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A Hybrid observer for frequency estimation of saturated multi-frequency signals

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***Abstract*— A hybrid observer to estimate the frequencies of a saturated multi-frequency signal is proposed. The observer is based on the Immersion and Invariance technique combined with a high-gain approach. This paper is a (semi-global) generalization of the result in [1] to the case of multiple frequencies.**

I. Introduction

The problem of estimating the frequencies of a signal has been extensively addressed via Fourier transform tools [2] and on-line methods, see [3], [4], [5], [6], [7], [8], [9] and references therein. The importance of such studies is well-known in several areas of engineering, such as mechanics, telecommunication and control. In the latter field on-line frequency estimators have been exploited, for example, to pursue output feedback stabilization of plants perturbed by unknown sinusoidal disturbances [10], [11], [12], [13].
In [1] the problem of estimating the frequency of a saturated single frequency signal has been addressed. The observer therein allows to estimate the single frequency of a sinusoidal signal when its amplitude exceed the sensor capabilities or when the sensor exhibits nonlinear characteristics.
To extend this result we consider the saturated multi-frequency signal

$$\zeta(t) = \mathrm{sat}_\sigma\left(\sum_{i=1}^{n} E_i \sin\left(\omega_i t + \phi_i\right)\right), \tag{1}$$

with known $n \geq 1$ and saturation threshold[1] $\sigma > 0$ where

$$\mathrm{sat}_\sigma(s) = \begin{cases} s & \text{if}|s| \leq \sigma, \\ \sigma \mathrm{sign}(s) & \text{elsewhere}, \end{cases}$$

and unknown angular frequencies $\omega_i > 0$, amplitudes E_i and phases ϕ_i, $i = 1, .., n$.
The difficulties of estimating the frequencies in the case $n \geq 1$ stands in the fact that the jump map, i.e. the map that relates the signal derivatives before and after the action of the saturation, depends on the unknown parameters ω_i's. In the case $n = 1$, this jump map does not depend on the unknown frequency, namely $\dot{y}(t_j^-) = -\dot{y}(t_{j+1}^+)$, where t_j is the instant at which $y(t)$ hits the saturation thresholds i.e. $|y(t_j - \varepsilon)| < \sigma$ and $|y(t_j + \varepsilon)| > \sigma$, and t_{j+1} is such that $|y(t_{j+1} - \varepsilon)| > \sigma$ and $|y(t_{j+1} + \sigma)| < \sigma$ for sufficiently small $\varepsilon > 0$.
The results in the paper can be used to deal with sensor saturation or sensor nonlinear behavior for large input signals. In the latter case, it is "advisable" to saturate the output of the sensor outside its linear range and use the result of Theorem 1. The benefits of this approach in case of quantized signal, which seems to suggest that low saturation levels (low σ) associated with fine quantization would be profitable for estimating the ω_i's are under investigation.
The paper is organized as follows: mathematical backgrounds are given in section Section II followed by the main result in Section III. Simulation results are given in Section IV and conclusions in Section V.

II. Mathematical background

To start with, we focus on the unsaturated signal $y(t)$ defined as

$$y(t) = \sum_{i=1}^{n} E_i \sin\left(\omega_i t + \phi_i\right), \tag{2}$$

which can be regarded as the output of a time invariant, neutrally stable, linear system in observer canonical form described by the equations

$$\bar{\Sigma}: \begin{cases} \dot{\eta} &= A_\theta \eta, \\ y &= H\eta, \end{cases}$$

with $y \in \mathbb{R}$, $\eta \in \mathbb{R}^{2n}$,

$$A_\theta = \left[\begin{array}{c|cccc} 0 & 1 & 0 & \dots & 0 \\ \hline -\theta_1 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \dots & \dots & \dots & \vdots \\ 0 & \dots & \dots & 0 & 1 \\ -\theta_n & 0 & \dots & 0 & 0 \end{array}\right] \in \mathbb{R}^{2n \times 2n},$$

$$H = \left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & 0 \end{array}\right] \in \mathbb{R}^{1 \times 2n},$$

and where the unknown parameters θ_i, are related to the angular frequencies w_i, $i = 1, .., n$, by the relation

$$\prod_{k=1}^{n}\left(s^2 + \omega_k^2\right) = s^{2n} + \theta_1 s^{2(n-1)} + \cdots + \theta_n. \tag{3}$$

System $\bar{\Sigma}$ can be written, in compact form, as

$$\Sigma: \begin{cases} \dot{y} &= Cx, \\ \dot{x} &= Ax - Q\Theta y, \end{cases} \tag{4}$$

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[1]The results can be easily extended to non-symmetric saturation maps.

with $x \in \mathbb{R}^{2n-1}$, $A \in \mathbb{R}^{(2n-1)\times(2n-1)}$, and $C \in \mathbb{R}^{1\times(2n-1)}$, where C and A are obtained eliminating the last column and the first row and column from A_θ and H, respectively, $\Theta = [\theta_1, \dots, \theta_n]^\top \in \mathbb{R}^n$, and $Q \in \mathbb{R}^{(2n-1)\times n}$ has zero elements except for $Q_{2j-1,j} = 1$, $j = 1, \dots, n$. Consider now the natural assumption that ensures the possibility of reconstructing the angular frequencies ω_i, $i = 1, \dots, n$, measuring $y(t)$.

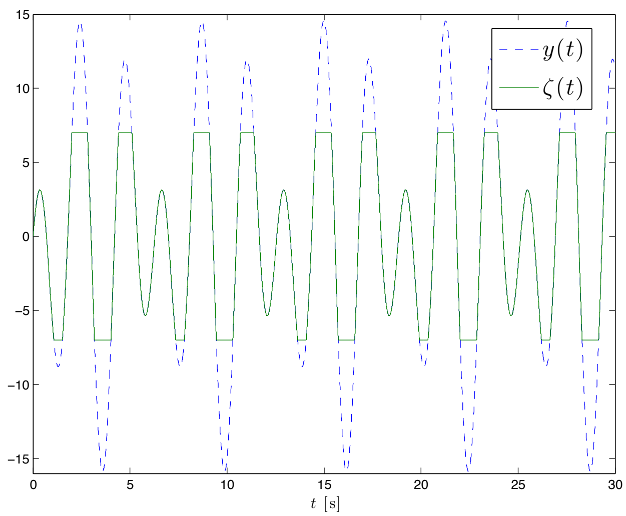

Fig. 1. The saturated signal $\zeta(t)$ (solid) and $y(t)$ (dashed) when $\sigma = 7$.

Assumption 1: For all $(i,j) \in \{1,\dots,n\}$, $i \neq j$, $0 < \omega_i \neq \omega_j$ and the initial conditions $x(0)$ and $y(0)$ of (4) are such that the signal $y(t)$ is rich of order $2n$, see [5].

To solve the estimation problem, similarly to [9], [1], and the general approach suggested in [14], we define the estimation error $z = [z_x', z_\theta']' \in \mathbb{R}^{3n-1}$ as

$$z = \beta(y, M, R, \xi) - h(R, x, \Theta), \tag{5}$$

where the maps $\beta(\cdot)$ and $h(\cdot)$ are selected as

$$\beta(y, M, R, \xi) = \begin{bmatrix} Ky + \gamma_1 M \\ \gamma_2 (CR)' y + \xi \end{bmatrix}, \tag{6}$$

$$h_R(x, \Theta) = \begin{bmatrix} x - \gamma_1 R\Theta \\ \frac{\gamma_1}{\gamma_3}\Theta \end{bmatrix}, \tag{7}$$

with $\xi(t) \in \mathbb{R}^n$, $M(t) \in \mathbb{R}^{2n-1}$ and the matrix[2] $R(t) \in \mathbb{R}^{2n-1\times n}$, and γ_i, $i \in \{1,2,3\}$ are positive constants. Note that the map $h_R(\cdot)$, parameterized by R is left invertible[3]

[2]The dimension of the observer might be reduced defining R as a function of M, as in [9]. However, to avoid burdening of notation, we do not take into account this possibility.

[3]A mapping $\phi_s(\cdot) : \mathbb{R}^{p_1} \to \mathbb{R}^{p_2}$ (parameterized by s) is left-invertible if there exists a mapping $\phi_R^L(\cdot) : \mathbb{R}^{p_2} \to \mathbb{R}^{p_1}$ such that $\phi_s^L(\phi_s(w)) = w$ for all $w \in \mathbb{R}^{p_1}$ (and for all s). This is equivalent to require that ϕ_s is injective uniformly in s.

hence the estimates of x and Θ are given by

$$\hat{x} = Ky + \gamma_1(M + R\Theta), \tag{8a}$$

$$\hat{\Theta} = \left(\gamma_2 (CR)' y + \xi\right) \gamma_3/\gamma_1. \tag{8b}$$

To obtain a globally asymptotically converging estimate of x and Θ we need to design the dynamics of M, R and ξ such that the manifold

$$\begin{aligned}\mathcal{M} \triangleq \{&(y, M, R, x, \theta, \xi) \in \mathbb{R} \times \mathbb{R}^{2n-1} \times \mathbb{R}^{2n-1\times n} \times \mathbb{R}^{2n-1} \\ &\times \mathbb{R}^n \times \mathbb{R}^n : [x', \theta']' = h_R^L(\beta(y, M, R, \xi))\},\end{aligned}$$

is forward invariant and globally attractive. To this aim we exploit the hybrid systems introduced in [15] to cope with the saturation nonlinearity. A few basic definitions [15] that are extensively used in the sequel are recalled.

Definition 1: A *compact hybrid time domain* is a set $\mathcal{T} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}_{\geq 0}$ given by:

$$\mathcal{T} = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j),$$

where $J \in \mathbb{N}_{\geq 0}$ and $0 = t_0 \leq t_1 \cdots \leq t_J$. A *hybrid time domain* is a set $\mathcal{T} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}_{\geq 0}$ such that, for each $(T, J) \in \mathcal{T}$, $\mathcal{T} \cap ([0,T] \times \{0, \dots, J\})$ is a compact hybrid time domain. □

Definition 2: A *hybrid trajectory* is a pair (dom χ, χ) consisting of a hybrid time domain dom χ and a function χ defined on dom χ that is continuously differentiable on (dom χ) $\cap (\mathbb{R}_{\geq 0} \times \{j\})$ for each $j \in \mathbb{N}_{\geq 0}$. □

Definition 3: For the hybrid system $\mathcal{H}$ given by the open state space $O \subset \mathbb{R}^n$, and the data (F, G, C, D) where $F : O \to \mathbb{R}^n$ is continuous, $G : O \to O$ is locally bounded, and C and D are subsets of O, a *hybrid arc* χ : dom $\chi \to O$ is a *solution to the hybrid system* $\mathcal{H}$ if $\chi(0,0) \in C \cup D$ and the following hold.

1) For all $j \in \mathbb{N}_{\geq 0}$, and for almost all $t \in I_j :=$ dom $\chi \cap (\mathbb{R}_{\geq 0} \times \{j\})$, we have $\chi(t,j) \in C$ and $\dot{\chi}(t,j) = F(\chi(t,j))$.
2) For all $(t,j) \in$ dom χ, such that $(t, j+1) \in$ dom χ, we have $\chi(t, j+1) = G(\chi(t,j))$ with $\chi(t,j) \in D$. □

The hybrid system model that we consider is of the form:

$$\begin{aligned}\dot{\chi}(t,j) &= F(\chi(t,j)) \qquad && \chi(t,j) \in C, \\ \chi(t_{j+1}, j+1) &= G(\chi(t_{j+1}, j)) && \chi(t_{j+1}, j) \in D.\end{aligned}$$

$F(\cdot)$ and $G(\cdot)$ are called flow map and jump map, respectively. In the sequel, as in [16], we omit the time arguments when possible and write:

$$\begin{aligned}\dot{\chi} &= F(\chi) \qquad && \chi \in C, \\ \chi^+ &= G(\chi) && \chi \in D,\end{aligned} \tag{9}$$

where we denoted $\chi(t_{j+1}, j+1)$ as χ^+ in the last equation.

Note that, unlike standard observer design in which flow maps are considered such that $\mathcal{M}$ is forward invariant and attractive, we have also to ensure that $\mathcal{M}$ is forward invariant with respect to jumps when $\hat{\Theta} = \Theta$.

III. Main result

We define two hybrid time sub-intervals $T_u(i) = [t_i, t_{i+1}]$ and $T_s(j) = [t_j, t_{j+1}]$, for some i and j such that $|y(t)| < \sigma$, $\forall t \in T_u(i)$, and $|y(t)| \geq \sigma$, $\forall t \in T_s(j)$. In the sequel, without loss of generality, we assume that $|y(t_0)| < \sigma$.
To prove Theorem 1 below we exploit the result presented in [17] and define $P(\rho) = P^\top(\rho) > 0$, $P(\rho) \in \mathbb{R}^{(2n-1)\times(2n-1)}$ such that

$$A'P(\rho) + P(\rho)A + 2C'C = -\rho P(\rho), \tag{10}$$

for some $\rho > 0$, where the matrix $P(\rho)$ can be evaluated from (10) fixing $\rho = 1$ and then defining

$$P(\rho)_{ij} = \frac{1}{\rho^{i+j-1}} P(1)_{ij}.$$

The classical definition, see [5], of Persistency of Excitation (PE) for non-autonomous linear systems is recalled next.

Definition 4: Consider an asymptotically stable and reachable linear time invariant system $\dot{\eta} = A\eta + Bu$, $\eta \in \mathbb{R}^m$, with scalar input u rich of order greater or equal than m. Then for all $t \geq 0$ there exist $\delta_T > 0$ and $\delta > 0$ such that

$$\int_t^{t+\delta_T} \eta(\tau)\eta'(\tau)d\tau \geq \delta I. \tag{11}$$

□

This means that the state of a reachable and asymptotically stable linear time invariant system excited by a persistently exciting input is persistently exciting. In this work we need inequality (11) to holds not just for some δ_T but for any $\delta_T > 0$ and some $\delta(\delta_T) > 0$, a property that follows immediately from (11) as stated in the following result.

Lemma 1: Under the assumptions of Definition 4, if inequality (11) holds, then it holds for any $\delta_T > 0$ and some $\delta(\delta_T) > 0$.

We state now the main theorem.

Theorem 1: Consider the extended hybrid dynamical system

$$\dot{\chi} = F_q(\chi), \quad \chi \in \mathcal{C}, \tag{12}$$
$$\chi^+ = G_q(\chi), \quad \chi \in \mathcal{D}, \tag{13}$$

with $\chi = (y, x, R, M, \xi, q, \tau) \in \mathcal{O} \subset \mathbb{R} \times \mathbb{R}^{2n-1} \times \mathbb{R}^{(2n-1)\times n} \times \mathbb{R}^{(2n-1)} \times \mathbb{R}^n \times \mathbb{N} \times \mathbb{R}_{\geq 0}$ and

$$F_1 = \begin{bmatrix} Cx \\ Ax - Q\Theta y \\ (A-KC)R - Qy/\gamma_1 \\ (A-KC)(M + Ky/\gamma_1) \\ W_u(y,R,M,\xi) \\ 0 \\ 1 \end{bmatrix}, \; F_0 = \begin{bmatrix} Cx \\ Ax - Q\Theta y \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{14a}$$

$$G_1 = \begin{bmatrix} y \\ x \\ R \\ M \\ \xi \\ 0 \\ 0 \end{bmatrix}, G_0 = \begin{bmatrix} y \\ x \\ \hat{\Phi}_2(\tau)R \\ \hat{\Phi}_2(\tau)M + \Big((\hat{\Phi}_2(\tau) - I)K + \hat{\Phi}_1(\tau)\Big)y/\gamma_1 \\ \xi + \gamma_2 y\Big(\Big(I - \hat{\Phi}_2(\tau)\Big)R\Big)' \\ 1 \\ 0 \end{bmatrix}, \tag{14b}$$

with $K = P(\rho)^{-1}C^\top$, and

$$\dot{\xi} = -\gamma_2 \Big((C\dot{R})'y + R'C'C(Ky + \gamma_1 M + \gamma_3 R(\gamma_2(CR)'y + \xi)\Big) \triangleq W_u(y,R,M,\xi), \tag{15}$$

$$e^{\hat{A}_\theta \tau} = \begin{bmatrix} \star & \star \\ \hat{\Phi}_1(\tau) & \hat{\Phi}_2(\tau) \end{bmatrix}, \tag{16}$$

with flow set $\mathcal{C} \triangleq \{\chi \in \mathcal{O} : (|y| \leq \sigma, q = 1) \vee (|y| \geq \sigma, q = 0)\}$, and jump set $\mathcal{D} \triangleq \{\chi \in \mathcal{O} : (|y| \geq \sigma, q = 1) \vee (|y| \leq \sigma, q = 0)\}$. Let Assumption 1 hold. Then the equilibrium $(z_x, z_\theta) = (0, 0)$ of the estimation error system is semi-globally in ρ, uniformly in (R, M, ξ), asymptotically stable. Furthermore, (R, M, ξ) are bounded. □

The observer proposed in Theorem 1 is such that the flow map of the estimation error when $t \in T_u(2i)$ ($|y(t)| \leq \sigma$) has a lower triangular structure, namely

$$\dot{z}_x = (A - KC)z_x, \tag{17a}$$
$$\dot{z}_\theta = -\gamma_2 R'C'Cz_x - \gamma_2\gamma_3 R'C'CRz_\theta. \tag{17b}$$

As a result, the parameter ρ can be determined numerically from the following conservative bound

$$\frac{c_\rho \left(c_\rho^\delta ||\bar{z}_x|| + 2(\sigma + c_x)s(||\bar{z}_\theta||)\sum_{j=0}^{\delta-1} c_\rho^j\right)}{e^{h_\rho \delta_T}} < \frac{\varepsilon ||\bar{z}_x||}{\lambda_{min}(P(\rho))}, \tag{18}$$

where $\varepsilon < 1$, $h_\rho = \rho - \gamma_2 c_\rho$, $c_\rho = \lambda_{max}(P(\rho))/\lambda_{min}(P(\rho))$ is the condition number of $P(\rho)$ in (10), and $c_x > 0$ is such that $||\hat{x}|| \leq c_x$ since y and x are bounded.

Note that by (4) and the definition of the jump and flow sets, Zeno solutions cannot exist. The implementation of the hybrid observer (12) requires the evaluation of (M, R, ξ, q, τ) to retrieve the estimates (8). When implementing (8), the jumps of the hybrid observer are triggered when the measured signal $y(t)$ reaches or leaves the selected threshold value σ.

Remark 1: In Theorem 1, it is possible to prove that the signal $z(t)$ and $(M(t), R(t), \xi(t), q(t))$ are bounded for every $\rho > 0$.

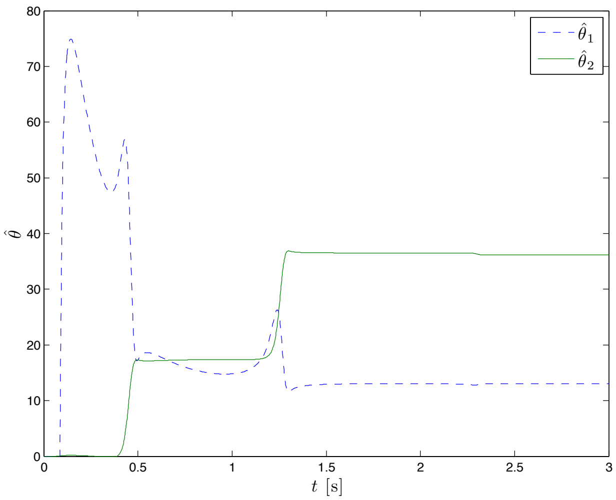

Fig. 2. Time histories of the parameter estimates without saturation ($\sigma > ||y(t)||_{\max}$). The estimates converge to $\Theta = [13, 36]'$.

IV. Numerical simulations

To illustrate the behavior of the proposed observer we consider the signal $y(t)$ generated by the system (2) with $\omega_1 = 2$ rad/s, $\omega_2 = 3$ rad/s, yielding $\theta_1 = 13$ and $\theta_2 = 36$, and $[y(0), x(0)'] = [0,\ 15,\ -15,\ 1]$.

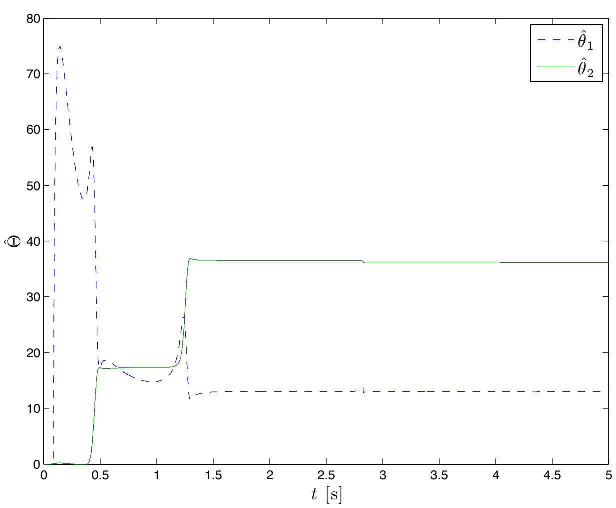

Fig. 3. Time histories of the parameter estimates with $\sigma = 10$ and $\lambda_i(A - KC) = -7$ for $i = 1, 2, 3$.

In the first two simulations the observer gain matrix $K = [21,\ 147,\ 343]'$ has been chosen such that the matrix

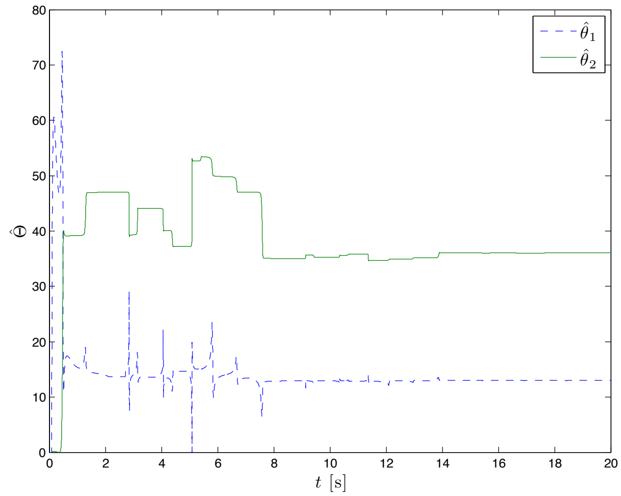

Fig. 4. Time histories of the parameter estimates with $\sigma = 10$ and $\lambda_i(A - KC) = -6$ for $i = 1, 2, 3$.

$A - KC$ has all poles in -7. This yields $||z_x(t, j)|| \leq e^{-7t} v(|z_x(t_j, j)|)$ with $(t, j) \in T_u$ and some class-$\mathcal{K}$ function $v(\cdot)$. The initial conditions of M, R and ξ are set equal to zero, whereas according to equation (17) we have selected $\gamma_2 = 0.2$ to limit the induced oscillations of z_x on z_θ, $\gamma_3 = 10^4$ to improve convergence of z_θ and $\gamma_1 = 0.1$ to limit the magnitude of the oscillations during the transients of z_x. In Fig. 2 the estimated Θ when the signal is not saturated, i.e. $\sigma > \max_{t \in \mathbb{R}_+}\{y(t)\}$, are depicted.

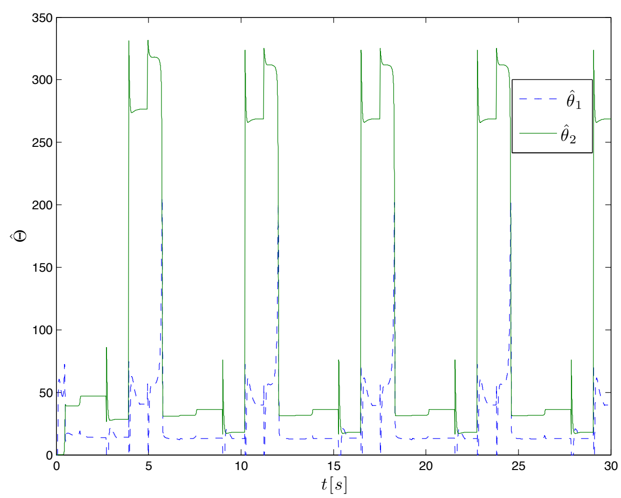

Fig. 5. Time histories of the parameter estimates with $\sigma = 10$ and $\lambda_i(A - KC) = -6$ when a purely continuous time observer is selected ($G_1 = G_0$).

In Fig. 3 the estimated Θ with a saturation level $\sigma = 10$, which corresponds to a cut of 30% of the maximal signal amplitude (see Fig. 1) is shown. Fig. 4 illustrates the degradation of performances when K is selected such that the eigenvalues of $A - KC$ are equal to -6, which is

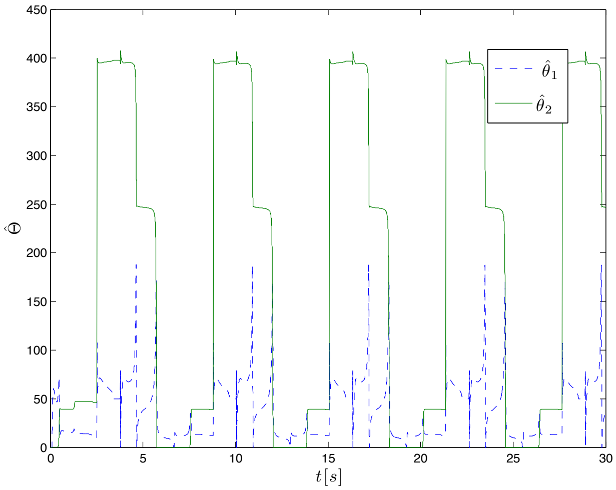

Fig. 6. Time histories of the parameter estimates with $\sigma = 14$ and $\lambda_i(A-KC) = -6$ when a purely continuous time observer is selected ($G_1 = G_0$).

equivalent to decrease ρ in (18). It is possibile to note in Fig. 5 the error induced by the saturation when the same K is considered, but with $G_1 = G_0$, i.e. when there are not jumps in the observer state. It is also interesting to note in Fig. 6 that with a larger saturation threshold $\sigma = 14$ the observer performances do not improve with respect to the previous case with $\sigma = 10$. This highlight the usefulness of the proposed method.

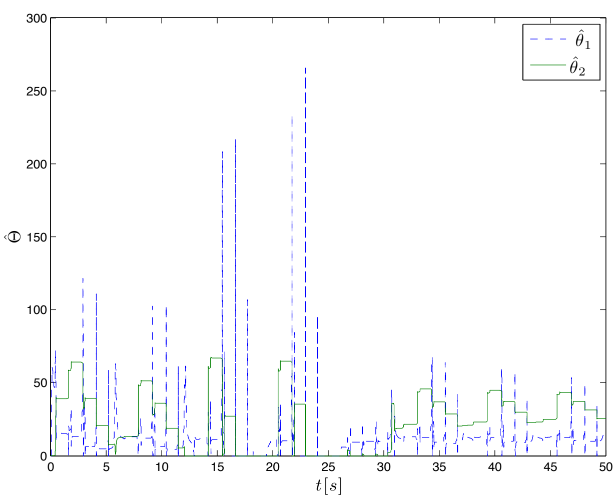

Fig. 7. Time histories of the parameter estimates with $\sigma = 7$ and $\lambda_i(A-KC) = -6$ for $i = 1, 2, 3$. In this case the gain matrix K of the observer is not adequate to ensure asymptotic convergence to zero of the estimation error.

According to Theorem 1 if $\sigma = 7$ (corresponding to less than 50% of the maximum amplitude of $y(t)$) and the gain matrix K is not adequately chosen to recover the error induced by the saturation during the time intervals T_u when $|y(t)| \leq \sigma$, a limit cycle appears as shown in Fig. 7, where

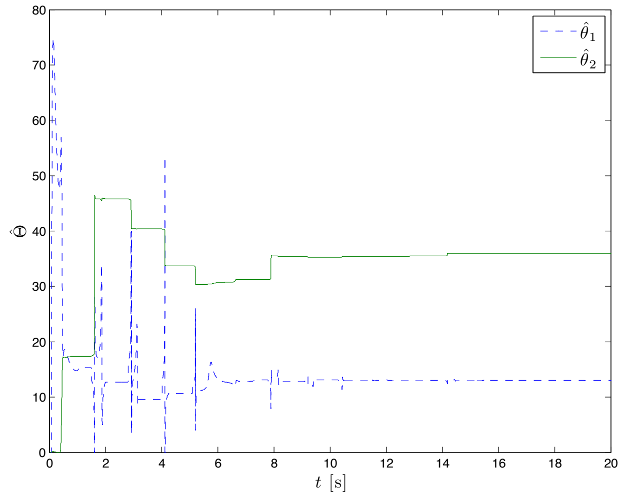

Fig. 8. Time histories of the parameter estimates with $\sigma = 7$, and $\lambda_i(A-KC) = -7$ for $i = 1, 2, 3$.

K is such that $\lambda_i(A - KC) = -6$, $i = 1, 2, 3$. In this case, a gain K such that $\lambda_i(A - KC) = -7$, $i = 1, 2, 3$, guarantees convergence to zero of the estimation error as shown in Fig. 8.

It is possible to see in Fig. 9 the total time $\sum T_u$ that monotonically decreases with the decrease of σ, whereas the number of jumps shows a maximum for $\sigma = 3$. Values of $\sum T_u$ and $\sum T_u/j_{\max}$ are given in Table I for two level of the saturation threshold σ.

σ	$\sum T_u$	$\sum T_u/j_{\max}$
7	13.8343	0.4323
10	18.1476	0.7562

TABLE I

UNSATURATED TIME AND ITS RATIO WITH THE NUMBER OF JUMPS OVER 25 SECONDS OF SIMULATION.

We recall that the angular frequencies ω_i's can be evaluated by $\hat{\Theta}$ finding the roots of $(\lambda^2 + \hat{\omega}_1^2)(\lambda^2 + \hat{\omega}_2^2) = \lambda^4 + \theta_1\lambda^2 + \theta_2$. As would be expected given the high-gain approach, further simulation results have shown the sensitivity of the estimation error to additive noise.

V. CONCLUSIONS

We have proposed a hybrid observer to estimate the n unknown frequencies of a multi-frequency saturated signal. The observer convergence relies on a high-gain type approach and observer state reset to recover the error induced by the saturation. The effectiveness of the approach is showed via numerical simulations and dependency of the observer performances by the matrix K and the saturation threshold σ have been discussed. Further refinements for the selections of ρ in Theorem 1 are necessary. The robustness of the approach to measurement noise, although theoretically inherited by

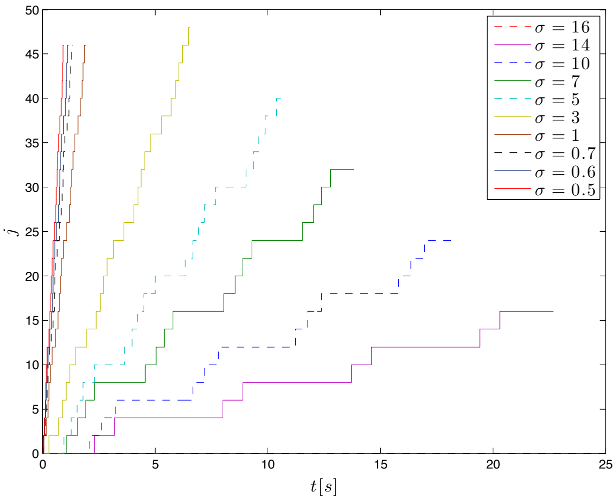

Fig. 9. Unsaturated hybrid time ($\sum T_u$) for different levels of σ with simulation time $t_f = 25$ s.

satisfaction of the standard assumptions in [15], needs to be improved, whereas the case in which σ is not known exactly can be easily addressed considering a lower bound.

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