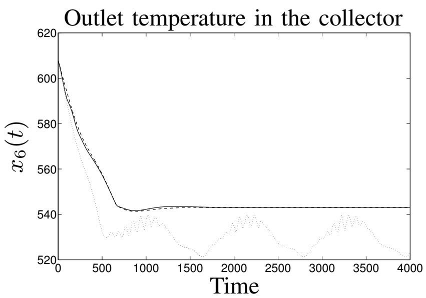

Fig. 5. State variable x_6 vs time: reference trajectory (dashed), perturbed trajectory (solid) and Standard MPC (dotted)

problem of obtaining robustness against unmodeled dynamics has been converted into an easier problem of achieving robustness against bounded disturbances by the addition of an output constraint. The bound on the disturbance and the output constraints are related by a design parameter c. If c is selected suitably, robustness against unstructured uncertainty can be achieved by using tube-based model predictive control. The controller possesses two degrees of freedom providing a useful flexibility in reducing the effect of disturbances as shown in the simulation experiments concerning the control of a solar collector plant. The on-line complexity of the resultant controller is comparable to that for conventional MPC. Off-line tuning and validation is required but this seems to be necessary for any design method for control of constrained, uncertain, nonlinear systems. The control procedure can be extended, using published results, to handle output MPC and additional disturbances.

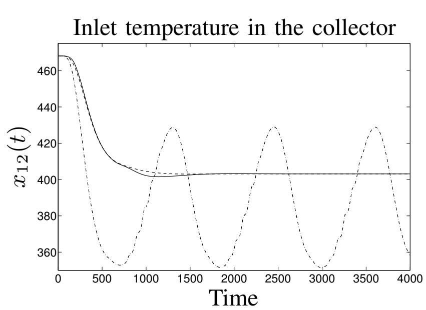

Fig. 6. State variable x_{12} vs time: reference trajectory (dashed), perturbed trajectory (solid) and Standard MPC (dotted)

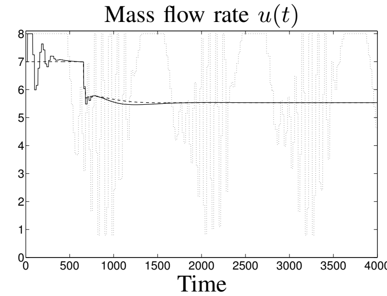

Fig. 7. Input variable u vs time: reference input (dashed), applied input (solid) and Standard MPC (dot-dashed)

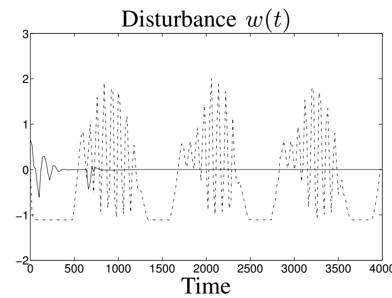

Fig. 8. Behavior of the disturbance w vs time for the Tube-based MPC (solid) and the Standard MPC (dot-dashed)

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