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Robust $\mathcal{H}_\infty$Output Feedback Control of Discrete-Time Networked Systems With Adaptive Quantizers

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Abstract— **Quantization effects are inevitable in networked control systems (NCSs). These quantization effects can be reduced by increasing the number of quantization levels. However, increasing the number of quantization levels may lead to network congestion, (i.e., the network needs to transfer more information than its capacity). In this paper, we investigate the problem of designing a robust $\mathcal{H}_\infty$output feedback controller for discrete-time networked systems with an adaptive quantization density or limited information. More precisely, the quantization density is designed to be a function of the network load condition which is modeled by a Markov process. A stability criterion is developed by using Lyapunov-Krasovskii functional and sufficient conditions for the existence of a dynamic quantized output feedback controller are given in terms of Bilinear Matrix Inequalities(BMIs). An iterative algorithm is suggested to obtain quasi-convex Linear Matrix Inequalities (LMIs) from BMIs. An example is presented to illustrate the effectiveness of the proposed design.**

Keywords: Networked Control Systems (NCSs), Adaptive quantization Density, Sector Bound Approach, Markovian Jump Systems(MJSs), Bernoulli Sequence.

I. Introduction

With emerging technologies such as shared digital wired and wireless networks, new research areas are opened in the field of networked control systems (NCSs). NCSs are distributed systems in which plants, sensors, controllers and actuators are spatially distributed and interconnected through communication networks. This development has greatly improved modularity, system flexibility and reduced processing cost. However, NCSs also bring many new challenges in control system design such as network-induced delays, packet dropouts, quantization errors, variable transmission intervals, network security and other communication constraints.

The issues of network induced time delays and packet dropouts have been considered by many researchers; see [1]-[11]. Many researchers have used Markov processes to model the randomness of the network-induced delays [3], [12], [13], [14], [18]. The Markov chain takes values, which corresponds to network-induced delays, in a finite set based on known probabilities. However, the main drawback of the aforementioned papers is that real communication networks are not able to send data with infinite precision. In communication networks, the length of each data packet is finite, therefore, in order to improve the performance of NCSs, the effect of data quantization must be incorporated into any controller design. In most studies, the quantization error is treated as an uncertainty in stability analysis and controller synthesis [5], [4], [15], [16], [17]. The size of the uncertainty depends on the quantization density and for a logarithm quantizer, the size of uncertainty can be bounded by a sector [19]. The main drawback of aforementioned papers is the quantization density is assumed to be fixed.

In [20], it has mentioned that quantization process is useful in NCSs . Quantizers with coarser quantization densities help in reducing the network congestion. Consequently, network-induced delays can be reduced because less information is transmitted. However, the coarser the quantization density the larger the quantization error. Hence, there exists a trade off between the quantization error and the network congestion or network-induced delay. In this paper, the quantization density is designed to be a function of the network load condition which is modeled by a Markov process. More precisely, when the network load is heavy, a coarser quantization density is used so that less number of information is transmitted. While in the lighter network load case, a finer quantization density is selected. In doing so, the network load condition can be sustained or maintained. Based on the Lyapunov-Krasovskii functional approach, stability criterion and the design procedures for an output feedback with an adaptive quantization density are given in terms of Bilinear Matrix Inequalities (BMIs) which are then converted into quasi-convex LMIs to be solved by using an iterative cone complementarity algorithm [25].

The main contributions of the paper can be summarized as follows:

- To ease the network congestion, the quantization density is designed to be a function of the network load which is modeled by a Markvo process. To the best of the authors' knowledge, this issue has never been investigated in the literatures.
- The controllers are parameterized by BMIs whose dimensions depend upon the dimension of the state variable of the open loop system and not on the number

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of the modes of the Markov chain. This decreases the computational burden and it has not been formally investigated in NCSs.

This paper is organized into five sections. In Section II, system description, quantization error modeling, packet dropouts and problem formulation are presented. Main results for stability criterion and controller synthesis are given in Section III. Conclusions are given in Section IV.

II. System Description and Definitions

A simple networked control system is shown in Figure 1. A class of uncertain discrete-time linear systems under

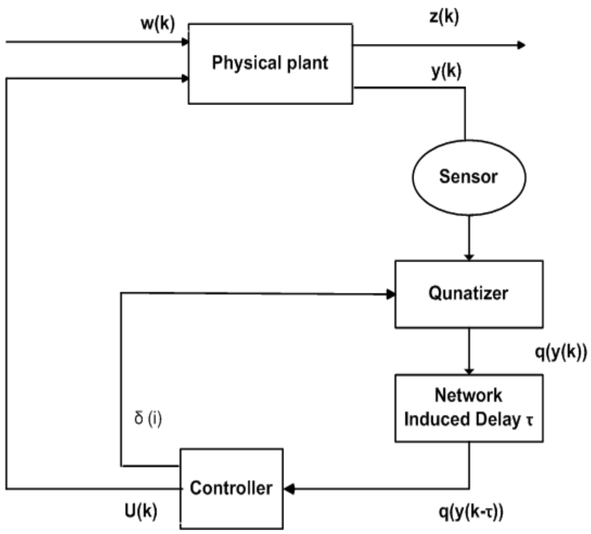

Fig. 1. Layout of the networked control systems with quantizer

consideration is described by the following model:

$$\begin{aligned} x(k+1) &= [A+\Delta A(k)]x(k)+[B_1+\Delta B_1(k)]w(k) \\ &\quad +[B_2+\Delta B_2(k)]u(k), \quad x(0)=0 \\ z(k) &= [C_1+\Delta C_1(k)]x(k)+[D_{11}+\Delta D_{11}(k)] \\ &\quad w(k)+[D_{12}+\Delta D_{12}(k)]u(k) \\ y(k) &= C_2 x(k) \end{aligned} \tag{1}$$

where $x(k) \in \Re^n, u(k) \in \Re^m, z(k) \in \Re^{m_1}, y(k) \in \Re^{m_2}$ are the state, input, controlled output and measured output, respectively. $w(k) \in \Re^{m_3}$ is the disturbance which belongs to $\mathcal{L}_2[0,\infty)$, the space of square summable vector sequence over $[0,\infty)$. The matrices A, B_1, B_2, C_1, D_{11}, D_{12} and C_2 are known matrices with appropriate dimensions. The matrix functions $\Delta A(k)$, $\Delta B_1(k)$, $\Delta B_2(k)$, $\Delta C_1(k)$, $\Delta D_{11}(k)$ and $\Delta D_{12}(k)$ represent the time-varying uncertainties in the system which satisfy the following assumption

Assumption 2.1:

$$\begin{bmatrix} \Delta A(k) & \Delta B_1(k) & \Delta B_2(k) \\ \Delta C_1(k) & \Delta D_{11}(k) & \Delta D_{12}(k) \end{bmatrix} = E_g F(k) H_g$$

$$E_g = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}, H_g = \begin{bmatrix} H_1 & H_2 & H_3 \end{bmatrix}$$

where E_g and H_g are known matrices which characterize the structure of the uncertainties. Furthermore, there always exist a positive-definite matrix $\mathcal{W}$ such that the following inequality holds:

$$F^T(k)\mathcal{W}F(k) \leq \mathcal{W}$$

where $F(k)$ is a time varying function.

Quantizer Modeling and Description:

The quantization process is useful in NCSs design. Network congestion and network-induced time delay can be reduced by using a coarser quantizer. Hence, by adjusting the quantization density, the network load capacity can be sustained.

Let $\{r_k, k\}$ be a discrete homogeneous Markov chain taking values in a finite set $\mathcal{S} = \{1, 2, \cdots, s\}$, with the following transition probability from mode i at k to mode j at time $k+1$

$$p_{ij} := \textbf{Prob}\{r_{k+1} = j | r_k = i\}$$

where $i, j \in \mathcal{S}$.

In this paper, the quantization density is designed to be a function of the network load condition. Hence, the quantization density, δ, is described as a finite state Markov process as $\delta = \delta(r_k)$. On the basis of this technique, a network load dependent quantizer is proposed as follows:

$$q(\nu, i) = \begin{cases} \rho^h(i) & \text{if } \frac{1}{1+\delta(i)}\rho^h(i) < \nu \leq \frac{1}{1-\delta(i)}\rho^h(i), \\ & \quad \nu > 0, h = 0, \pm 1, \pm 2, \cdots \\ 0, & \text{if } \nu = 0 \\ -q(-\nu, i) & \text{if } \nu < 0 \end{cases} \tag{2}$$

where $0 < \rho(i) < 1$ is the quantization density of $q(\cdot,\cdot)$, and $\delta(i)$ is related to $\rho(i)$ by

$$\delta(i) = \frac{1-\rho(i)}{1+\rho(i)} \tag{3}$$

The associated quantized set $\mathcal{U}$ is given by

$$\mathcal{U} = \{\pm\rho^h(i), h = 0, \pm 1, \pm 2, \cdots\} \bigcup \{0\}. \tag{4}$$

Now define the quantization error as

$$e(k, i) = q(v(k), i) - v(k) = \Delta_q(k, i)v(k), \tag{5}$$

where $v(k)$ is signal to be quantized, $q(v(k), i)$ is the quantized signal. It has been shown in [19] that $\Delta_q(k, i) \in [-\delta(i), \delta(i)]$.

The measured output $y(k)$, which is used as feedback information for controller, may not be available all the time due to packet dropouts. To compensate these packet dropouts, a stochastic variable following Bernoulli sequence is used. We are interested in the following mode dependent dynamic output feedback control law:

$$\begin{aligned} \hat{x}(k+1) &= \alpha(k)\{A_c(i)\hat{x}(k) + B_c(i)q(y(k-\tau(k)), i)\} \\ &\quad +(1-\alpha(k))A_{cf}(i)\hat{x}(k) \\ u(k) &= C_c(i)\hat{x}(k) \end{aligned} \tag{6}$$

where $A_c(i)$, $A_{cf}(i)$, $B_c(i)$, $C_c(i)$ are controller matrices. $\alpha(k) \in\in 0, 1$ is random variable following Bernoulli random distribution:

$$\alpha(k) = \begin{cases} 1, & \text{if feedback information is available} \\ 0, & \text{Without feedback signal} \end{cases}$$

Assume that $\alpha(k)$ has probability:

$$Prob\{\alpha(k) = 1\} = E\{\alpha(k)\} = \alpha, Prob\{\alpha(k) = 0\} = 1-\alpha$$

where $0 \leq \alpha \leq 1$ is a constant and

$$E\{\alpha(k) - \alpha\} = 0, \beta^2 \equiv E\{(\alpha(k) - \alpha)^2\} = \alpha(1 - \alpha)$$

where $E(\cdot)$ is the expectation operator and β^2 is the variance. $\tau(k)$ is the time varying delay satisfying:

$$0 < \underline{\tau} \leq \tau(k) \leq \bar{\tau}$$

where $\underline{\tau}$ and $\bar{\tau}$ are known constants. It is worth mentioning that in NCSs, there exist various type of delays such as sensor to controller delays $\tau_{sc}(k)$; controller to actuator delays $\tau_{ca}(k)$ and processing delays $\tau_c(k)$. However, these delays can all be lumped together [22]:

$$\tau(k) = \tau_{sc}(k) + \tau_{ca}(k) + \tau_c(k)$$

Using (5), the closed loop system of (1) with (6) is given as follows:

$$\begin{aligned} \zeta(k+1) &= [A_{cl1}(i) + \alpha(k)A_{cl2}(i) + (1 - \alpha(k))A_{cl3}(i) \\ &\quad +\bar{E}_1 F(k)\bar{H}_1(i)]\zeta(k) + [\bar{B}_1 + \bar{E}_1 F(k)H_2] \\ &\quad w(k) + \alpha(k)B_{cl}(i)(1 + \Delta_q(k, i))\bar{C}_2\zeta(k - \tau(k)) \\ z(k) &= [C_{cl}(i) + E_2 F(k)\bar{H}_1(i)]\zeta(k) + \\ &\quad [D_{11} + E_2 F(k)H_2]w(k). \end{aligned} \tag{7}$$

where $\zeta(k) = [x(k) \;\; \hat{x}(k)]^T$,

$$\begin{aligned} A_{cl1}(i) &= \begin{bmatrix} A & B_2C_c(i) \\ 0 & 0 \end{bmatrix}, A_{cl2}(i) = \begin{bmatrix} 0 & 0 \\ 0 & A_c(i) \end{bmatrix}, \\ A_{cl3}(i) &= \begin{bmatrix} 0 & 0 \\ 0 & A_{cf}(i) \end{bmatrix}, B_{cl}(i) = \begin{bmatrix} 0 \\ B_c(i) \end{bmatrix}, \\ \bar{B}_1 &= \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \bar{C}_2 = \begin{bmatrix} C_2 & 0 \end{bmatrix}, \bar{E}_1 = \begin{bmatrix} E_1 \\ 0 \end{bmatrix} \\ \bar{H}_1(i) &= \begin{bmatrix} H_1 & H_3C_c(i) \end{bmatrix}, C_{cl}(i) = \begin{bmatrix} C_1 & D_{12}C_c(i) \end{bmatrix} \end{aligned}$$

Equation (7) can be simplified as:

$$\begin{aligned} \zeta(k+1) &= [A_{cl}(i) + \bar{E}_1 F(k)\bar{H}_1(i)]\zeta(k) + [\bar{B}_1 + \bar{E}_1 F(k) \\ &\quad H_2]w(k) + \alpha B_{cl}(i)(1 + \Delta_q(k, i))\bar{C}_2\zeta(k - \\ &\quad \tau(k)) + (\alpha(k) - \alpha)\{(A_{cl2}(i)\zeta(k) - A_{cl3}(i) \\ &\quad \zeta(k) + B_{cl}(i)(1 + \Delta_q(k, i))\zeta(k - \tau(k))\} \\ z(k) &= [C_{cl}(i) + E_2 F(k)\bar{H}_1(i)]\zeta(k) \\ &\quad +[D_{11} + E_2 F(k)H_2]w(k). \end{aligned} \tag{8}$$

where

$$A_{cl}(i) = A_{cl1}(i) + \alpha A_{cl2}(i) + (1 - \alpha)A_{cl3}(i)$$

The problem under study is formulated as follows.

Problem Formulation:

Given a prescribed $\gamma > 0$ and quantization densities $\rho(i)$, design a dynamic output feedback controller of the form (6) such that

1) the system , given in (7) with (6) and $w(k) = 0$ is stochastically stable, i.e, there exists a constant $0 < \alpha_1 < \infty$ such that

$$E\left\{\sum_{\ell=0}^{\infty} \zeta^T(\ell)\zeta(\ell)\right\} < \alpha_1 \tag{9}$$

for all $\zeta(0)$ and r_0.

2) Under the zero-initial condition, the controlled output $z(k)$ satisfies $\mathcal{H}_\infty$performance:

$$E\left\{\sum_{k=0}^{\infty} z^T(k)z(k)|r_0\right\} < \gamma^2 \sum_{k=0}^{\infty} w^T(k)w(k) \tag{10}$$

for all nonzero $w(k)$.

The following lemma plays an important role in the derivation of the main results.

Lemma 2.1: Let $\bar{x}(k) = x(k+1) - x(k)$ and $\tilde{\zeta}(k) = \Big[\zeta^T(k) \;\; \zeta^T(k - \tau(k)) \;\; w^T(k) \;\; \zeta^T(k)\bar{H}_1^T(i)F^T(k)$ $\zeta^T(k - \tau(k))\bar{C}_2^T\Delta q^T(k) \quad w^T(k)H_2^T F^T(k)\Big]^T \in \Re^l$, then for any matrices $R \in \Re^{n\times n}$, $M \in \Re^{n\times l}$ and $Z \in \Re^{l\times l}$ satisfying

$$\begin{bmatrix} R & M \\ M^T & Z \end{bmatrix} \geq 0 \tag{11}$$

the following inequality holds

$$\sum_{i=k-\bar{\tau}}^{k-1} \bar{x}^T(i)R\bar{x}(i) \leq \tilde{\zeta}^T(k)\Big\{\Upsilon_1 + \Upsilon_1^T + \bar{\tau}Z\Big\}\tilde{\zeta}(k) \tag{12}$$

where $\Upsilon_1 = M^T[\text{diag}\{I, 0\} \;\; \text{diag}\{-I, 0\} \;\; 0 \;\; 0 \;\; 0 \;\; 0]$. Proof is given in our previous work [21].$\nabla\nabla\nabla$

III. Stability Analysis and Synthesis of NCSs

Stability criterion for uncertain discrete-time systems with random quantization densities are given in the following theorem.

Theorem 3.1: For given controller matrices $A_c(i)$, $A_{cf}(i)$, $B_c(i)$ and $C_c(i)$and quantization densities $\rho(i)$ where $i = 1, \cdots, s$, $\gamma > 0$, if there exist sets of positive-definite matrices $P(i)$, $R_1(i)$, R_1, $W_1(i)$, $W_2(i)$,$W_3(i)$,Q, $Z(i)$ and matrices $M(i)$ satisfying the following inequalities

$$R_1 > R_1(i) \tag{13}$$

$$\begin{bmatrix} R_1(i) & M(i) \\ * & Z(i) \end{bmatrix} \geq 0 \tag{14}$$

$$\begin{aligned} &\Lambda(i) + \Gamma_1^T(i)\tilde{P}(i)\Gamma_1(i) + \Gamma_2^T(i)\bar{\tau}R_1\Gamma_2(i) + \\ &\Upsilon_1(i) + \Upsilon_1^T(i) + \bar{\tau}Z(i) + \Gamma_3^T(i)\Gamma_3(i) < 0 \end{aligned} \tag{15}$$

where

$$
\begin{aligned}
\Gamma_1(i) &= \left[\begin{array}{cccccc} \bar{A}_{cl}(i) & \bar{B}_{cl}(i)\bar{C}_2 & \bar{B}_1 & \bar{E}_1 & \bar{B}_{cl}(i)\bar{C}_2 & \bar{E}_1 \end{array}\right] \\
\Gamma_2(i) &= \left[\begin{array}{cccccc} \bar{A} & 0 & \bar{B}_1 & \bar{E}_1 & 0 & \bar{E}_1 \end{array}\right] \\
\Gamma_3(i) &= \left[\begin{array}{cccccc} C_{cl}(i) & 0 & D_{11} & E_2 & 0 & E_2 \end{array}\right] \\
\Lambda(i) &= \text{diag}\Big\{\Big((\bar{\tau}-\underline{\tau}+1)Q + \bar{H}_1^T(i)W_1(i)\bar{H}_1(i) - P(i)\Big), (\delta^2(i)W_2(i), -Q), \Big(H_2^TW_3(i)H_2 - \gamma^2 I\Big), -W_1(i), -W_2(i), -W_3(i)\Big\} \\
\tilde{P}(i) &= \sum_{j=1}^{s} p_{ij}P(j) \\
\Upsilon_1(i) &= M^T(i)[\text{diag}\{I,0\} \ \ \text{diag}\{-I,0\} \ \ 0 \ \ 0 \ \ 0 \ \ 0] \\
\bar{A} &= \left[\begin{array}{cc} A-I & B_2C_c(i) \\ 0 & 0 \end{array}\right] \\
\bar{A}_{cl}(i) &= A_{cl}(i) + \beta\{A_{cl2}(i) - A_{cl3}(i)\}, \\
\bar{B}_{cl}(i) &= (\alpha+\beta)B_{cl}(i).
\end{aligned} \quad (16)
$$

Then the closed-loop system is stochastically stable with the prescribed $\mathcal{H}_\infty$performance.

Proof: The system (8) can be written as

$$
\begin{aligned}
\zeta_{k+1} &= \Gamma_1(r_k)\tilde{\zeta}_k \\
z_k &= \Gamma_3(r_k)\tilde{\zeta}_k
\end{aligned} \quad (17)
$$

where $\zeta(k+1) = \zeta_{k+1}$, $\Gamma_1(r_k)$ and $\Gamma_3(r_k)$ are given in (16) and $\tilde{\zeta}_k$ is defined in Lemma 2.1

Select the L-K candidate functional for the closed loop system as:

$$V(\zeta_k, r_k) = V_1(\zeta_k, r_k) + V_2(\zeta_k, r_k) + V_3(\zeta_k, r_k) \quad (18)$$

with

$$V_1(\zeta_k, r_k) = \zeta_k^T P(r_k)\zeta_k \quad (19)$$

$$V_2(\zeta_k, r_k) = \sum_{\ell=-\bar{\tau}}^{-1}\sum_{j=k+\ell}^{k-1} \bar{x}_j^T R_1 \bar{x}_j \quad (20)$$

$$V_3(\zeta_k, r_k) = \sum_{\ell=k-\tau(k)}^{k-1} \zeta_\ell^T Q\zeta_\ell + \sum_{\ell=-\bar{\tau}+2}^{-\underline{\tau}+1}\sum_{j=k+\ell-1}^{k-1} \zeta_j^T Q\zeta_j \quad (21)$$

First forward difference of $V(\zeta_k, r_k)$ is given as follows:

$$\Delta V(\zeta_k, r_k) = \Delta V_1(\zeta_k, r_k) + \Delta V_2(\zeta_k, r_k) + \Delta V_3(\zeta_k, r_k) \quad (22)$$

with

$$
\begin{aligned}
\Delta V_1(\zeta_k, r_k) &= \zeta_{k+1}^T\tilde{P}(r_k)\zeta_{k+1} - \zeta_k^TP(r_k)\zeta_k \\
&= \tilde{\zeta}_k^T\Gamma_1^T(r_k)\tilde{P}(r_k)\Gamma_1(r_k)\tilde{\zeta}_k - \zeta_k^TP(r_k)\zeta_k
\end{aligned} \quad (23)
$$

$$\Delta V_2(\zeta_k, r_k) \le \bar{x}_k^T\bar{\tau}R_1\bar{x}_k - \sum_{\ell=k-\bar{\tau}}^{k-1}\bar{x}_\ell^T R_1\bar{x}_\ell \quad (24)$$

and

$$\Delta V_3(\zeta_k, r_k) \le (\bar{\tau}-\underline{\tau}+1)\zeta_k^TQ\zeta_k - \zeta_{k-\tau(k)}^TQ\zeta_{k-\tau(k)}. \quad (25)$$

Using Lemma 2.1 and $\bar{x}_k = x_{k+1} - x_k = \Gamma_2(r_k)\tilde{\zeta}_k$, we have

$$\Delta V_2(\zeta_k, r_k) \le \tilde{\zeta}_k^T\Big\{\Gamma_2^T(r_k)\bar{\tau}R_1\Gamma_2(r_k) + \Upsilon_1(r_k) + \Upsilon_1^T(r_k) + \bar{\tau}Z(r_k)\Big\}\tilde{\zeta}_k$$

where $\Gamma_2(r_k)$ is given in (16).

Therefore,

$$\Delta V(\zeta_k, r_k) \le -\zeta_k^T\Big(P(r_k) - (\bar{\tau}-\underline{\tau}+1)Q\Big)\zeta_k - \zeta_{k-\tau(k)}^TQ\zeta_{k-\tau(k)} + \tilde{\zeta}_k^T\Big\{\Gamma_1^T(r_k)\tilde{P}(r_k)\Gamma_1(r_k) + \Gamma_2^T(r_k)\bar{\tau}R_1\Gamma_2(r_k) + \Upsilon_1(r_k) + \Upsilon_1^T(r_k) + \bar{\tau}Z(r_k)\Big\}\tilde{\zeta}_k \quad (26)$$

Using Assumption 2.1; and adding and subtracting $\zeta_k^T\bar{H}_1^T(r_k)F_k^TW_1(rk)F_k\bar{H}_1(r_k)\zeta_k$, $\zeta_{k-\tau(k)}^T\Delta_q(k,i)^TW_2(r_k)\Delta_q(k,i)\zeta_{k-\tau(k)}$, $w_k^TH_2^TF_k^TW_3(r_k)F_kH_2w_k$, $z_k^Tz_k$ and $\gamma^2w_k^Tw_k$ to and from (26), we obtain

$$
\begin{aligned}
\Delta V(\zeta_k, r_k) \le{} & -\zeta_k^T\Big(P(r_k) - (\bar{\tau}-\underline{\tau}+1)Q - \bar{H}_1^T(r_k) F_k^TW_1(rk)F_k\bar{H}_1(r_k)\Big)\zeta_k - \zeta_{k-\tau(k)}^T \\
& \left(Q - \delta^2(r_k)W_2(r_k)\right)\zeta_{k-\tau(k)} + \tilde{\zeta}_k^T\Big\{ \Gamma_1^T(r_k)\tilde{P}(r_k)\Gamma_1(r_k) + \Gamma_2^T(r_k)\bar{\tau}R_1\Gamma_2(r_k) \\
& + \Upsilon_1(r_k) + \Upsilon_1^T(r_k) + \bar{\tau}Z(r_k) + \Gamma_3^T(r_k) \Gamma_3(r_k)\Big\}\tilde{\zeta}_k - z_k^Tz_k + \gamma^2w_k^Tw_k - w_k^T\Big(\gamma^2 I \\
& - H_2^TW_3(r_k)H_2\Big)w_k - \zeta_{k-\tau(k)}^T\delta^2(i) W_2(r_k)\zeta_{k-\tau(k)} - \zeta_k^T\bar{H}_1^T(r_k)F_k^TW_1(r_k)F_k \\
& \bar{H}_1(r_k)\zeta_k - w_k^TH_2^TF_k^TW_3(r_k)F_kH_2w_k
\end{aligned} \quad (27)
$$

Using (16), (27) can be rewritten as

$$\Delta V(\zeta_k, r_k) \le \tilde{\zeta}_k^T\Big\{\Lambda(r_k) + \Gamma_1^T(r_k)\tilde{P}(r_k)\Gamma_1(r_k) + \Gamma_2^T(r_k)\bar{\tau}R_1\Gamma_2(r_k) + \Upsilon_1(r_k) + \Upsilon_1^T(r_k) + \bar{\tau}Z(r_k) + \Gamma_3^T(r_k)\Gamma_3(r_k)\Big\}\tilde{\zeta}_k - z_k^Tz_k + \gamma^2w_k^Tw_k \quad (28)$$

Using (15), we have

$$\Delta V(\zeta_k, r_k) \le -z_k^Tz_k + \gamma^2w_k^Tw_k \quad (29)$$

Taking expectation and sum from 0 to ∞ on both sides of (29) yields

$$E\{V(\zeta_\infty, r_\infty)\} - E\{V(\zeta_0, r_0)\} \le -E\Big\{\sum_{\ell=0}^{\infty} z_\ell^Tz_\ell\Big\} + \gamma^2\sum_{\ell=0}^{\infty} w_\ell^Tw_\ell$$

As initial conditions, as given in problem formulation, are considered zero i.e, $V(\zeta_0, r_0) = 0$, so

$$E\Big\{\sum_{\ell=0}^{\infty} z_\ell^T z_\ell\Big\} \le \gamma^2 \sum_{\ell=0}^{\infty} w_\ell^T w_\ell \quad (30)$$

If $w(k) = 0$, $\forall k \ge 0$ closed-loop system should be stochastically stable. From (28) and (15), we learn that

$$V(\zeta_{(k+1)}, r_{(k+1)}) - V(\zeta_k, r_k) \le -\beta \tilde{\zeta}_k^T \tilde{\zeta}_k \quad (31)$$

where $\beta = \inf\{\Lambda(r_k)_{\min}[-\mathcal{M}(r_k)], i \in S\}$ with

$$\begin{aligned} \mathcal{M} &= \Lambda(r_k) + \Gamma_1^T(r_k)\tilde{P}(r_k)\Gamma_1(r_k) + \Gamma_2^T(r_k)\bar{\tau}R_1\Gamma_2(r_k) \\ &+ \Upsilon_1(r_k) + \Upsilon_1^T(r_k) + \bar{\tau}Z(r_k) + \Gamma_3^T(r_k)\Gamma_3(r_k) \quad (32) \end{aligned}$$

Summing from 0 to ∞ and by taking expectation on both sides of (31) gives

$$\begin{aligned} E\{V(\zeta_\infty, r_\infty)\} - E\{V(\zeta_0, r_0)\} &\le -\beta_1 E\Big\{\sum_{k=0}^{\infty} \tilde{\zeta}_\ell^T \tilde{\zeta}_\ell\Big\} \\ &\le -\beta_1 E\Big\{\sum_{k=0}^{\infty} \zeta_\ell^T \zeta_\ell\Big\} \end{aligned} \quad (33)$$

Re-arranging (33), we have

$$\begin{aligned} E\Big\{\sum_{k=0}^{\infty} \zeta_\ell^T \zeta_\ell\Big\} &\le \frac{1}{\beta_1}E\{V(\zeta_0, r_0)\} - \frac{1}{\beta_1}E\{V(\zeta_\infty, r_\infty)\} \\ &\le \alpha_1 \quad (34) \end{aligned}$$

where $\alpha_1 = \frac{1}{\beta_1}E\{V(\zeta_0, r_0)\} < \infty$. This shows that the closed loop system is stable and (9) holds. $\nabla\nabla\nabla$

The following theorem provides procedures for designing a quantized output feedback controller.

Theorem 3.2: For given $\gamma > 0$ and quantization densities $\rho(i)$, if there exist positive symmetric matrices $X(i) > 0$, $Y(i) > 0$, $\mathcal{Y}(i) > 0$, $W_1(i)$, $\mathcal{W}_1(i)$, $\mathcal{W}_2(i)$, $W_2(i)$, $W_3(i)$, $\mathcal{Q}$, Q, $N(i)$, $\mathcal{N}(i)$, R_1, $\mathcal{R}_1$, $R_1(i)$, $S(i,j)$, $\bar{Z}(i)$ and matrices $\mathcal{A}(i)$, $\mathcal{A}_f(i)$, $\mathcal{B}(i)$, $\mathcal{C}(i)$, $J(i)$, $\bar{M}(i)$ satisfying the following inequalities for all $i, j \in \mathcal{S}$

$$R_1 > R_1(i) \quad (35)$$

$$\begin{bmatrix} N(i) & \bar{M}(i) \\ * & \bar{Z}(i) \end{bmatrix} \ge 0 \quad (36)$$

$$\begin{bmatrix} S(i,j) & J^T(i) \\ * & Y(j) \end{bmatrix} > 0 \quad (38)$$

$$\begin{bmatrix} R_1(i) & \mathcal{T}^T(i) \\ * & \mathcal{N}(i) \end{bmatrix} > 0 \quad (39)$$

$$\mathcal{Y}(i)Y(i) = I,\ \mathcal{Q}Q = I,\ \mathcal{N}(i)N(i) = I,\ \mathcal{R}_1R_1 = I. \quad (40)$$

$$\tilde{W}_1(i)W_1(i) = I, \tilde{W}_2(i)W_2(i) = I, \quad (41)$$

where

$$\begin{aligned} \bar{\Lambda}(i) &= \text{diag}\left\{ \begin{array}{c} -\begin{bmatrix} Y(i) & I \\ I & X(i) \end{bmatrix}, -Q, \\ \big(H_2^T W_3(i) H_2 - \gamma^2 I\big), \\ -W_1(i), -W_2(i), -W_3(i) \end{array} \right\} \\ \bar{\Upsilon}(i) &= \bar{M}^T(i)[\text{diag}\{I, 0\}\ \ \text{diag}\{-I, 0\}\ \ 0\ \ 0\ \ 0\ \ 0] \\ \bar{\Gamma}_1(i) &= \begin{bmatrix} \check{A}_{cl}(i) & \check{B}_{cl}(i)\bar{C}_2 & \check{B}_1 & \check{E}_1 & \check{B}_{cl}(i)\bar{C}_2 & \check{E}_1 \end{bmatrix} \\ \bar{\Gamma}_2(i) &= \sqrt{\bar{\tau}}\begin{bmatrix} \check{A}(i) & 0 & \bar{B}_1 & \bar{E}_1 & 0 & \bar{E}_1 \end{bmatrix} \\ \bar{\Gamma}_3(i) &= \begin{bmatrix} \check{C}_{cl}(i) & 0 & D_{11} & E_2 & 0 & E_2 \end{bmatrix} \\ \bar{\Gamma}_4(i) &= \begin{bmatrix} (\sqrt{\bar{\tau} - \underline{\tau} + 1})T(i) & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \bar{\Gamma}_5(i) &= \begin{bmatrix} \check{H}_1(i) & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \bar{\Gamma}_6(i) &= \begin{bmatrix} 0 & \delta(i) & 0 & 0 & 0 & 0 \end{bmatrix} \\ \tilde{\Xi}(i) &= \begin{bmatrix} -(\tilde{S}(i) - J(i) - J^T(i)) & I \\ I & \tilde{X}(i) \end{bmatrix} \\ \tilde{X}(i) &= \sum_{j=1}^{s} p_{ij}X(j), \tilde{S}(i) = \sum_{j=1}^{s} p_{ij}S(i,j) \\ \check{A}_{cl} &= \begin{bmatrix} AY(i) + B_2\mathcal{C}(i) & A \\ (\alpha+\beta)\mathcal{A}(i) + (1-\alpha-\beta)\mathcal{A}_f(i) & \tilde{X}(i)A \end{bmatrix} \\ \check{B}_{cl} &= \begin{bmatrix} 0 \\ (\alpha+\beta)\mathcal{B}(i) \end{bmatrix}, \check{B}_1 = \begin{bmatrix} B_1 \\ \tilde{X}(i)B_1 \end{bmatrix} \\ \check{E}_1 &= \begin{bmatrix} E_1 \\ \tilde{X}(i)E_1 \end{bmatrix}, \check{A} = \begin{bmatrix} (A-I)Y(i) & B_2\mathcal{C} \\ 0 & 0 \end{bmatrix} \\ \check{C}_{cl}(i) &= [C_1Y(i) + D_{12}\mathcal{C}(i)\ \ C_1], T(i) = \begin{bmatrix} Y(i) & I \\ Y(i) & 0 \end{bmatrix} \\ \check{H}_1(i) &= [H_1Y(i) + H_3\mathcal{C}\ \ H_1], \mathcal{T}(i) = \begin{bmatrix} 0 & \mathcal{Y}(i) \\ I & -I \end{bmatrix}. \end{aligned} \quad (42)$$

Then the closed-loop system is stochastically stable with the prescribed $\mathcal{H}_\infty$ performance. Furthermore, a suitable controller is given as follows

$$\begin{aligned} A_c(i) &= \Big(\textstyle\sum_{j=1}^{s} p_{ij}Y^{-1}(j) - \tilde{X}(i)\Big)^{-1} \\ &\quad \Big(\mathcal{A}(i) - \tilde{X}(i)(AY(i) + B_2\mathcal{C}(i))\Big)Y^{-1}(i) \\ A_{cf}(i) &= \Big(\textstyle\sum_{j=1}^{s} p_{ij}Y^{-1}(j) - \tilde{X}(i)\Big)^{-1} \\ &\quad \Big(\mathcal{A}_f(i) - \tilde{X}(i)(AY(i) + B_2\mathcal{C}(i))\Big)Y^{-1}(i) \\ B_c(i) &= \Big(\textstyle\sum_{j=1}^{s} p_{ij}Y^{-1}(j) - \tilde{X}(i)\Big)^{-1}\mathcal{B}(i) \\ C_c(i) &= \mathcal{C}(i)Y^{-1}(i). \end{aligned} \quad (43)$$

Proof: The proof is omitted due to the space limitation. $\nabla\nabla\nabla$

Using the cone complementary algorithm [25], the feasibility problem formulated by (35)-(41) which is not a convex problem can be converted into the following nonlinear minimization problem: Minimize $Tr\Big(Y(i)\mathcal{Y}(i) + Q(i)\mathcal{Q}(i) + N(i)\mathcal{N}(i) + R_1\mathcal{R}_1 + W_1(i)\mathcal{W}_1(i) + W_2(i)\mathcal{W}_2(i)\Big)$

Subject to (35)-(39) and

$$\begin{bmatrix} \bar{\Lambda}(i)+\bar{\Upsilon}_1(i)+\bar{\Upsilon}_1^T(i)+\bar{\tau}\bar{Z}(i) & \bar{\Gamma}_1^T(i) & \bar{\Gamma}_2^T(i) & \bar{\Gamma}_3^T(i) & \bar{\Gamma}_4^T(i) & \bar{\Gamma}_5^T(i) & \bar{\Gamma}_6^T(i) \\ * & -\tilde{\Xi}(i) & 0 & 0 & 0 & 0 & 0 \\ * & * & -\mathcal{R}_1 & 0 & 0 & 0 & 0 \\ * & * & * & -I & 0 & 0 & 0 \\ * & * & * & * & -\mathcal{Q} & 0 & 0 \\ * & * & * & * & * & -\tilde{W}_1(i) & 0 \\ * & * & * & * & * & * & -\tilde{W}_2(i) \end{bmatrix} < 0 \quad (37)$$

$$\begin{bmatrix} Y(i) & I \\ I & \mathcal{Y}(i) \end{bmatrix} \geq 0, \begin{bmatrix} Q & I \\ I & \mathcal{Q} \end{bmatrix} \geq 0,$$
$$\begin{bmatrix} N(i) & I \\ I & \mathcal{N}(i) \end{bmatrix} \geq 0, \begin{bmatrix} R_1 & I \\ I & \mathcal{R}_1 \end{bmatrix} \geq 0,$$
$$\begin{bmatrix} W_1(i) & I \\ I & \tilde{W}_1(i) \end{bmatrix} \geq 0 \begin{bmatrix} W_2(i) & I \\ I & \tilde{W}_2(i) \end{bmatrix} \geq 0. \quad (44)$$

Algorithm to solve this optimization problem is omitted due to space limitations.

IV. CONCLUSIONS

A novel method for designing a robust $\mathcal{H}_\infty$output feedback control for discrete-time networked systems with an adaptive quantization density or limited information is proposed. The quantization density is designed to be a function of the network load condition which is modeled by a Markov process. Bernoulli random sequences are used to model packet dropouts in the network. Stability criterion and controller design are developed using the Lyapunov-Krasovskii functional approach. The design procedures for a robust $\mathcal{H}_\infty$quantized dynamic output feedback controller are given in terms of BMIs. A cone complementarity algorithm is suggested to convert BMIs into quasi-convex LMIs. Through a simulation example of an LTI system, effectiveness of proposed design is verified as well.

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