2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Asynchronous regulation of service speed in inventory-production systems with time-varying positive demand

Davide Giglio, Simona Sacone, and Silvia Siri

Abstract— **This paper deals with an inventory-production system in which raw parts are transformed into processed parts, in order to satisfy a time-varying positive demand over a given time horizon. The production resource is capacitated whereas the inventory is unbounded; the external demand is known and is expressed as a piecewise constant function changing at asynchronous time instants. The objective is to find the optimal service speed pattern, assumed to be a piecewise constant function, which minimizes setup, production, and holding costs; hence, the decisions concern both the values of the service speed and the (asynchronous) time instants at which it changes. The optimization problem defined for this class of systems has a parametric structure and includes both nonlinear and combinatorial aspects. In the paper, some structural properties of any optimal solution are firstly proven, and then a solution procedure that allows finding an optimal solution in polynomial times is provided.**

I. INTRODUCTION

The paper addresses a production planning problem for an inventory-production system transforming raw parts into processed parts, in order to satisfy a deterministic external demand over a given time horizon. This demand is a positive piecewise constant function changing at asynchronous time instants. The system considered in this paper can be related to the class of lot sizing models, which has been studied for some decades [1]. Lot sizing is one of the most difficult problems in production planning and the relevant decisions concern when and how much to produce in order to minimize setup, production, and holding costs. Lot sizing models can be classified according to different characteristics, such as the planning horizon (finite or infinite), the number of production levels (single-level or multi-level), the number of products (single-item or multi-item), the capacity constraints (capacitated or uncapacitated), the demand type (static or dynamic, deterministic or probabilistic, dependent or independent), and so on [2]. The first study on lot sizing models dates back to the fifties [3], for the uncapacitated case; since then, a lot of research work has been done by introducing new modelling aspects and by providing various solution algorithms.

An important distinction in the class of lot sizing models concerns the demand satisfaction. In some cases the demand must be satisfied in time (model without backlogging) by producing and, if necessary, storing products [4]; in other cases the demand can be satisfied by backlogging to subsequent periods or it can be outsourced [5]. The model considered in this paper belongs to the class of capacitated lot sizing problems without backlogging, with dynamic and deterministic demand; anyway, it is highly different from the models present in the literature and belonging to this class, since it presents very peculiar characteristics. Specifically, the main novelty of this model concerns timing aspects; in the approaches present in the literature on lot sizing problems the system is either studied and observed continuously or at discrete-time points, corresponding to continuous-time (see for instance [6], [7]) or discrete-time approaches (as in [8], [9]); few works consider a discretized planning horizon in which the time intervals have different lengths [10].

In the present model, the processes representing external demand and production are piecewise constant functions changing in correspondence with asynchronous time instants, i.e. not equally spaced over the time horizon, and some decisions on timing must be taken, as in [11], [12]. More specifically, the decisions concern not only the values of the production speed over time, but also the time instants at which these values change. The decisions on timing make the considered planning problem more complicated than in the case of discrete-time horizon with fixed and known time intervals and, in fact, this planning problem cannot be formulated and solved via mixed-integer programming, as generally done in this field [13]. On the contrary, the optimization problem treated in this paper has a parametric structure and includes non linearities and combinatorial aspects.

As already introduced, the objective of this work is to find the optimal service speed in order to satisfy a positive time-varying external demand over a finite horizon, considering that the production speed cannot exceed a given maximum value, and minimizing setup, production, and holding costs. In the next section, the considered class of production-inventory systems is introduced; the optimization problem is formalized in Section III, and in Section IV the structural properties which characterize the optimal solutions of that problem are given and proven; the procedure which provides an optimal solution of the optimization problem is in section V, and an example is in section VI; in the last section, some conclusions are drawn.

II. THE INVENTORY-PRODUCTION SYSTEM

The inventory-production system considered in this paper consists of a production resource which transforms raw parts (assumed unlimited, that is, always available) into processed parts, an inventory where processed parts are stored, and a departure process which withdraws processed parts from the inventory and delivers them, in accordance with an external

The authors are with the Department of Communication, Computer and Systems Science (DIST), University of Genova, Via Opera Pia 13, 16145 – Genova, Italy. Corresponding author is Davide Giglio. E-mail: davide.giglio@unige.it.

demand (which is assumed to be completely satisfied). Parts stored in the inventory are considered as continuous entities; in this connection, let $x(t)$ be the level of the inventory at time t.

The external demand $e(t)$ is assumed to be a piece-wise constant function of time defined in the interval $[0, T]$, being T the considered time horizon. The values of $e(t)$ represent the rates at which processed parts are withdrawn from the production system (withdrawal rates). The external demand is therefore characterized by the asynchronous time instants at which the withdrawal rate changes, namely θ_k, $k = 1, \dots, Q-1$, and by the withdrawal rates themselves, namely e_k, $k = 1, \dots, Q$, being e_k the rate in time interval $[\theta_{k-1}, \theta_k)$, with $\theta_0 = 0$ and $\theta_Q = T$. In this paper it is assumed $e_k > 0$ $\forall k = 1, \dots, Q$. An example of external demand is illustrated in Fig. 1.

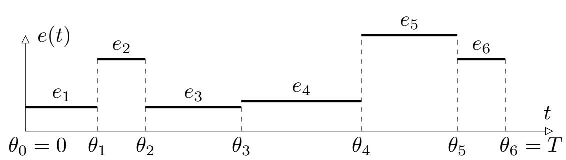

Fig. 1. Example of external demand $e(t)$.

The production resource transforms parts at a certain speed; in this connection, let $u(t)$ be the service speed at time t, that is the number of parts processed by the system per time unit. In the proposed system, the service speed is modelled as a piece-wise constant function of t (see Fig. 2), where u_h, $h = 1, \dots, P$, is the speed within time interval $[\tau_{h-1}, \tau_h)$; moreover, the service speed is upper-bounded by value U. The speed changes at asynchronous time instants τ_h, $h = 1, \dots, P-1$ (then, $u_{h+1} \neq u_h$). It is assumed, of course without loss of generality, $\tau_0 = 0$ and $\tau_P = T$.

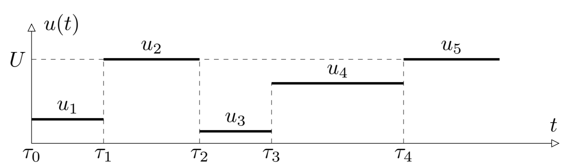

Fig. 2. Example of Production speed $u(t)$.

The state of the system is represented by the level $x(t)$ of the inventory. Such a level obviously increases when the service speed is greater than the withdrawal rate, and decreases when the service speed is lower than the withdrawal rate. In this paper, it is assumed that the inventory is not upper-bounded and is always nonnegative (no backlog allowed). $x(t)$ can be written as

$$\begin{aligned} x(t) &= x(0) + \textstyle\int_0^t u(t)\mathrm{d}t - \int_0^t e(t)\mathrm{d}t = \\ &= x(0) + \textstyle\sum_{h=1}^{p+1} u_h(\tau_h - \tau_{h-1}) - u_{p+1}(\tau_{p+1} - t) \\ &\quad - \textstyle\sum_{k=1}^{q+1} e_k(\theta_k - \theta_{k-1}) + e_{q+1}(\theta_{q+1} - t) \end{aligned} \tag{1}$$

with $p : \tau_p \leq t < \tau_{p+1}$ and $q : \theta_q \leq t < \theta_{q+1}$. Since both the service speed and the external demand are piece-wise constant functions of t, $x(t)$ is a piece-wise linear function.

III. OPTIMIZATION OF SERVICE SPEED

As discussed in the Introduction, the objective of this paper is to state and solve a finite-horizon optimization problem which provides a piece-wise constant function representing the optimal production speed over time that minimizes setup, production, and holding costs. Being the production driven by the external demand, the optimization horizon is set equal to the time interval in which the external demand is known, that is, T.

The decision variables are τ_h, $h = 1, \dots, P-1$, and u_h, $h = 1, \dots, P$. It is worth noting that also the number of "discontinuity points" (that is, the number of time instants at which the production rate changes) is matter of optimization. Then, value P is considered as a decision variable.

The setup cost is

$$\mathcal{C}_\mathrm{S} = \textstyle\sum_{h=0}^{P-1} \sigma y_h \tag{2}$$

being σ the cost to be paid when the production resource passes from an idle state to a working state, and where

$$\begin{aligned} y_0 &= \begin{cases} 0 & \text{if } u_1 = 0 \\ 1 & \text{if } u_1 > 0 \end{cases} \\ y_h &= \begin{cases} 0 & \text{if } u_h > 0 \\ 1 & \text{if } u_h = 0 \end{cases} \qquad h = 1, \dots, P-1 \end{aligned} \tag{3}$$

is a binary variable whose value, for a certain $h \in \{0, \dots, P-1\}$, is equal to 1 if a setup is required at τ_h. It is assumed that the resource is initially idle (at $t = 0$) and that no setup is required at the end of the production process (at $t = T$).

The production cost is

$$\mathcal{C}_\mathrm{P} = \psi \textstyle\sum_{h=1}^{P} u_h(\tau_h - \tau_{h-1}) \tag{4}$$

being ψ the unitary cost for the processing of parts.

The holding cost is

$$\begin{aligned} \mathcal{C}_\mathrm{H} &= \varphi \textstyle\int_0^T x(t)\mathrm{d}t = \\ &= \varphi\big[x_0 T + \textstyle\sum_{h=1}^{P} \frac{1}{2} u_h(\tau_h - \tau_{h-1})^2 \\ &\quad + \textstyle\sum_{h=2}^{P} \sum_{p=1}^{h-1} u_p(\tau_p - \tau_{p-1})(\tau_h - \tau_{h-1}) \\ &\quad - \textstyle\sum_{k=1}^{Q} \frac{1}{2} e_k(\theta_k - \theta_{k-1})^2 \\ &\quad - \textstyle\sum_{k=2}^{Q} \sum_{q=1}^{k-1} e_q(\theta_q - \theta_{q-1})(\theta_k - \theta_{k-1})\big] \end{aligned} \tag{5}$$

being $x_0 = x(0)$ the initial inventory level, and φ the unitary cost of inventory.

In the proposed model, no backlog is allowed. Then, the inventory level must be non-negative at each time instant, that is, $x(t) \geq 0$, $0 \leq t \leq T$. However, since $x(t)$ is a piece-wise linear function and assuming $x_0 \geq 0$, it is sufficient to check the non-negativity at τ_h, $h = 1, \dots, P$, and at θ_k, $k = 1, \dots, Q$, which are the asynchronous time instants in which the slope of $x(t)$ changes. In these time instants, the inventory level is provided by

$$\begin{aligned} x(\tau_h) &= x_0 + \textstyle\sum_{p=1}^{h} u_p(\tau_p - \tau_{p-1}) \\ &\quad - \textstyle\sum_{k=1}^{Q} \rho_{h,k} e_k(\theta_k - \theta_{k-1}) \\ &\quad - \textstyle\sum_{k=1}^{Q} (\rho_{h,k-1} - \rho_{h,k}) e_k(\tau_h - \theta_{k-1}) \end{aligned} \tag{6}$$

$$x(\theta_k) = x_0 + \sum_{h=1}^{P}(1-\rho_{h,k})u_h(\tau_h - \tau_{h-1}) \\ + \sum_{h=1}^{P}(\rho_{h,k} - \rho_{h-1,k})u_h(\theta_k - \tau_{h-1}) \\ - \sum_{q=1}^{k} e_q(\theta_q - \theta_{q-1}) \tag{7}$$

where

$$\rho_{h,k} = \begin{cases} 0 & \text{if } \tau_h \le \theta_k \\ 1 & \text{if } \tau_h > \theta_k \end{cases} \tag{8}$$

is a binary variable which defines, for any pair (h,k), $h = 0,\dots,P$ and $k = 0,\dots,Q$, the relative position between τ_h and θ_k. Since it has been assumed $\tau_0 = \theta_0 = 0$ and $\tau_P = \theta_Q = T$, it turns out

$$\begin{cases} \rho_{0,k} = 0 & \forall\, k = 0,\dots,Q \\ \rho_{h,0} = 1 & \forall\, h = 1,\dots,P \\ \rho_{P,k} = 1 & \forall\, k = 1,\dots,Q-1 \\ \rho_{h,Q} = 0 & \forall\, h = 1,\dots,P \end{cases} \tag{9}$$

The optimization problem for the considered production system can be defined as follows.

Problem 1: Given the initial inventory level $x_0 \ge 0$, given the maximum production speed U, given the time horizon T, and given the external demand $e(t)$ (expressed by θ_k, $k = 0,\dots,Q$, with $\theta_0 = 0$ and $\theta_Q = T$, and e_k, $k = 1,\dots,Q$), find

$$\min_{\substack{P \\ \tau_h,\, h=1,\dots,P-1 \\ u_h,\, h=1,\dots,P \\ y_h,\, h=0,\dots,P-1 \\ \rho_{h,k},\, h=1,\dots,P-1,\, k=1,\dots,Q-1}} \mathcal{C}_\mathrm{S} + \mathcal{C}_\mathrm{P} + \mathcal{C}_\mathrm{H} \tag{10}$$

being $\mathcal{C}_\mathrm{S}$, $\mathcal{C}_\mathrm{P}$, and $\mathcal{C}_\mathrm{H}$ provided by (2), (4), and (5), respectively, with $\tau_0 = 0$, $\tau_P = T$, subject to

$$x(\tau_h) \ge 0, \quad h = 1,\dots,P \tag{11}$$
$$x(\theta_k) \ge 0, \quad k = 1,\dots,Q \tag{12}$$
$$\tau_h > \tau_{h-1}, \quad h = 1,\dots,P \tag{13}$$
$$0 \le u_h \le U, \quad h = 1,\dots,P \tag{14}$$
$$\tau_h - \theta_k + M(1-\rho_{h,k}) > 0, \quad \begin{matrix} h = 1,\dots,P-1 \\ k = 1,\dots,Q-1 \end{matrix} \tag{15}$$
$$\theta_k - \tau_h + M\rho_{h,k} \ge 0, \quad \begin{matrix} h = 1,\dots,P-1 \\ k = 1,\dots,Q-1 \end{matrix} \tag{16}$$
$$|u_h - u_{h+1}| > 0, \quad h = 1,\dots,P-1 \tag{17}$$
$$u_1 + N(1-y_0) > 0 \tag{18}$$
$$-u_1 + N y_0 \ge 0 \tag{19}$$
$$u_h + N y_h > 0, \quad h = 1,\dots,P-1 \tag{20}$$
$$-u_h + N(1-y_h) \ge 0, \quad h = 1,\dots,P-1 \tag{21}$$
$$P \in \mathbb{N}_{>0} \tag{22}$$
$$\rho_{h,k} \in \{0,1\}, \quad \begin{matrix} h = 1,\dots,P-1 \\ k = 1,\dots,Q-1 \end{matrix} \tag{23}$$
$$y_h \in \{0,1\}, \quad h = 0,\dots,P-1 \tag{24}$$

where $x(\tau_h)$ and $x(\theta_k)$ are respectively provided by (6) and (7) (taking into account (9)), and M and N are positive numbers that are sufficiently large to guarantee that, in an optimal solution, the values of binary variables $\rho_{h,k}$ and y_h are in accordance with (8) and (3), respectively.

Problem 1 has a parametric structure since P belongs to the set of decision variables; moreover, it includes non linearities and combinatorial aspects. The following theorem provides necessary and sufficient conditions for the existence of a solution for Problem 1.

Theorem 1: Necessary and sufficient conditions for the existence of a solution for Problem 1 are

$$x_0 + U(\theta_k - \theta_0) \ge \sum_{q=1}^{k} e_q(\theta_q - \theta_{q-1}) \tag{25}$$

for any $k = 1,\dots,Q$. □

Proof: Assume that condition (25) is not satisfied for a certain $k \in \{1,\dots,Q\}$. Then, it turns out $x(\theta_k) < 0$, which violates constraint (12). For this reason (25) are necessary conditions. Assume now that conditions (25) are satisfied for all $k \in \{1,\dots,Q\}$. Then, the specific solution $S^\star$ for Problem 1, characterized by $P = 1$, $u_1 = U$, and $y_0 = 1$ (this solution refers to the case in which parts are produced at the constant speed U throughout the interval $[0,T]$) respects all constraints and then it is a feasible solution of Problem 1. For this reason (25) are sufficient conditions. ■

IV. PROPERTIES OF THE OPTIMAL SOLUTIONS

The following theorems provide some structural properties of any optimal solution of Problem 1.

Theorem 2: In any optimal solution of Problem 1, the number of parts to be produced is $n_\mathrm{P} = \max\{0, \sum_{k=1}^{Q} e_k(\theta_k - \theta_{k-1}) - x_0\}$. □

Proof: In the considered problem, there is no advantage in producing more parts than those which are necessary to satisfy exactly the external demand, since the production of more parts would result in an equal or higher setup cost, a higher production cost, and a higher holding cost. Moreover, it is not possible to produce less parts then the necessary since the external demand is assumed to be satisfied. Then, in any optimal solution of Problem 1, the number of parts to be produced is always equal to the total demand minus the initial inventory; obviously, if the initial inventory is higher than the total demand, no parts are produced. ■

Theorem 2 also allows determining the inventory at the final time instant depending on the inventory at the initial time instant. As a matter of fact, if $x_0 \le \sum_{k=1}^{Q} e_k(\theta_k - \theta_{k-1})$, then, $x(T) = 0$; otherwise, $x(T) = x_0 - \sum_{k=1}^{Q} e_k(\theta_k - \theta_{k-1})$. A further consequence of the above theorem is reported in the following result.

Corollary 1: In any optimal solution of Problem 1, the production cost is always equal to $\psi\, n_\mathrm{P}$. □

Proof: The number of parts to be produced is n_P, as stated by Theorem 2; then, $\sum_{h=1}^{P} u_h(\tau_h - \tau_{h-1}) = n_\mathrm{P}$. Hence, according to (4), the production cost is $\psi\, n_\mathrm{P}$. ■

Theorem 3: A necessary condition for the existence of an idle period in any optimal solution of Problem 1, that is, $u_h = 0$ for some $[\tau_{h-1}, \tau_h) \subseteq [0,T]$, is $x_0 > 0$. If $x_0 > 0$ then there is one idle period at most; when present, it is at the beginning of the considered time horizon. □

Proof: First of all, it is proven that, in any optimal solution, an idle period cannot exist in the middle of the production process. To this end, assume the existence of a feasible solution in which n_P parts are produced (in accordance with Theorem 2) and in which an idle period exists in the interval $[\tau_h, \tau_{h+1})$ (assume for simplicity $[\tau_h, \tau_{h+1}) \subset [\theta_{k-1}, \theta_k)$). Let S' be such solution and let $x'(t)$ be the level of inventory at t when S' is adopted; moreover, let u'_h, with $0 < u'_h \leq U$ be the service speed in the interval $[\tau_{h-1}, \tau_h)$. Consider now another solution in which n_{P} parts are produced, no idle period is present in $[\tau_h, \tau_{h+1})$, and the service speeds in the intervals $[0, \tau_{h-1})$ and $[\tau_{h+1}, T]$ are equal to those in S'. Let S'' be such solution and let $x''(t)$ be the level of inventory at t when S'' is adopted; moreover, let $u''_h > 0$ and $u''_{h+1} > 0$ be, respectively, the service speed in $[\tau_{h-1}, \tau_h)$ and in $[\tau_h, \tau_{h+1})$. Since the resource produces, both in S' and in S'', the same number of parts, it is assumed that the two speeds u''_h and u''_{h+1} satisfy the condition $u'_h(\tau_h - \tau_{h-1}) = u''_h(\tau_h - \tau_{h-1}) + u''_{h+1}(\tau_{h+1} - \tau_h)$. Then, assuming u''_{h+1} to be a positive value arbitrarily small (that is, less than or equal to U), it turns out $u''_h < u'_h \leq U$. Thus, S'' is a feasible solution. However, the solution S' yields an extra cost equal to $\sigma + \frac{1}{2}\varphi u''_{h+1}(\tau_{h+1} - \tau_h)(\tau_{h+1} - \tau_{h-1})$, with respect to the cost yielded by S''. This means that a solution in which an idle period is present in the middle of the production process cannot be optimal.

Consider now the possibility of having an idle time at the beginning of the considered interval $[0, T]$. It is evident that no idle period exists in the case $x_0 = 0$, since the production must start immediately in order to satisfy the positive external demand. This means that $x_0 > 0$ is a necessary condition for the existence of an idle period in any optimal solution of Problem 1. Consider the case $x_0 > 0$. First of all, if $x_0 \geq \sum_{k=1}^{Q} e_k(\theta_k - \theta_{k-1})$ no parts are produced and then the whole time horizon $[0, T]$ is an idle period. Moreover, if $x_0 = \sum_{q=1}^{k}(e_q - U)(\theta_q - \theta_{q-1})$ for some $k \in \{1, \ldots, Q\}$, then it is necessary to produce at speed U within the time horizon $[0, \theta_k)$ in order to satisfy (25); thus, also in this case, no idle period is present in an optimal solution of Problem 1. This means that $x_0 > 0$ is not a sufficient condition for the existence of an idle period in an optimal solution of Problem 1.

Assume now $x_0 > 0$, $x_0 < \sum_{k=1}^{Q} e_k(\theta_k - \theta_{k-1})$, and $x_0 > \sum_{q=1}^{k}(e_q - U)(\theta_q - \theta_{q-1})$ for all $k = 1, \ldots, Q$, and let us prove that, in any optimal solution, an idle period exists at the beginning of the considered time horizon. To prove that an idle period exists at the beginning of the considered time horizon, assume the existence of a feasible solution in which n_{P} parts are produced and in which no idle period exists. Let S' be such solution and let $x'(t)$ and τ'_h, $h = 1, 2, \ldots$, be, respectively, the level of inventory at t and the time instants at which the service speed changes, when S' is adopted; moreover, let u'_1, with $0 < u'_1 < U$, be the service speed at the beginning, that is, in the interval $[\theta_0, \tau'_1)$ (condition $x_0 > \sum_{q=1}^{k}(e_q - U)(\theta_q - \theta_{q-1})$ for all $k = 1, \ldots, Q$ allows choosing, at the beginning, a service speed u'_1 lower than the maximum speed U). Consider now another solution in which n_{P} parts are produced and an idle period is present at the beginning. Let S'' be such solution and let $x''(t)$ and τ''_h, $h = 1, 2, \ldots$, be, respectively, the level of inventory at t and the time instants at which the service speed changes, when S'' is adopted. In this solution, it is assumed that the idle period ends at τ''_1, and $\tau''_2 = \tau'_1$; moreover, the service speed in the interval $[\tau''_1, \tau''_2)$ is such that $x''(\tau''_2) = x'(\tau'_1)$, and the speeds in $[\tau''_2, T]$ are equal to those in S'. In order to have $x''(\tau''_2) = x'(\tau'_1)$, the speed u''_2 and the time instant τ''_1 must satisfy the condition $u'_1\tau'_1 = u''_2(\tau''_2 - \tau''_1)$; since $u'_1 < U$, then a pair (τ''_1, u''_2), with $0 < \tau''_1 < \tau'_1$ and $0 < u''_2 \leq U$, certainly exists. Thus, S'' is a feasible solution. However, the solution S' yields an extra cost equal to $\frac{1}{2}\varphi u''_2\tau''_1(\tau''_2 - \tau''_1)$, with respect to the cost yielded by S''. This means that a solution in which no idle period exists at the beginning cannot be optimal in the case $x_0 > 0$, $x_0 < \sum_{k=1}^{Q} e_k(\theta_k - \theta_{k-1})$, and $x_0 > \sum_{q=1}^{k}(e_q - U)(\theta_q - \theta_{q-1})$ for all $k = 1, \ldots, Q$.

In conclusion, since it has been proven that an idle period cannot exist in the middle of the production process, then in the case $x_0 > 0$ there is one idle period at the beginning of the considered time horizon at most. ■

Remark 1: With reference to the proof of Theorem 3, the time instant at which the idle period ends, namely $\tilde{\tau}$, can be determined as the farthest (from 0) time instant for which the solution remains feasible (such time instant will be determined inside the solution procedure proposed in the following section). □

Corollary 2: In any optimal solution of Problem 1, the setup cost is paid one time at most, that is $\sum_{h=0}^{P-1} y_h \leq 1$.□

Proof: According to Theorem 3, the production resource passes from an idle to a working state only one time if $x_0 < \sum_{k=1}^{Q} e_k(\theta_k - \theta_{k-1})$, and remains in the idle state otherwise. Then the setup cost is paid one time at most. ■

Theorem 4: In any optimal solution of Problem 1, the production speed in the generic interval $[\theta_{k-1}, \theta_k)$ of the working period is either e_k or U, in case $e_k < U$, and U, in case $e_k \geq U$. □

Proof: Consider firstly the case $e_k < U$. In this connection, consider a feasible solution S' and, with reference to generic interval $[\theta_{k-1}, \theta_k)$ of the working period (in which the withdrawal rate is e_k), let S' be characterized by two changes of production speed, namely τ_{h-1} and τ_h with $[\tau_{h-1}, \tau_h) \subseteq [\theta_{k-1}, \theta_k)$ and by the speed values $u_{h-1} = e_k$, $u_h = \tilde{u}$, and $u_{h+1} = e_k$.

Assume $0 < \tilde{u} < e_k$. Since the external demand has been assumed to be always satisfied, at time instant τ_{h-1} a number of parts equal to $(e_k - \tilde{u})(\tau_h - \tau_{h-1})$ must be available in the inventory; since $u_{h-1} = e_k$, such parts must have been produced before time instant θ_{k-1}. This yields an extra cost of inventory of at least $\frac{1}{2}\varphi(e_k - \tilde{u})(\tau_h - \tau_{h-1})(\tau_h + \tau_{h-1} - 2\theta_{k-1})$, with respect to the cost yielded by a solution in which the production speed is e_k in the whole interval $[\theta_{k-1}, \theta_k)$, and no parts are produced in advance. Then, in this case, S' cannot be optimal.

Assume now $e_k < \tilde{u} < U$. In this case, two situations must be considered. In the first situation, the production at speed $\tilde{u} > e_k$ is justified by the need of producing and storing parts, in order to satisfy future external demands that are characterized by a withdrawal rate higher than the maximum allowable speed U. Then, in the solution S', a fraction of the production effort in the interval $[\tau_{h-1}, \tau_h)$

is dedicated to this aim; more specifically, the number of parts produced and stored is $(\tilde{u}-e_k)(\tau_h-\tau_{h-1})$. Consider now another feasible solution, namely S'', in which the production speed in the interval $[\theta_{k-1},\theta_k)$ is e_k in $[\theta_{k-1},\hat{\tau})$, with $\hat{\tau}=\theta_k-\frac{(\tilde{u}-e_k)(\tau_h-\tau_{h-1})}{U-e_k}$, and U in $[\hat{\tau},\theta_k)$. It is evident that the number of parts produced in $[\theta_{k-1},\theta_k)$ is the same in both solutions, and then S'' is feasible being S' feasible. With respect to the considered interval, the solution S' yields an extra cost equal to $\frac{1}{2}\varphi(\tilde{u}-e_k)(\tau_h-\tau_{h-1})(2\theta_k-\tau_h-\tau_{h-1})-\frac{1}{2}\varphi(U-e_k)(\theta_k-\hat{\tau})^2$, with respect to the cost yielded by S''. Then, in this situation, S' cannot be optimal. In the second situation, there is no need of producing and storing parts in advance and then the speed $\tilde{u}>e_k$ is not justified by feasibility issues. This yields an extra cost of at least $\frac{1}{2}\varphi(\tilde{u}-e_k)(\tau_h-\tau_{h-1})(2\theta_k-\tau_h-\tau_{h-1})$, with respect to the cost yielded by a solution in which the production speed is e_k in the whole interval $[\theta_{k-1},\theta_k)$. Then, also in this situation, S' cannot be optimal.

Summarizing, when $e_k<U$ in the interval $[\theta_{k-1},\theta_k)$, there is no advantage in producing parts at a speed different from e_k or U.

Consider now the case $e_k\geq U$. As before, consider a feasible solution S', and let S' be now characterized by the presence of a time interval $[\tau_{h-1},\tau_h)\subseteq[\theta_{k-1},\theta_k)$ with the speed values $u_{h-1}=U$, $u_h=\tilde{u}<U$, and $u_{h+1}=U$. Since the external demand has been assumed to be always satisfied, at time instant θ_{k-1} a number of parts equal to $(e_k-U)(\theta_k-\theta_{k-1})+(U-\tilde{u})(\tau_h-\tau_{h-1})$ must be available in the inventory. Consider now another feasible solution, namely S'', in which the production speed in the interval $[\theta_{k-1},\theta_k)$ is always U. In this case the number of parts that must be available in the inventory at θ_{k-1} is equal to $(e_k-U)(\theta_k-\theta_{k-1})$ only. Then, the solution S' yields, with respect to the cost yielded by S'', an extra cost of at least $\frac{1}{2}\varphi(U-\tilde{u})(\tau_h-\tau_{h-1})(\tau_h+\tau_{h-1}-2\theta_{k-1})$. Then, S' cannot be optimal.

This means that, when $e_k\geq U$ in the interval $[\theta_{k-1},\theta_k)$, there is no advantage in producing parts at a speed different from U. ∎

It is worth finally observing that the above theorems and corollaries allow stating that neither the setup cost nor the production cost influence the solution of Problem 1 (as a consequence, the solution is not affected by the unitary costs σ, ψ, and φ).

V. The Solution Procedure

An optimal solution of Problem 1 is provided by the following theorem.

Theorem 5: An optimal solution of Problem 1 can be obtained through the following procedure:

1) determine the set of values $\{\xi_k, k=0,\dots,Q\}$ ("residual stocks") through the forward procedure

$$\xi_k=\xi_{k-1}-e_k(\theta_k-\theta_{k-1}) \tag{26}$$

$k=1,\dots,Q$, initialized by $\xi_0=x_0$;

2) determine the set of values $\{X_k, k=0,\dots,Q\}$ ("safety stocks") through the backward procedure

$$X_k=\max\left\{0, X_{k+1}+(e_{k+1}-U)(\theta_{k+1}-\theta_k)\right\} \tag{27}$$

$k=Q-1,\dots,0$, initialized by $X_Q=0$;

3) apply the following algorithm which provides the optimal values of decision variables P, τ_h, $h=1,\dots,P-1$, u_h, $h=1,\dots,P$, y_h, $h=0,\dots,P-1$, and $\rho_{h,k}$, $h=1,\dots,P-1$, $k=1,\dots,Q-1$:

if $(\xi_Q\geq 0)$ **then**
  $y_0=0$, $u_1=0$, $P=1$
else
  $h=0$, $k=1$
  if $(\xi_0>0)\wedge(\xi_0>X_0)$ **then**
    $y_0=0$
    while $\neg\big\{\big[(\xi_{k-1}\geq 0)\wedge(X_{k-1}\leq\xi_{k-1})\wedge(\xi_k<0)\big]\vee\big[(X_{k-1}\leq\xi_{k-1})\wedge(X_k>\xi_k)\wedge(\xi_k>0)\big]\big\}$ **do**
      $k=k+1$
    end while
    if $\neg\big[(\xi_{k-1}=0)\vee(X_{k-1}=\xi_{k-1})\big]$ **then**
      if $(\xi_k<0)\wedge(X_{k-1}=0)\wedge(X_k=0)$ **then**
        $u_1=0$, $\tau_1=\theta_{k-1}+\frac{\xi_{k-1}}{e_k}$, $y_1=1$
        if $k>1$ **then**
          for j = 1 **to** $k-1$ **do** $\rho_{1,j}=1$
        if $k<Q$ **then**
          for j = k **to** $Q-1$ **do** $\rho_{1,j}=0$
        $u_2=e_k$, $h=2$
      else if $(\xi_k<0)\wedge(X_{k-1}=0)\wedge(X_k>0)\wedge\big(\frac{X_k}{U-e_k}<\frac{-\xi_k}{e_k}\big)$ **then**
        $u_1=0$, $\tau_1=\theta_{k-1}+\frac{\xi_{k-1}}{e_k}$, $y_1=1$
        if $k>1$ **then**
          for j = 1 **to** $k-1$ **do** $\rho_{1,j}=1$
        if $k<Q$ **then**
          for j = k **to** $Q-1$ **do** $\rho_{1,j}=0$
        $u_2=e_k$, $\tau_2=\theta_k-\frac{X_k}{U-e_k}$, $y_2=0$
        if $k>1$ **then**
          for j = 1 **to** $k-1$ **do** $\rho_{2,j}=1$
        if $k<Q$ **then**
          for j = k **to** $Q-1$ **do** $\rho_{2,j}=0$
        $u_3=U$, $h=3$
      else if $\big[(\xi_k<0)\wedge(X_{k-1}=0)\wedge(X_k>0)\wedge\big(\frac{X_k}{U-e_k}\geq\frac{-\xi_k}{e_k}\big)\big]\vee\big[(X_{k-1}<\xi_{k-1})\wedge(\xi_k<0)\wedge(X_{k-1}>0)\big]\vee\big[(\xi_k>0)\big]$ **then**
        $u_1=0$, $\tau_1=\theta_k-\frac{X_k-\xi_k}{U}$, $y_1=1$
        if $k>1$ **then**
          for j = 1 **to** $k-1$ **do** $\rho_{1,j}=1$
        if $k<Q$ **then**
          for j = k **to** $Q-1$ **do** $\rho_{1,j}=0$
        $u_2=U$, $h=2$
      end if
      $k=k+1$
    end if
  end if
  for i = k **to** Q **do**
    if $(X_{i-1}=0)\wedge(X_i=0)$ **then**
      if $(i\geq 2)$ **then**
        $\tau_h=\theta_{i-1}$, $y_h=0$
        if $i>2$ **then**
          for j = 1 **to** $i-2$ **do** $\rho_{h,j}=1$
        if $i<Q$ **then**
          for j = $i-1$ **to** $Q-1$ **do** $\rho_{h,j}=0$
      else if $(i=1)$ **then**

50: $y_h = 1$
51: **end if**
52: $u_{h+1} = e_i$, $h = h + 1$
53: **else if** $(X_{i-1} = 0) \wedge (0 < X_i < (U - e_i)(\theta_i - \theta_{i-1}))$ **then**
54: **if** $(i \geq 2)$ **then**
55: $\tau_h = \theta_{i-1}$, $y_h = 0$
56: **if** $i > 2$ **then**
57: **for** j = 1 **to** $i - 2$ **do** $\rho_{h,j} = 1$
58: **if** $i < Q$ **then**
59: **for** j = $i - 1$ **to** $Q - 1$ **do** $\rho_{h,j} = 0$
60: **else if** $(i = 1)$ **then**
61: $y_h = 1$
62: **end if**
63: $u_{h+1} = e_i$, $\tau_{h+1} = \theta_i - \frac{X_i}{U - e_i}$, $y_{h+1} = 0$
64: **if** $i > 1$ **then**
65: **for** j = 1 **to** $i - 1$ **do** $\rho_{h+1,j} = 1$
66: **if** $i < Q$ **then**
67: **for** j = i **to** $Q - 1$ **do** $\rho_{h+1,j} = 0$
68: $u_{h+2} = U$, $h = h + 2$
69: **else if** $(X_{i-1} > 0) \vee (X_i = (U - e_i)(\theta_i - \theta_{i-1}))$ **then**
70: **if** $(i \geq 2) \wedge \left[(X_i = (U - e_i)(\theta_i - \theta_{i-1})) \wedge (e_{i-1} < U)\right]$ **then**
71: $\tau_h = \theta_{i-1}$, $y_h = 0$
72: **if** $i > 2$ **then**
73: **for** j = 1 **to** $i - 2$ **do** $\rho_{h,j} = 1$
74: **if** $i < Q$ **then**
75: **for** j = $i - 1$ **to** $Q - 1$ **do** $\rho_{h,j} = 0$
76: $u_{h+1} = U$, $h = h + 1$
77: **else if** $(i = 1)$ **then**
78: $y_h = 1$, $u_{h+1} = U$, $h = h + 1$
79: **end if**
80: **end if**
81: **end for**
82: $P = h$
83: **end if** □

Sketch of the Proof: First of all, consider the two sets of values which are determined at steps 1) and 2) of the procedure. The "residual stock" ξ_k, $k \in \{0, \ldots, Q\}$, represents the portion of the initial inventory level which is still available at time instant θ_k, after having satisfied the external demands until that time instant; negative values of ξ_k mean that the initial inventory has been entirely consumed and then it is necessary to produce parts to satisfy the demand. Instead, the "safety stock" X_k, $k \in \{0, \ldots, Q\}$, represents the number of parts that the production resource must guarantee at θ_k in order to satisfy the future demands, from θ_k onwards; in other words, in order to meet the future demands, it is necessary that the inventory contains, at θ_k, at least X_k parts; $X_k = 0$ means that the demand in the following interval, namely $[\theta_k, \theta_{k+1})$, can be satisfied with the production from θ_k onwards.

It has been mentioned in Remark 1 that, in any optimal solution of Problem 1, the time instant at which the idle period ends is the farthest (from 0) time instant for which the solution remains feasible; in other words, the idle period, if present, lasts as much as possible. It is also possible to show (the proof is not reported due to the lack of space) that such a time instant is

$$\tau_1 = \theta_{k-1} + \frac{\xi_{k-1}}{e_k} \tag{28}$$

if $(\xi_{k-1} \geq 0) \wedge (\xi_k < 0) \wedge (X_{k-1} = 0) \wedge \left(\frac{X_k}{U - e_k} < \frac{-\xi_k}{e_k}\right)$, or

$$\tau_1 = \theta_k - \frac{X_k - \xi_k}{U} \tag{29}$$

otherwise, where $k \in \{1, \ldots, Q\}$ is such that $[(\xi_{k-1} \geq 0) \wedge (X_{k-1} \leq \xi_{k-1}) \wedge (\xi_k < 0)] \vee [(X_{k-1} \leq \xi_{k-1}) \wedge (X_k > \xi_k) \wedge (\xi_k > 0)]$.

On the basis of this result, it is possible to show (again, the proof is not reported) that, in any optimal solution of Problem 1, the optimal service speed in the time interval in which the idle period (if present) ends, namely $[\theta_{k-1}, \theta_k)$, with $k \in \{1, \ldots, Q\}$ such that $[(\xi_{k-1} > 0) \wedge (X_{k-1} < \xi_{k-1}) \wedge (\xi_k < 0)] \vee [(X_{k-1} < \xi_{k-1}) \wedge (X_k > \xi_k) \wedge (\xi_k > 0)]$, is equal to:

- in case $(\xi_k < 0) \wedge (X_{k-1} = 0) \wedge (X_k = 0)$:

$$\begin{cases} u_1 = 0 & \text{in } [\theta_{k-1}, \tau_1) \\ u_2 = e_k & \text{in } [\tau_1, \theta_k) \end{cases} \tag{30}$$

  and $\tau_1 = \theta_{k-1} + \frac{\xi_{k-1}}{e_k}$ is the time instant at which the production starts;
- in case $(\xi_k < 0) \wedge (X_{k-1} = 0) \wedge (X_k > 0) \wedge \left(\frac{X_k}{U - e_k} < \frac{-\xi_k}{e_k}\right)$:

$$\begin{cases} u_1 = 0 & \text{in } [\theta_{k-1}, \tau_1) \\ u_2 = e_k & \text{in } [\tau_1, \tau_2) \\ u_3 = U & \text{in } [\tau_2, \theta_k) \end{cases} \tag{31}$$

  and $\tau_1 = \theta_{k-1} + \frac{\xi_{k-1}}{e_k}$ is the time instant at which the production starts, whereas $\tau_2 = \theta_k - \frac{X_k}{U - e_k}$ is the time instant at which the service speed changes to U;
- in case $(\xi_k < 0) \wedge (X_{k-1} = 0) \wedge (X_k > 0) \wedge \left(\frac{X_k}{U - e_k} \geq \frac{-\xi_k}{e_k}\right)$ or in case $(X_{k-1} < \xi_{k-1}) \wedge (\xi_k < 0) \wedge (X_{k-1} > 0)$ or in case $(\xi_k > 0)$:

$$\begin{cases} u_1 = 0 & \text{in } [\theta_{k-1}, \tau_1) \\ u_2 = U & \text{in } [\tau_1, \theta_k) \end{cases} \tag{32}$$

  and $\tau_1 = \theta_k - \frac{X_k - \xi_k}{U}$ is the time instant at which the production starts.

Moreover, as corollary of Theorem 4, it can be also shown that, in any optimal solution of Problem 1, the optimal service speed in the generic interval of the working period, namely $[\theta_{k-1}, \theta_k)$, for any $k \in \{1, \ldots, Q\}$ such that $(\xi_{k-1} \leq 0) \vee (\xi_{k-1} = X_{k-1})$, is equal to:

- in case $(X_{k-1} = 0) \wedge (X_k = 0)$:

$$u_{h+1} = e_k \tag{33}$$

  and $\tau_h = \theta_{k-1}$ is the time instant at which the service speed changes to e_k (when $k \geq 2$) or the time instant at which the production starts (when $k = 1$);
- in case $(X_{k-1} = 0) \wedge (0 < X_k < (U - e_k)(\theta_k - \theta_{k-1}))$:

$$\begin{cases} u_{h+1} = e_k & \text{in } [\theta_{k-1}, \tau_{h+1}) \\ u_{h+2} = U & \text{in } [\tau_{h+1}, \theta_k) \end{cases} \tag{34}$$

  and $\tau_h = \theta_{k-1}$ is the time instant at which the service speed changes to e_k (when $k \geq 2$) or the time instant at which the production starts (when $k = 1$), whereas $\tau_{h+1} = \theta_k - \frac{X_k}{U - e_k}$ is the time instant at which the service speed changes to U;

- in case $(X_{k-1} > 0)$ or in case $(X_k = (U - e_k)(\theta_k - \theta_{k-1}))$:

$$u_{h+1} = U \tag{35}$$

and $\tau_h = \theta_{k-1}$ is the time instant at which the service speed changes to U (when $k \geq 2$ and if $(X_k = (U - e_k)(\theta_k - \theta_{k-1})) \wedge (e_{k-1} < U)$; otherwise no change of service speeds occurs within the considered interval) or the time instant at which the production starts (when $k = 1$);

being $h \in \{0, \ldots, P - 1\}$ the number of service speed changes occurring before θ_{k-1}.

The algorithm at step 3 of the procedure applies, in an algorithmic fashion, these results, thus providing an optimal solution of Problem 1.

More specifically, rows 1÷2 are relative to the case in which the initial inventory is sufficient to satisfy entirely the external demand and then no production is required (the resource stays idle during the whole time horizon); rows 3÷40 verifies if an idle period exists at the beginning or not and, in positive case, the time interval $[\theta_{k-1}, \theta_k)$, $k \in \{1, \ldots, Q\}$, during which the idle period ends is determined and the production speed is set according to (30)÷(32) (except in the case the idle period ends at θ_{k-1}); finally, rows 41÷83 takes into account all the intervals $[\theta_{k-1}, \theta_k)$ corresponding to a working period and sets the production speed in accordance with (33)÷(35). ■

Remark 2: Necessary and sufficient conditions for the existence of a solution for Problem 1, that have been provided in Theorem 1, can be now simply written as $x_0 \geq X_0$.

Remark 3: The procedure in Theorem 5 allows finding the optimal solution of Problem 1 in polynomial time, being $O(Q^2)$ the complexity of the procedure, and having considered Q as the size of the problem to be solved (number of changes of withdrawal rate in the external demand).

VI. EXAMPLE

Consider an inventory-production system characterized by a maximum speed $U = 20$ parts/hour and an initial inventory level $x_0 = 7$ parts. The external demand $e(t)$ is the piecewise constant function illustrated in Fig. 3.

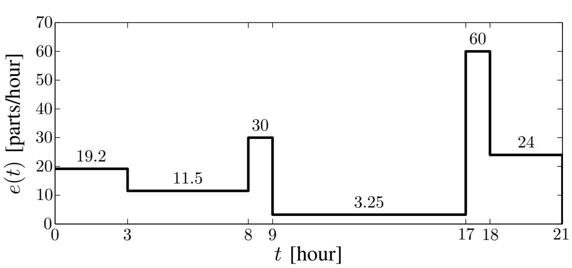

Fig. 3. Example – Pattern of the external demand $e(t)$.

By applying Theorem 5, the optimal service speed $u^\circ(t)$ illustrated in Fig. 4 is determined. It is characterized by the following values of the decision variables: $P^\circ = 6$, $\tau_1^\circ = 0.8854$, $\tau_2^\circ = 3$, $\tau_3^\circ = 6.8235$, $\tau_4^\circ = 9$, $\tau_5^\circ = 13.8955$, $u_1^\circ = 0$, $u_2^\circ = 19.2$, $u_3^\circ = 11.5$, $u_4^\circ = 20$, $u_5^\circ = 3.25$, $u_6^\circ = 20$.

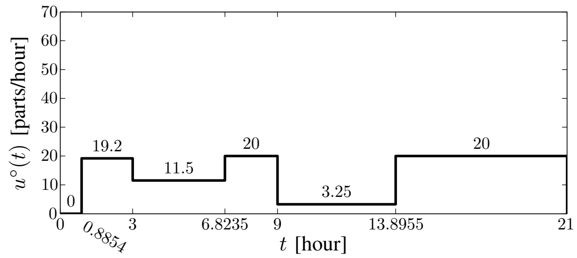

Fig. 4. Example – Optimal service speed $u^\circ(t)$.

VII. CONCLUSIONS

In this paper, we have proposed a procedure which allows finding in polynomial time the solution of a finite-horizon optimization problem for an inventory-production system that must satisfy a positive demand changing at asynchronous time instants. The decisions on such system regard the optimal pattern of the service speed (in terms of values and asynchronous time instants in which these values change) in order to minimize setup, production, and holding costs. Present and future research is devoted to exploit this solution procedure in order to find optimal (closed-loop) control strategies, functions of the system state, to be used when some perturbations affect the system.

REFERENCES

[1] H.C. Bahl, L.P. Ritzman and J.N.D. Gupta. Determining lot sizes and resource requirements: a review, *Operations Research*, vol. 35, 1987, pp 329-345.

[2] B. Karimi, S.M.T. Fatemi Ghomi and J.M. Wilson. The capacitated lot sizing problem: a review of models and algorithms, *Omega*, vol. 31, 2003, pp 365-378.

[3] H. Wagner and T. Whitin. Dynamic version of the economic lot-size model, *Management Science*, vol. 5, 1958, pp 89-96.

[4] H. Stadtler. Multi-level single machine lot-sizing and scheduling with zero lead times, *European Journal of Operational Research*, vol. 209, 2011, pp 241-252.

[5] F. Chu and C. Chu. Polynomial algorithms for single-item lot-sizing models with bounded inventory and backlogging or outsourcing, *IEEE Transactions On Automation Science And Engineering*, vol. 4, 2007, pp 233-251.

[6] S. Benjaafar, M. ElHafsi and T. Huang. Optimal control of a production-inventory system with both backorders and lost sales, *Naval Research Logistics*, vol. 57, 2010, pp 252-265.

[7] E. Khmelnitsky and Y. Gerchak. Optimal control approach to production systems with inventory-level-dependent demand, *IEEE Transactions On Automatic Control*, vol. 47, 2002, pp 289-292.

[8] F. Chu and C. Chu. Single-item dynamic lot-sizing models with bounded inventory and outsourcing, *IEEE Transactions On Systems, Man, And Cybernetics Part A*, vol. 38, 2008, pp 70-77.

[9] D.X. Shaw and A.P.M. Wagelmans. An algorithm for single-item capacitated economic lot sizing with piecewise linear production costs and general holding costs, *Management Science*, vol. 44, 1998, pp 831-838.

[10] C.H. Bannister and R.J. Kaye. A rapid method for optimization of linear systems with storage, *Operations Research*, vol. 39, 1991, pp 220-232.

[11] D. Giglio, R. Minciardi, S. Sacone and S. Siri. Optimization of inventory levels and production effort in Hybrid Inventory-Production (HIP) systems, *in IEEE Conference on Automation Science and Engineering*, 2008, pp 8-15.

[12] D. Giglio, R. Minciardi, S. Sacone and S. Siri. Optimal control of manufacturing processes in Hybrid Inventory-Production systems, *in European Control Conference*, 2009, pp 2560-2565.

[13] G. Belvaux and L.A. Wolsey. Modelling practical lot-sizing problems as mixed-integer programs, *Management Science*, vol. 47, 2001, pp 9931007.