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On the Optimal Sending Rate for Networked Control Systems with a Shared Communication Medium

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Abstract— **In this work, we consider the optimal sending rate for Networked Control Systems (NCS) with different communication schemes. Therefore, we compare time-triggered control with a Time Division Multiple Access (TDMA) or a Frequency Division Multiple Access (FDMA) communication protocol and event-based control with an ALOHA or Carrier Sense Multiple Access (CSMA) communication protocol. Our main interest is the optimal load and optimal performance of such systems. Nevertheless, we are also interested how the optimal performance scales with the number of agents, whether or not it is possible to add a new agent to an already running system, and the effects of an overloaded communication system.**

Index Terms— **Networked Control Systems; Communication Systems**

I. Introduction

Due to the steady increase in microcontroller and communication systems, more and more control loops are closed by a packet based communication network. Such a control system, where the loop is closed by a communication network is consequently called Networked Control System (NCS). Up to now, the communication system is most often modeled as an iid loss process or time delay, independent of the network load and communication protocol. For a correct analysis of NCS, it is necessary to use more realistic network models. Therefore, we analyze time-triggered and event-based control with different communication protocols. In doing so, the loss probability and delay are no more given in advance and constant; instead, they depend on the network load.

The benefits of event-based control have been demonstrated in [1], by comparing event-based and time-triggered control of first order systems. The authors showed that for event-based control less events are necessary to achieve the same performance as time-triggered control. Consequently, many works using event-based control for NCS followed, e.g., [2]–[10].

However, in real communication systems, packets might get delayed or lost. Thus, event-based control with packet losses was analyzed in [8]. Although the loss probability depends on the utilization, the exact interevent-times are not taken into account. Based on this work, we analyzed the performance of event-based control with a shared communication medium in [11], [12]. In these works, we were mainly concerned with the exact loss probability of event-based control with an ALOHA communication system and showed that it is crucial to include the communication strategy when comparing time-triggered and event-based control. In the present paper, we continue to analyze this interaction between control and communication strategy by considering the performance of NCS with a large number of agents and different communication protocols.

A similar problem to the one studied in this work was also studied in [6] and [10]. A simulation based approach was used in [6] to compare time-triggered and event-based control with different communication protocols. As in the present paper, the impulsive control of an integrator system with a CSMA communication protocol is considered analytically in [10]. However, there are some important differences, which are discussed in Sec. II, Sec. III, and Sec. V. Despite these differences, we get the same performance for event-based control with CSMA for a large number of agents, but different sending rates.

The remainder of this work is outlined as follows. The problem setup is given in Sec. II. In Sec. III, we take a detailed look at the interevent times of event-based control and show that the interevent times of all agents together converge to a Poisson process as the number of agents goes to infinity. In Sec. IV, we derive the optimal performance of time-triggered control and in Sec. V, the optimal performance of event-based control with ALOHA and CSMA communication protocols. The different approaches are compared in Sec. VI. Finally, the paper is concluded in Sec. VII.

II. Preliminaries

Our model of the underlying communication medium itself is relatively simple. Each packet takes the same time to be transmitted, the *packet duration* τ. Moreover, we assume that packets are lost if and only if two or more users are sending simultaneously. Based on this simple model of the communication medium itself, the loss probability and delay for different communication protocols can be derived, see Sec. V and, e.g., [13], [14] for more details. Consequently, the loss probability and delay are not assumed to be given, instead they depend on the network load.

Moreover, we assume that there is a sharp separation between control and communication. When the control unit detects an event, it generates a packet, which contains the information how the state of the system must be changed, and passes it to the communication unit to transmit. Consequently, when the packet arrives, it will be outdated and does not reset the state to the origin. In contrast, in [10] it is assumed that a packet arrival always resets the state to zero.

The control problem of this work is similar to the one of [1], [8], [10]–[12]. There are N agents, each has to control

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a system with integrator dynamic:

$$dx_i = u_i dt + dv_i, \qquad (1)$$

where $x_i \in \mathbb{R}$ is the state of system i, the disturbance $v_i(t) \in \mathbb{R}$ is a Wiener process with unit incremental variance and v_i and v_j are mutually independent. The control signal $u_i(t) \in \mathbb{R}$ is a sequence of impulses, each impulse resets the state to the origin. For simplicity of notation, we assume that all agents are identical and drop the subindex i in the remainder of this work. In event-based control, the impulses are generated whenever the state x exceeds a bound, i.e., when $x \leq \underline{\Delta}$ or $x \geq \overline{\Delta}$. In order to guarantee that the scheme works properly even when packets are delayed or lost, the bounds must be shifted at each event generation and packet arrival, see [8], [11]. If a bound is exceeded, the bounds are shifted by the *boundary increment* Δ such that the state is exactly between the new bounds. Hence, the distribution of the time between two events, the so called *interevent* or *interarrival time*, does not depend on lost or delayed packets, i.e., the interevent times are independent and identically distributed and the load does not depend on the number of previous packet losses. Note that this is crucial for a proper analysis of the problem since it allows us to use the results from renewal theory.

In [1], it was shown that for event-based control the mean interevent time T_{eb} can be easily derived from the boundary increments as $T_{\text{eb}} = \Delta^2$. However, for the analysis of communication systems, it is more appropriate to use the interarrival rate $\lambda := 1/T_{\text{eb}} = 1/\Delta^2$ mainly due to the fact that the accumulated rate of the users is just the sum of the individual rates, i.e., $\lambda_\Sigma = \sum_i \lambda_i$. Moreover, the offered load ρ is defined as the product of the packet duration τ and the interarrival rate λ, i.e., $\rho := \tau\lambda = \tau/\Delta^2$. Since there are N identical agents, the network load $\rho_\Sigma = \sum_i \rho_i$ becomes $\rho_\Sigma = N\rho$.

As in [1], [8], [10]–[12], the cost of time-triggered and event-based control is given by the variance of the state:

$$J = \limsup_{M\to\infty} \frac{1}{M} \int_0^M \mathrm{E}[x(t)^2]dt. \qquad (2)$$

In [1], time-triggered and event-based control without packet loss and delay was compared. It was shown that the costs of time-triggered and event-based control are

$$J_{\text{tt}} = \frac{T_{\text{tt}}}{2}, \qquad J_{\text{eb}} = \frac{T_{\text{eb}}}{6},$$

where T_{tt} is the sampling time and T_{eb} the mean interevent time.

Not surprisingly, loss and delay increase the cost, as shown in [8], [11], [12]. For event-based control, the additional cost due to packet loss is

$$J_{\text{eb},p} = \frac{p}{\rho(1-p)}\tau, \qquad (3)$$

where p is the packet loss probability, see [8], [11], [12]. For time-triggered and event-based control, the additional cost due to delay is

$$J_d = d, \qquad (4)$$

where d is the mean delay, see [11]. Moreover, in [11], [12] we observed that the cost scales directly with the packet duration τ. Thus, we use the normalized cost, i.e., the cost J divided by the packet duration τ, for comparing the different approaches. To sum up, the normalized cost of event-based control with loss and delay is

$$\frac{J_{\text{eb}}}{\tau} = \frac{1}{6\rho} + \frac{p}{\rho(1-p)} + \frac{d}{\tau}. \qquad (5)$$

III. Interevent Times of Event-Based Control

In this section, we first give the interevent time distribution of event-based control and then show that the superposition of the arrival processes of all agents together converges to a Poisson process for $N \to \infty$. This convergence is crucial since it allows us to use standard results from communication theory. However, this is only due to our specific choice of the control and communication scheme and does not necessarily hold for the setup of [10], where it is assumed that the interevent times are generated by a Poisson process.

In [8], the Probability Density Function (PDF) $f(t|\Delta)$ of the interarrival times of event-based control is given as:

$$f(t|\Delta) = \Delta\sqrt{\frac{2}{\pi t^3}} \sum_{k=-\infty}^{\infty} (4k+1) e^{-\frac{(4k+1)^2\Delta^2}{2t}}. \qquad (6)$$

Note that (6) can be written as a function of a normalized PDF $f(t|1)$:

$$f(t|\Delta) = \frac{1}{\Delta^2} f(\frac{t}{\Delta^2}|1) = \lambda f(\lambda t|1). \qquad (7)$$

In order to analyze the case of an infinite number of agents, we use the Palm-Khintchine Theorem and the following two assumptions, definition and short discussion, from [15].

Assumption 1: For all N sufficiently large,

$$\lambda_{1,N} + \cdots + \lambda_{N,N} = \lambda_\Sigma < \infty, \qquad (8)$$

where $\lambda_{j,N}$ is the sending rate of agent j for the case that there are N agents.

Assumption 2: Given $\epsilon > 0$, for each $t > 0$ and N sufficiently large,

$$F_{j,N}(t) \leq \epsilon, \qquad j = 1, \ldots, N, \qquad (9)$$

where $F_{j,N}$ is the Cumulative Distribution Function (CDF) of agent j for the case that there are N agents.

Definition 1: For each N define

$$L_{0,N}(t) = L_{1,N}(t) + \cdots + L_{N,N}(t), \qquad (10)$$

where $L_{j,N}(t)$ is a stochastic process, which counts the number of events of agent j that occur by time t for the case that there are N agents.

Assumption 2 asserts that as N increases, the processes being combined have renewals very infrequently. Assumption 1 shows that $L_{0,N+1}(t)$ is not formed by adding another process to $L_{0,N}(t)$. As N increases, the processes being combined are changed so that (at least for large N) the asymptotic rate at which renewals occur is a constant.

Theorem 1: [Palm-Khintchine [15, Theorem 5.15]] Under Assumptions 1 and 2, as $N \rightarrow \infty$, $\{L_{0,N}(t); t \geq 0\}$ approaches a Poisson process.

Based in the Palm-Khintchine Theorem, we can show that the arrival process of all agents together converges to a Poisson process.

Theorem 2: Suppose all N agents together send with a certain rate $\lambda_\Sigma < \infty$, i.e., each agent sends with the rate λ_Σ/N. As $N \rightarrow \infty$, the superposition of the arrival processes of all agents approaches a Poisson process with rate λ_Σ.

Proof: We use Thm. 1 to proof Thm. 2. Since Assumption 1 is part of Thm. 2, it remains to show that Assumption 2 holds. From (7) and the definition of the CDF, it follows that $F_{j,N}(t) = \int_0^t f(x|\Delta)dx = \int_0^t \lambda f(\lambda x|1)dx = \lambda_\Sigma/N \int_0^t f(\lambda_\Sigma x/N|1)dx$. Since $f(t|1) \rightarrow 0$ for $t \rightarrow 0$, there exists an N such that $F_{j,N}(t) \leq \epsilon$ and we see that Assumption 2 holds. ■

Unfortunately, in real applications, we have to deal with a finite number of agents. Nevertheless, for the remainder of this work, we approximate the interevent times of event-based control by a Poisson process even when the number of agents is finite. Note that in [11], [12], we analyzed event-based control with unslotted and slotted ALOHA for a finite number of agents without this approximation.

IV. Time-Triggered Control

In time-triggered control, each system is sampled with a constant sampling time T_{tt}, which is determined during the design of the controller. Consequently, it is possible to use a deterministic communication system like TDMA, where the sending time of each agent is assigned in advanced. This gives a good performance if the number of agents is known in advance but makes this approach very inflexible.

A. Time Division Multiple Access (TDMA)

In this approach, each agent sends its packets in preassigned time slots. Note that $\rho = \tau/T_{\text{tt}}$ and the delay is just the packet duration, i.e., $d = \tau$. Thus, we get for the normalized cost

$$\frac{J_{\text{tt}}}{\tau} = \frac{1}{2\rho} + 1, \quad \rho \leq \frac{1}{N}. \tag{11}$$

Obviously, this cost is minimal for $\rho = 1/N$ and becomes

$$\frac{J^*_{\text{tt}}}{\tau} = \frac{1}{2}N + 1. \tag{12}$$

Note that the minimal cost is affine in the number of agents.

B. Frequency Division Multiple Access (FDMA)

In FDMA, the frequency is shared between the agents, i.e., each agent gets $1/N$-th of the bandwidth. Although the sampling time is not changed, the packet duration is increased, i.e., the packet duration becomes $N\tau$ instead of τ. Thus, the normalized cost is

$$\frac{J_{\text{tt}}}{\tau} = \frac{1}{2\rho} + N, \quad \rho \leq \frac{1}{N}. \tag{13}$$

Again, this cost is minimal for $\rho = 1/N$, and becomes

$$\frac{J^*_{\text{tt}}}{\tau} = \frac{N}{2} + N = \frac{3}{2}N. \tag{14}$$

Again, the minimal cost is affine in the number of agents. Moreover, note that it is larger than the one with TDMA.

V. Event-Based Control

In this section, we analyze the performance of event-based control with different communication protocols, which allow arbitrary access times. Therefore, we first analyze event-based control with slotted and unslotted ALOHA, where the users start to send without checking the medium, see [13], [14], [16]. Then, we will analyze event-based control with different versions of the family of Carrier Sense Multiple Access (CSMA) protocols, as described in [13], [14], [17]. Here, the user senses the medium and only sends if the medium is idle. If the medium is busy, the user either waits until the medium becomes idle (1-persistent CSMA), or waits a random time (non-persistent CSMA), before retrying to send.[1] In all these approaches lost packets are retransmitted to get a reliable communication channel. In the original analysis of these protocols it is assumed that all packets, i.e., the new packets, the delayed as well as the retransmitted packets are generated by a Poisson process. However, we believe devoutly that retransmitting or artificially delaying packets does not make sense for event-based control. Consequently, we assume that lost packets are not retransmitted. Moreover, in non-persistent CSMA, the packets are not buffered for later transmissions if the medium is sensed busy. Instead, the packet is dropped, the bounds are shifted and we wait for the next event to occur. However, for 1-persistent CSMA, we assume that the order of arrival is kept, i.e., all packets are buffered in a global queue. In contrast, in [10], it is assumed that the next packet is chosen randomly.

A. Unslotted ALOHA

First, we look at the minimal cost of networked event based control with unslotted ALOHA. Since in unslotted ALOHA the agents start sending at arbitrary times, the delay is just the packet duration, i.e., $d = \tau$. Assuming a Poisson arrival process, the loss probability for unslotted ALOHA is

$$p^u = 1 - e^{-2\rho_\Sigma} = 1 - e^{-2\rho N}, \tag{15}$$

see, e.g. [13], [14]. Based on this approximation of the loss probability, we can derive the optimal load for event-based control with unslotted ALOHA.

Lemma 1: For event-based control with unslotted ALOHA, the minimal normalized cost J^{u*}_{eb}/τ is achieved when the network load ρ^{u*}_Σ fulfills

$$5 + 6(2\rho^{u*}_\Sigma - 1)e^{2\rho^{u*}_\Sigma} = 0. \tag{16}$$

Numerically, this is $\rho^{u*}_\Sigma \approx 0.2445$.

[1] Note that there also exists p-persistent CSMA, which roughly works as follows. Case 1 (idle medium): The user sends with probability p. Case 2 (busy medium): The user waits until the medium becomes idle and then continues as in Case 1 (idle medium).

Proof: By using (15) and $d = \tau$ in (5), we get

$$\frac{J^u_{\text{eb}}}{\tau} = \frac{1}{6\rho} + \frac{1 - e^{-2\rho N}}{\rho e^{-2\rho N}} + 1 = \frac{6e^{2\rho N} - 5}{6\rho} + 1. \tag{17}$$

Now, (16) follows by checking the first and second derivative. ■

Interestingly, the optimal network load does not depend on the number of agents. However, the optimal load of each agent, i.e., ρ^{u*}_Σ/N depends on the number of agents. Consequently, an agent can not know its optimal load if it does not know the number of agents. Nevertheless, this could be solved by looking at the loss probability, which turns out to be independent from the number of agents.

Lemma 2: For event-based control with unslotted ALOHA, the minimal normalized cost J^{u*}_{eb}/τ is achieved when the loss probability p^{u*} fulfills

$$5p^{u*} + 6\ln(1 - p^{u*}) + 1 = 0. \tag{18}$$

Numerically, this is $p^{u*} \approx 0.3867$.

Proof: We get (18) by using (15) in (16). ■

Finally, the following theorem gives the minimal cost of event-based control with unslotted ALOHA and shows that it is affine in the number of agents.

Theorem 3: For event-based control with unslotted ALOHA, the minimal normalized cost is

$$\frac{J^*}{\tau} = 2e^{2\rho^{u*}_\Sigma} N + 1 \approx 3.2612N + 1. \tag{19}$$

Proof: This theorem follows by using (15) in (17). ■

B. Slotted ALOHA

In slotted ALOHA, the packet loss probability is reduced by restricting the times when the users are allowed to start sending. As the name suggests, time is divided into slots and the users are only allowed to start sending at the begin of a slot. In doing so, packet loss is traded against delay. In this work, we assume that the slot length equals the packet duration and thus the mean waiting time is half the packet duration. Thus, we get for the mean delay $d = 1.5\tau$ (mean waiting time + packet duration). Assuming a Poisson arrival process, the loss probability for slotted ALOHA is

$$p^s = 1 - e^{-\rho_\Sigma} = 1 - e^{-\rho N}. \tag{20}$$

See, e.g. [13], [14]. Based on this approximation of the loss probability, we can derive the optimal load for event-based control with slotted ALOHA. Since the proofs are similar to the ones of slotted ALOHA they are omitted.

Lemma 3: For event-based control with slotted ALOHA, the minimal normalized cost J^{s*}_{eb}/τ is achieved when the network load ρ^{s*}_Σ fulfills

$$5 + 6(\rho^{s*}_\Sigma - 1)e^{\rho^{s*}_\Sigma} = 0. \tag{21}$$

Numerically, this is $\rho^{s*}_\Sigma \approx 0.4889$.

Note that the optimal network load with slotted ALOHA is twice the optimal network load with unslotted ALOHA, i.e., $\rho^{s*}_\Sigma = 2\rho^{u*}_\Sigma$.

Lemma 4: For event-based control with slotted ALOHA, the minimal normalized cost J^{s*}_{eb}/τ is achieved when the loss probability p^{s*} fulfills

$$5p^{s*} + 6\ln(1 - p^{s*}) + 1 = 0. \tag{22}$$

Numerically, this is $p^{s*} \approx 0.3867$.

Surprisingly, the optimal loss probability for slotted and unslotted ALOHA are identical, i.e., $p^{u*} = p^{s*}$ although the optimal network load is different.

Theorem 4: For event-based control with slotted ALOHA, the minimal normalized cost is

$$\frac{J^{s*}_{eb}}{\tau} = e^{\rho^{s*}_\Sigma} N + 1.5 \approx 1.6306N + 1.5. \tag{23}$$

Again, the minimal cost is affine in the number of agents. Moreover, since the optimal entire load of slotted ALOHA is twice the one of unslotted ALOHA, we see that the cost with unslotted ALOHA is twice the cost with slotted ALOHA, except for the additional cost due to the delay.

C. 1-Persistent CSMA with an Infinite Queue

Since we assume that the order of arrival is kept, we model the 1-persistent CSMA protocol by a queueing system. As stated in Sec. III, we approximate the arrival process by a Poisson process. Moreover, since the packet duration is constant, we have a deterministic service time. Finally, since all agents use the same communication medium, we have one server. To sum up, our 1-persistent CSMA scheme can be modeled by an M/D/1 queue, see, e.g., [18] for an introduction to queueing theory. Using such a queueing system, the mean delay, i.e., the waiting time plus the processing time, is

$$d = \tau \frac{2 - \rho_\Sigma}{2(1 - \rho_\Sigma)}. \tag{24}$$

Since the queue is lossless ($p = 0$), we have to require $0 \le \rho_\Sigma < 1$ for a stable queue.

Using (24) in (5), the normalized cost becomes

$$\frac{J}{\tau} = \frac{1}{6\rho} + \frac{2 - \rho_\Sigma}{2(1 - \rho_\Sigma)} = \frac{N}{6\rho_\Sigma} + \frac{2 - \rho_\Sigma}{2(1 - \rho_\Sigma)}. \tag{25}$$

Again, we are interested in the load ρ^{i*}_Σ, which gives the minimal cost.

Lemma 5: For event-based control with 1-persistent CSMA with an infinite queue, the minimal normalized cost J^{i*}_{eb}/τ is achieved when the network load ρ^{i*}_Σ fulfills

$$\rho^{i*}_\Sigma = \begin{cases} 1/2 & \text{for } N = 3, \\ \frac{N - \sqrt{3N}}{N-3} & \text{for } N \neq 3. \end{cases} \tag{26}$$

Proof: The optimal load is found by checking the first and second derivative of (25). ■

By using ρ^{i*}_Σ in (25), we get the following lemma.

Lemma 6: For event-based control with 1-persistent CSMA with an infinite queue, the minimal normalized cost J^{i*}_{eb}/τ is achieved when the mean delay fulfills

$$d^{i*} = \left(\frac{N + \sqrt{3N} - 6}{2(-3 + \sqrt{3N})} \right) \tau. \tag{27}$$

Now, we get the minimal cost by using (26) and (27) in (5).

Theorem 5: For event-based control with CSMA with an infinite queue, the minimal normalized cost is

$$\frac{J_{eb}^{i*}}{\tau} = \frac{N-3}{6(1-\sqrt{3/N})} + \frac{\sqrt{N}+\sqrt{3}-6/\sqrt{N}}{2(-3/\sqrt{N}+\sqrt{3})} \tag{28}$$

$$\approx \frac{N}{6} + \frac{\sqrt{N}}{2\sqrt{3}} \qquad \textit{for large } N. \tag{29}$$

From (29), we see that the minimal normalized cost is not affine in N. However, since the dominating term $N/6$ does not grow as fast as the one for time-triggered control, we get a lower cost than for the time-triggered case. Moreover, in [10] the minimal cost of event-based control with CSMA converge to $J_{\text{eb}}^{i*} = N\tau/6$ for $N \to \infty$, although a slightly different problem setup is used there.

Unfortunately, the optimal network load ρ_Σ^{i*}, the optimal load of each agent ρ^{i*} as well as the optimal delay d^{i*} depend on the number of agents N. Consequently, in order to achieve the optimal performance, the agents have to know N. Moreover, suppose we have N optimally sending agents, adding some agents might then overload the queue, resulting in an infinite cost. Consequently, it is possible but dangerous to add new agents to an already running system.

D. 1-Persistent CSMA with Finite Queue

Since a finite queue is not realistic and might be risky, we have a look at 1-persistent CSMA with a finite queue in this section. In [20], the loss probability and mean delay for a finite M/D/1 queue of size M (waiting room + processing unit) is given by

$$p = 1 - \frac{b_{M-1}}{1+\rho_\Sigma b_{M-1}}, \qquad d = \left(M - \frac{\sum_{k=0}^{M-1} b_k - M}{\rho_\Sigma b_{M-1}}\right)\tau,$$

where the coefficients b_n are

$$b_n = \sum_{k=0}^{n} \frac{(-1)^k}{k!}(n-k)^k e^{(n-k)\rho_\Sigma}\rho_\Sigma^k. \tag{30}$$

Obviously, it is difficult to get exact analytical results based on these equations. Thus, we restrict our discussion on the cost for low and hight loads.

If the load is low, then the finite queue behaves like the infinite queue, i.e., there is no loss and the delay is similar to the one of the infinite queue. Consequently, the cost is similar to the one with an infinite queue.

The other interesting case is a heavily overloaded system, i.e., $\rho \to \infty$. In order to derive the additional cost due to loss and delay, it is crucial to note that $b_i < b_j$ for $i < j$ and that b_n is increasing with increasing ρ. Thus, we have

$$\frac{J_p}{\tau} = \frac{1+\rho N b_{M-1} - b_{M-1}}{\rho b_{M-1}} \to N \qquad \text{for } \rho \to \infty.$$

Moreover, observe that

$$\frac{\sum_{k=0}^{M-1} b_k - M}{N\rho b_{M-1}} \to 1 \qquad \text{for } \rho \to \infty$$

and thus

$$\frac{J_d}{\tau} \to M \qquad \text{for } \rho \to \infty.$$

TABLE I

THE MINIMAL NORMALIZED COST OF THE DIFFERENT SCHEMES.

control scheme	communication scheme	minimal normalized cost J^*/τ
time-triggered	TDMA	$0.5N+1$
	FDMA	$1.5N$
event-based	unslotted ALOHA	$2e^{2\rho_\Sigma^{u*}}N+1 \approx 3.26N+1$
	slotted ALOHA	$e^{\rho_\Sigma^{s*}}N+1.5 \approx 1.63N+1.5$
	non-persistent CSMA	$N+1$
	1-persistent CSMA with infinite queue	$\frac{N-3}{6(1-\sqrt{3/N})} + \frac{\sqrt{N}+\sqrt{3}-6/\sqrt{N}}{2(-3/\sqrt{N}+\sqrt{3})}$ $\approx \frac{1}{6}N + \frac{\sqrt{N}}{2\sqrt{3}}$ for large N

To sum up, if the load is low and the queue size is not too small, then the cost is similar to the cost of event-based control with 1-persistent CSMA with an infinite queue. If the system becomes heavily overloaded, i.e., $\rho \to \infty$ the cost remains finite and converges to the sum of the number of agents and the queue size.

E. Non-persistent CSMA (Erlang's Loss Model)

As already noted, we do not buffer packets and retry to send them after a random time if the medium is busy. Instead, we drop the packet and shift the bounds in order to generate a new, meaningful packet. Due to this assumption, we model the non-persistent CSMA as a queueing system without a waiting room, i.e., $M = 1$, which is known as Erlang's loss model. Thus, there is no queueing delay and the loss probability is

$$p = \frac{\rho_\Sigma}{1+\rho_\Sigma}, \tag{31}$$

see, e.g. [18]. By using (31) in (5) and $d = \tau$, we get for the normalized cost

$$\frac{J}{\tau} = \frac{1}{\rho} + N + 1 = \frac{N}{\rho_\Sigma} + N + 1. \tag{32}$$

Note that the cost is decreasing with increasing ρ_Σ. Consequently, the minimal cost is achieved for $\rho \to \infty$.

Theorem 6: For event-based control with non-persistent CSMA, the minimal normalized cost is

$$\frac{J_{eb}^{e*}}{\tau} = N + 1. \tag{33}$$

Note that $\rho \to \infty$ is achieved when $\Delta \to 0$, i.e., the boundary increments are so small, that each agent is trying to send all the time. As a consequence thereof, the systems are sampled randomly and the shared medium is fully utilized. Note that the cost is twice the cost of time-triggered control if the cost due to the delay, which is equal, is not taken into account.

VI. COMPARING THE DIFFERENT SCHEMES

In this section, we finally compare the different control and communication schemes. Therefore, Table I shows how the different minimal costs scale with the number of agents N. Except for event-based control with 1-persistent CSMA with an infinite queue, the minimal cost is affine in the number of agents, making it easy to compare the different costs.

Moreover, Fig. 1 shows the normalized costs for a system with $N = 8$ agents. The red lines show the costs of time-triggered control with a TDMA and FDMA communication system. The green lines show the costs of event-based

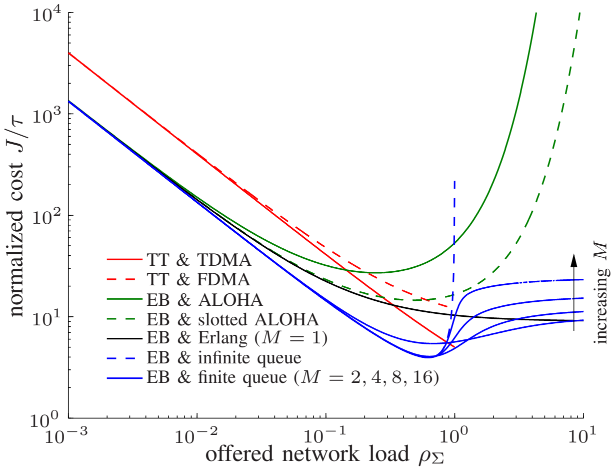

Fig. 1. The normalized cost for $N = 8$ agents.

control with the two versions of ALOHA. The dashed blue line shows the cost of event-based control with 1-persistent CSMA with an infinite queue, whereas the solid blue lines show the costs of event-based control with a finite queue. Finally, the black line shows the cost of event-based control with non-persistent CSMA.

If the network load is relatively low (ρ_Σ between 10^{-3} and 10^{-2} in Fig. 1), the additional costs due to loss and delay are not significant. Consequently, the cost is determined by the sampling scheme and does not depend on the communication protocol. Here, the cost can be decreased by increasing the network load. If the network load is further increased ($\rho_\Sigma > 10^{-2}$ in Fig. 1), then the additional costs due to loss and delay become significant and the effects of the different communication protocols become dominant.

Event-based control with 1-persistent CSMA and an infinite queue gives the minimal cost of all analyzed control and communication schemes but might be risky. If the offered network load ρ_Σ exceeds one, the cost becomes infinite. Fortunately, the cost remains finite if the queue is finite. If the network load is not too high, we see no significant difference in Fig. 1 between the cost with an infinite queue and the cost with a properly chosen finite queue.

Note that, except for the non-persistent CSMA, the network load should always be limited. Obviously, $\rho_\Sigma = 1$ is a hard limit for time-triggered control. On the other hand, the limit for the event-based control is soft and depends on the protocol. Increasing the network load beyond the optimal one is possible but should be avoided because the cost due to loss and delay become significant.

VII. Conclusion

In this work, we analyzed the performance of impulsive control of an integrator system with different communication systems with a shared medium. In doing so, we saw that the performance depends significantly on the control strategy (time-triggered vs. event-based) and the communication protocol. Consequently, there is evidence to suggest that the communication protocol must be taken into account when analyzing and designing networked control systems, making these steps even more interesting and challenging.

Obviously, the analysis of this work should be extended to scalar systems and non impulsive control. Moreover, event-based control with further communication protocols needs to be analyzed.

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