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A new state observer for switched linear systems using a non-smooth optimization approach

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Abstract— **A new continuous state observer is derived for discrete-time linear switched systems under the assumptions that neither the continuous state nor the discrete state are known. A specificity of the proposed observer is that, contrary to the state-of-art, it does not require an explicit estimation of the discrete state. The key idea of the method consists in minimizing a non-smooth ℓ_2-norm-based weighted cost functional, constructed from the matrices of all the subsystems regardless of when each of them is active. In the light of some recent development in the literature of compressed sensing, the minimized cost functional has the ability to promote sparsity in a way that makes prior knowledge/estimation of the discrete mode sequence unnecessary.**

I. INTRODUCTION

For a dynamic system, the state usually refers to a vector of signals that encodes, from a modeling perspective, the full information about the past of that system. There are many practical engineering situations in which an accurate estimate of the state is desirable. Recovering the full state from partial observations has many quite obvious advantages. For example, this can help get around the necessity of instrumenting the system with some possibly expensive state sensors. Another application of state estimation is in fault detection. In effect, comparing a model-based estimate of the state to its measured version can bring out model inconsistencies thereby enabling the detection of changes in the system whose nominal behavior is described by that model. Also, in state feedback control systems, a complete knowledge of the state is required for the implementation of the controller. Depending on whether past, present or future states are estimated, the terminologies smoothing, filtering or prediction are respectively used.

Prior work. The state estimation problem has been extensively investigated for the classes of linear and nonlinear dynamic systems. For the class of hybrid systems, the interest of researchers in the state estimation problem is more recent, though a number of relevant approaches have already been published in the existing literature [7], [2], [3], [13], [15], [23], [5]. The earliest works on state estimation for hybrid systems were dedicated to the class of Jump Markov Linear Systems [1], [23], [12], [13]. In these systems the switching mechanism is a first-order Markov chain and can be determined from the observations. For such systems the estimation problem can be tackled in a stochastic framework using a particle filtering approach [13] or a bank of Kalman filters [23].

The main challenge of hybrid state estimation lies in the fact that the discrete mode is unknown and also needs to be inferred from the input-output measurements along with the continuous state. For switched systems in particular, there is not necessarily a model of the switching law that can be learnt from data. This is because the class of switched systems can cover a variety of switching mechanisms; the switches can in this case be e.g. exogenous, deterministic, state-driven, event-driven, time-driven or even totally random. The work reported in [3] which is one of the first proposed approaches for switched linear systems, assumes that the discrete mode sequence is available so that the problem reduces to the synthesis of a classical Luenberger type of observer for each linear subsystem. In [2], the discrete mode is assumed unknown and a receding horizon procedure is presented. A moving horizon approach was also followed in [15], [20], [21] to derive a state smoother for the class of piecewise affine systems.

Our approach. In this paper we develop a new state observer for discrete-time linear switched systems. Since the discrete state is also unknown, a very natural approach would be to solve a mixed integer-continuous optimization problem for both the discrete and continuous states. Most of the existing hybrid state estimators for switched linear systems follow this idea. Unfortunately this may be computationally heavy and sometimes intractable. The main feature of the method introduced in this paper is that it approximates the mixed integer-continuous optimization problem with a non-smooth but still continuous optimization one. Moreover this last problem is convex and therefore solvable by efficient and well-documented techniques [8], [16]. Thanks to the interesting properties of the proposed particular cost functional, we can get around the necessity of explicitly estimating the system discrete state. At each time t, the continuous state is computed in two steps : 1) a prediction step during which the state estimate is predicted based on the system matrices and the previous estimate ; this is performed through the minimization of an ℓ_2-norm criterion involving all the subsystems' matrices and 2) a correction step during which the measurements at time t are used to refine the predicted state.

Outline. We start with presenting some preliminary concepts in Section II. We distinguish between two cases: the case when the modes are instantaneously discernible, is treated first in Section III; the more general situation where the

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modes are discernible over a finite data-window, is dealt with in Section IV. Numerical experiments are depicted in Section V. Some concluding remarks are provided in Section VI.

II. Preliminaries

We consider a discrete-time switched linear system described in state space form by

$$\begin{cases} x(t+1) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \\ y(t) &= C_{\sigma(t)}x(t) + D_{\sigma(t)}u(t) \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^{n_u}$, $y(t) \in \mathbb{R}^{n_y}$ are respectively the state, the input and the output of the system at time $t \in \mathbb{N}$. $\sigma(t) \in Q$ is the discrete mode (state), that is, the index of the subsystem which is active at time t ; Q is the finite set of discrete modes with cardinality $|Q| = s$. For each discrete mode $\sigma \in Q$, $A_\sigma, B_\sigma, C_\sigma, D_\sigma$ are the matrices (of appropriate dimensions) which are associated with that mode.

Given the system matrices $A_\sigma, B_\sigma, C_\sigma, D_\sigma$, $\sigma \in Q$, and the past input-output observations $(u(\tau), y(\tau))$, $\tau = 0, \ldots, t-1$, the problem discussed in this paper is concerned with the estimation of the continuous state $x(t)$ at time t under the assumption that the discrete state sequence is unknown. We more specifically try to answer the question of whether it is possible to reconstruct the continuous state without an explicit knowledge/estimate of the discrete mode sequence.

For the need of convergence analysis, we shall associate with system (1) its uncontrolled version,

$$z(t+1) = A_{\sigma(t)}z(t), \quad z(0) = x(0) \tag{2}$$

where the switching $\sigma(\cdot)$ is the same as the one in system (1).

Assumption 1: System (2) is uniformly exponentially stable for any initial state $z(0)$ and under any switching path. That is, there exist (independently of the initial state and the switching sequence) some numbers λ and c, with $\lambda \in]0\ 1[$ and $0 < c < \infty$, such that

$$\|z(t)\|_2 \leq c\lambda^t \|z(0)\|_2 \quad \forall t.$$

Note from [18] that uniform exponential stability of system (2) is equivalent to Bounded Input-Bounded Output (BIBO) stability of system (1). It is also equivalent to the existence of a finite number k such that $\|A_{i_1} \cdots A_{i_k}\|_2 < 1$ for any selection $(i_1, \ldots, i_k)$ of indices in Q. Let K denote the smallest such number k and define, for future use,

$$\rho = \max_{\substack{(i_1,\ldots,i_q) \in Q^q \\ q \leq K}} \left\|A_{i_1} \cdots A_{i_q}\right\|_2. \tag{3}$$

Intuitively, for any hybrid state estimation method to be successful, there is a reasonable need to assume that the discrete modes are discernible in a certain sense. Some algebraic characterizations of mode discernibility and continuous state observability have been proposed in the literature, e.g., in [22], [4], [24], [14]. Such characterizations are non-trivial and are based on rank tests on some complex structured matrices. Here, due to the non-algebraic nature of the estimation scheme, we will express mode discernibility in simpler terms. With the notation $g_i(t) = y(t) - C_i x(t) - D_i u(t)$, we define a somewhat strong notion of mode discernibility (instantaneous) which is data-dependent, as[1]

$$R = \min_{\substack{t \geq 0 \\ i \neq \sigma(t)}} \|g_i(t)\|_2 > 0 \tag{4}$$

being large in a certain sense to be specified later. If $[C_i\ D_i] \neq [C_j\ D_j]$ for $i \neq j$, then the assumption (4) holds generically with respect to the state and the input. This means that the property $\|g_i(t)\|_2 > 0$, $i \neq \sigma(t)$, holds everywhere in the state-input space $\Omega \subset \mathbb{R}^{n+n_u}$ of the system, except possibly on a set of measure zero, which is precisely the algebraic set defined by $\cup_{i\neq j} \ker([C_i - C_j\ \ D_i - D_j])$, with $\ker(\cdot)$ referring to the kernel space. Assumption (4) does not hold absolutely when the state-input space Ω intersects $\cup_{i\neq j} \ker([C_i - C_j\ D_i - D_j])$. It always holds if $\operatorname{rank}([C_i - C_j\ \ D_i - D_j]) = n + n_u$ for any (i, j) such that $i \neq j$. Requiring R to be strictly positive may appear severely restrictive in general. A more relaxed discernibility assumption will therefore be introduced in Section IV.

III. Observer design when the modes are instantaneously discernible

We first consider the case when the discrete modes are discernible directly from one instantaneous output. A more general situation will be studied in Section IV.

Most conventional approaches for hybrid observer design are based, at least in a first step, on an attempt to recover explicitly the discrete state sequence. The motivation for such a treatment of the state estimation problem is that linear techniques can directly be carried over to hybrid systems once the discrete state is known, see e.g. [3]. However, finding simultaneously both the discrete and continuous states is a problem that is partly combinatorial (mixed integer-continuous programming). Even worse, deciding the discrete state becomes much problematic when the data are contaminated by noise.

This paper proposes a conceptually different approach in that it is able to overcome the need of explicitly recovering the discrete state before proceeding with the estimation of the continuous state. Instead, the designed observer relies on an appropriately weighted continuous and non-smooth optimization. To discuss clearly the foundation of the method, it is perhaps interesting to start with a batch mode estimation.

A. *Off-line state estimation*

We first consider a batch estimation of the state. The idea is based on the following observation. If we denote by $\{x(t)\}_{t=1}^N$ the true state sequence, then the vector sequence

[1] Strictly speaking, it is not necessary that $\|g_i(t)\|_2$, $i \neq \sigma(t)$, be strictly positive for any $t \geq 0$. This just needs to hold most of the time. In that respect, the seemingly strong assumption (4) is motivated by clarity of presentation.

$\{h_i(t), i = 1, \ldots, s,\ t = 1, \ldots, N\}$ with

$$h_i(t) = \begin{bmatrix} x(t+1) \\ y(t) \end{bmatrix} - \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \qquad (5)$$

forms a sparse sequence of vectors that is, a great number of elements of this sequence are equal to zero. More precisely, if at any time t there is only one $i \in \{1, \ldots, s\}$ satisfying the system equation (1), then exactly N vectors of the sequence $\{h_i(t)\}_{i,t}$ are equal to zero. Note that sparsity of the vector sequence $\{h_i(t)\}_{i,t}$ is, in principle, equivalent to sparsity of the scalar sequence $\{\|h_i(t)\|\}_{i,t}$ for any norm $\|\cdot\|$ and in particular for the ℓ_2 norm. Capitalizing on this fundamental observation, we may find the estimate $\{\hat{x}(t)\}$ of the state sequence so as to minimize the cost functional

$$J\big(\hat{x}(1), \ldots, \hat{x}(N)\big) = \sum_{t=1}^{N-1} \sum_{i=1}^{s} w_i(t) \left\| \begin{bmatrix} \hat{x}(t+1) \\ y(t) \end{bmatrix} - \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ u(t) \end{bmatrix} \right\|_2 \qquad (6)$$

where the $w_i(t)$ are some known weights and N denotes the number of samples. The choice of criterion (6) calls for some comments. First, note that the criterion is constructed as a sum-of-norms instead of a sum-of-squared-norms as is usually the case in estimation literature. This is motivated by the fact that the non-smooth sum-of-norms has the property of promoting the obtention of a solution $\{\hat{x}(t)\}$ such that the vector sequence $\{\hat{h}_i(t)\}_{i,t}$ is sparse [11], [19], [6]. While minimizing a sum-of-squared-norms cost function (which corresponds to the classical least squares) tends to make small the average error vector, the cost function in (6) can tolerate the presence of a few gross errors, that is, the terms $\|h_i(t)\|_2$ are minimized in a somewhat discriminatory manner: those errors which can be set zero are placed to zero, the others can be left large. This implicit discrimination ability can be reinforced here by an appropriate choice of the weights $w_i(t)$. A procedure similar to the one presented in [10] can be used to iteratively minimize (6). In this case, the weights are updated at each iteration. One drawback however of the cost functional (6) is that it treats the states as independent variables.

B. State observer

In this section, we construct a state observer (i.e., an online estimator for the continuous state) for switched linear systems based on the concept presented above. At any time the continuous state is inferred from the prior estimate and the newly available input-output measurements.

More specifically the proposed recursive state estimator operates in two steps as follows :

Prediction step. At time $t-1$, predict the state at time t based on $u(t-1), \hat{x}(t-1), y(t-1)$,

$$\hat{x}(t|t-1) = \operatorname*{arg\,min}_{\eta \in \mathbb{R}^n} \sum_{i=1}^{s} w_i(t-1) \left\|\eta - A_i\hat{x}(t-1) - B_i u(t-1)\right\|_2 \qquad (7)$$

where

$$w_i(t-1) = \frac{w_i^o(t-1)}{\sum_{i=1}^{s} w_i^o(t-1)} \qquad (8)$$

$$w_i^o(t-1) = \Big[\left\|y(t-1) - C_i\hat{x}(t-1) - D_i u(t-1)\right\|_2 + \varepsilon\Big]^{-1}. \qquad (9)$$

Update step. At time t refine the prediction $\hat{x}(t|t-1)$ using the newly available measurement $(u(t), y(t))$,

$$\hat{x}(t|t) = \operatorname*{arg\,min}_{\eta \in \mathbb{R}^n} \Big[\|\eta - \hat{x}(t|t-1)\|_2^2 + \gamma \sum_{i=1}^{s} w_i'(t) \left\|y(t) - C_i\eta - D_i u(t)\right\|_2\Big], \qquad (10)$$

where the weights are re-calculated using the input $u(t)$, the state $\hat{x}(t|t-1)$ and the output $y(t)$ as

$$w_i'(t) = \frac{w_i^1(t)}{\sum_{i=1}^{s} w_i^1(t)} \qquad (11)$$

$$w_i^1(t) = \Big[\left\|y(t) - C_i\hat{x}(t|t-1) - D_i u(t)\right\|_2 + \varepsilon\Big]^{-1}. \qquad (12)$$

Here, $\gamma > 0$ is a regularization constant which controls the importance of the two terms involved in the cost function (10) to be minimized; ε is a small positive number whose primary role here is to prevent division by zero. The notation $\hat{x}(t|t-1)$ refers to the one step ahead prediction of the estimate $\hat{x}(t)$ at time $t-1$. Also, notice indeed that $\hat{x}(t|t) = \hat{x}(t)$. The weights (8)-(9) in the prediction step are constructed such that the contribution of submodel i to the determination of the state at time t is all the more important as submodel i is likely to be active at $t-1$. The same holds similarly for the weights (11)-(12) involved in the update step. Note that the non-smooth optimization problems (7) and (10) can be numerically implemented using e.g. the CVX toolbox [16], [8].

In the sequel we will study the convergence of the predicted state $\hat{x}(t|t-1)$ given in Eq. (7) toward the true state $x(t)$. This analysis is carried out under the assumption that the switching sequence $\{\sigma(t)\}$, the input sequence $\{u(t)\}$ and the initial state $x(0)$ allow for distinguishing between the different modes.

Lemma 1: Under Assumption 1, the dynamical system[2]

$$\hat{x}(t) = \operatorname*{arg\,min}_{\eta \in \mathbb{R}^n} \sum_{i=1}^{s} w_i(t-1) \left\|\eta - A_i\hat{x}(t-1) - B_i u(t-1)\right\|_2 \qquad (13)$$

arising from (7), is BIBO stable.

Proof: As defined in (13), $\hat{x}(t)$ corresponds to the solution of the so-called Fermat-Weber problem. From the related literature [9], it is known that $\hat{x}(t)$ as defined in (13),

[2] Eq. (13) defines in fact a difference (convex) inclusion system. It should, in a strict sense, be read as $\hat{x}(t) \in \arg\min_{\eta \in \mathbb{R}^n} \sum_{i=1}^{s} w_i(t-1) \left\|\eta - A_i\hat{x}(t-1) - B_i u(t-1)\right\|_2$

satisfies

$$\hat{x}(t) = \sum_{i=1}^{s} v_i(t-1)\big(A_i\hat{x}(t-1) + B_i u(t-1)\big).$$

where the weights v_i satisfy $v_i(t-1) \geq 0$ for all i, t and $v_1(t-1) + \cdots + v_s(t-1) = 1$. Letting $b(t) = \sum_{i=1}^{s} v_i(t-1)B_i u(t-1)$ and $A(t-1) = \sum_{i=1}^{s} v_i(t-1)A_i$, the expression of $\hat{x}(t)$ simplifies to

$$\hat{x}(t) = A(t-1)\hat{x}(t-1) + b(t).$$

Note that $A(t-1)$ lies in the convex hull of the set of matrices $\{A_1, \ldots, A_s\}$. It can thus be concluded from Proposition 1 in [17] and Assumption 1 that the above system is asymptotically stable (when $b(t)$ is identically zero). Moreover, by observing that $v_i(t-1) \leq 1$, it is easy to see that if the input u is bounded, then $\|b(t)\|_2$ is bounded above by $\max_i \|B_i\|_2 \sup_{t\geq 0} \|u(t)\|_2$. In conclusion, the system (13) is BIBO stable. That is, if the input sequence $\{u(t)\}$ is bounded, the state $\{\hat{x}(t)\}$ is also bounded. ■

The convergence result is now stated as follows.

Theorem 1: Suppose that Assumption 1 holds and let the state estimation error be defined as $e(t) = \hat{x}(t|t-1) - x(t)$. Assume that ε is chosen sufficiently small to satisfy

$$\frac{s-2}{s-1}\varepsilon \leq \alpha R \tag{14}$$

where $0 < \alpha < \frac{1}{s-1}$, $s \geq 2$ and R is defined as in (4). If there is a finite T such that

$$\|e(T-1)\|_2 < \frac{R}{\mu} \tag{15}$$

where $\mu > (\mu_0 + \delta)\max(1, \rho)$, with ρ defined as in (3) and

$$\begin{aligned}
\mu_0 &= \frac{1 + \alpha(s-1)^2}{1 - \alpha^2(s-1)^2} M \\
\delta &= \sqrt{\mu_0^2 - \frac{m(s-1)^2}{1 - \alpha^2(s-1)^2}}, \\
m &= \min_{i\neq j} \lambda_{\min}\big(\frac{1}{(s-1)^2} C_i^\top C_i - C_j^\top C_j\big), \\
M &= \max_i \|C_i\|_2,
\end{aligned}$$

then the error $e(t)$ converges asymptotically to 0. The notation $\lambda_{\min}(\cdot)$ refers here to the minimum eigenvalue. Note that as defined above, the number m is necessarily negative.

To prove the theorem, we will need the following technical lemma.

Lemma 2: Let Assumption 1 hold and define the error $e(t) = \hat{x}(t|t-1) - x(t)$. Then $e(t)$ obeys the model

$$e(t) = A_{\sigma(t-1)} e(t-1) + \hat{f}_{\sigma(t-1)}(t-1), \tag{16}$$

where $\sigma(t-1)$ represents the true discrete mode of system (1) at time $t-1$ and $\{\hat{f}_{\sigma(t-1)}(t-1)\}$ forms a bounded vector sequence.

Moreover, if the weights satisfy

$$w^o_{\sigma(t-1)}(t-1) > \sum_{i\neq\sigma(t-1)} w^o_i(t-1) \tag{17}$$

then $\hat{f}_{\sigma(t-1)}(t-1) = 0$, i.e.

$$e(t) = A_{\sigma(t-1)} e(t-1). \tag{18}$$

Proof: We first prove Eq. (16). To this end, notice that problem (7) can be reformulated as

$$\begin{aligned}
&\min_{\hat{x}(t|t-1), \hat{f}_1(t-1), \ldots, \hat{f}_s(t-1)} \sum_{i=1}^{s} w_i(t-1)\big\|\hat{f}_i(t-1)\big\|_2, \\
&\quad \text{s.t. } \hat{x}(t|t-1) = A_i\hat{x}(t-1) + B_i u(t-1) + \hat{f}_i(t-1), \\
&\qquad i = 1, \ldots, s.
\end{aligned}$$

The terms $\hat{f}_i(t-1)$ represent some auxiliary variables. In particular,

$$\hat{x}(t|t-1) = A_{\sigma(t-1)}\hat{x}(t-1) + B_{\sigma(t-1)}u(t-1) + \hat{f}_{\sigma(t-1)}(t-1).$$

By subtracting in $\hat{x}(t|t-1)$, the expression

$$x(t) = A_{\sigma(t-1)}x(t-1) + B_{\sigma(t-1)}u(t-1)$$

of the true state, we get Eq. (16). Boundedness of the sequence $\{\hat{f}_{\sigma(t-1)}(t-1)\}$ follows from Lemma 1. Assumption 1 then implies immediately that the estimation error (16) is bounded, see e.g. [18].

We now prove (18). For this purpose, we start by noting that, with a set of vectors $f_i(t-1) \in \mathbb{R}^n$, $i \in Q$, appropriately chosen, one can always write the true state as $x(t) = A_i x(t-1) + B_i u(t-1) + f_i(t-1)$ for any $i \in Q$. In this case, note in passing that $f_{\sigma(t-1)}(t-1) = 0$. By setting $\widetilde{\eta} = \eta - x(t)$, we can write[3]

$$\begin{aligned}
&\eta - A_i\hat{x}(t-1) - B_i u(t-1) = \\
&= \eta - x(t) + x(t) - A_i\hat{x}(t-1) - B_i u(t-1) \\
&= \widetilde{\eta} + A_i x(t-1) + B_i u(t-1) + f_i(t-1) \\
&\qquad\qquad - A_i\hat{x}(t-1) - B_i u(t-1) \\
&= \widetilde{\eta} - A_i e(t-1) + f_i(t-1).
\end{aligned}$$

Hence, by changing the optimization variable in problem (7) to $\widetilde{\eta} = \eta - x(t)$, where $x(t)$ is the true state, the following holds

$$\begin{aligned}
e(t) &= \arg\min_{\widetilde{\eta}\in\mathbb{R}^n} J_{t-1}(\widetilde{\eta}) = \\
&\sum_{i=1}^{s} w_i(t-1)\,\|\widetilde{\eta} - A_i e(t-1) + f_i(t-1)\|_2
\end{aligned}$$

or equivalently, $J_{t-1}(\widetilde{\eta}) \geq J_{t-1}(e(t))$ for any $\widetilde{\eta} \in \mathbb{R}^n$. In particular,

$$\begin{aligned}
&0 \leq J_{t-1}\big(A_{\sigma(t-1)}e(t-1)\big) - J_{t-1}(e(t)) = \\
&= \sum_{i=1}^{s} w_i(t-1)\,\big\|(A_{\sigma(t-1)} - A_i)e(t-1) + f_i(t-1))\big\|_2 \\
&\quad - \sum_{i=1}^{s} w_i(t-1)\,\|e(t) - A_i e(t-1) + f_i(t-1))\|_2
\end{aligned}$$

[3]Note that for the sake of simplicity, we have used here the notation $\hat{x}(t-1)$ instead of $\hat{x}(t-1|t-2)$.

$$= \sum_{i \neq \sigma(t-1)}^{s} w_i(t-1) \left\| (A_{\sigma(t-1)} - A_i) e(t-1) + f_i(t-1)) \right\|_2$$
$$- \sum_{i \neq \sigma(t-1)}^{s} w_i(t-1) \left\| e(t) - A_i e(t-1) + f_i(t-1)) \right\|_2$$
$$- w_{\sigma(t-1)}(t-1) \left\| e(t) - A_{\sigma(t-1)} e(t-1) \right\|_2$$
$$\leq \sum_{i \neq \sigma(t-1)}^{s} w_i(t-1) \left\| e(t) - A_{\sigma(t-1)} e(t-1) \right\|_2$$
$$- w_{\sigma(t-1)}(t-1) \left\| e(t) - A_{\sigma(t-1)} e(t-1) \right\|_2$$
$$= \left(1 - 2w_{\sigma(t-1)}(t-1)\right) \left\| e(t) - A_{\sigma(t-1)} e(t-1) \right\|_2.$$

Note that we have used the identity $\|x\|_2 - \|y\|_2 \leq \|x-y\|_2$ in deriving the last inequality. The above inequality implies that if $2w_{\sigma(t-1)}(t-1) > 1$ that is, if (17) is satisfied, then $e(t) = A_{\sigma(t-1)} e(t-1)$. ■

Proof: The proof of the theorem is a consequence of Lemma 2. Our method of proof consists of two steps: (i) derive a sufficient condition for (17) to hold, (ii) use the asymptotic stability condition of system (2) to conclude.

Part (i). Let $t = T$. For (17) to be true for $t = T$, it suffices that

$$w^o_{\sigma(t-1)}(t-1) > (s-1) w^o_i(t-1) \tag{19}$$

for all $i \neq \sigma(t-1)$. By writing $y(t-1) = C_i x(t-1) + D_i u(t-1) + g_i(t-1)$, it follows that

$$w^o_i(t-1) = \frac{1}{\|y(t-1) - C_i \hat{x}(t-1) - D_i u(t-1)\|_2 + \varepsilon}$$
$$= \frac{1}{\|C_i e(t-1) - g_i(t-1)\|_2 + \varepsilon}$$

Exploiting this, it is easy to see that Eq. (19) is equivalent to

$$\frac{1}{s-1} \|C_i e(t-1) - g_i(t-1)\|_2 >$$
$$\left\|C_{\sigma(t-1)} e(t-1)\right\|_2 + \frac{s-2}{s-1}\varepsilon. \tag{20}$$

Squaring this inequality leads to

$$e(t-1)^\top \Big(\frac{1}{(s-1)^2} C_i^\top C_i - C_{\sigma(t-1)}^\top C_{\sigma(t-1)}\Big) e(t-1)$$
$$+ \frac{1}{(s-1)^2} \|g_i(t-1)\|^2$$
$$- \frac{2}{(s-1)^2} e(t-1)^\top C_i^\top g_i(t-1)$$
$$- 2\varepsilon \frac{s-2}{s-1} \left\|C_{\sigma(t-1)} e(t-1)\right\|_2 - \left(\frac{s-2}{s-1}\right)^2 \varepsilon^2 > 0. \tag{21}$$

In order to find a lower bound of the expression lying on the left-hand side of the inequality symbol, let us emphasize the following set of inequalities

$$e(t-1)^\top \Big(\frac{1}{(s-1)^2} C_i^\top C_i - C_{\sigma(t-1)}^\top C_{\sigma(t-1)}\Big) e(t-1)$$
$$\geq m \|e(t-1)\|_2^2,$$
$$- \frac{2}{(s-1)^2} g_i(t-1)^\top C_i e(t-1) \geq - \frac{2M}{\mu (s-1)^2} \|g_i(t-1)\|_2^2,$$
$$- 2\varepsilon \frac{s-2}{s-1} \left\|C_{\sigma(t-1)} e(t-1)\right\|_2 \geq - \frac{2\alpha M}{\mu} \|g_i(t-1)\|_2^2,$$
$$- \left(\frac{s-2}{s-1}\right)^2 \varepsilon^2 \geq -\alpha^2 \|g_i(t-1)\|_2^2.$$

Now reference to inequality (21) shows that for (17) to hold, it suffices that

$$m \|e(t-1)\|_2^2 + K(\alpha,\mu) \|g_i(t-1)\|_2^2 > 0. \tag{22}$$

with $K(\alpha,\mu) = \frac{1}{(s-1)^2} - \alpha^2 - \Big(\frac{1}{(s-1)^2} + \alpha\Big)\frac{2M}{\mu}$. Note that here it necessarily holds that $m \leq 0$. This follows from the simple fact that if $\lambda_{\min}\big(\frac{1}{(s-1)^2} C_i^\top C_i - C_j^\top C_j\big) > 0$, then $\lambda_{\min}\big(\frac{1}{(s-1)^2} C_j^\top C_j - C_i^\top C_i\big) < 0$.

With $m \leq 0$, it follows from (15) and (22) that

$$\big(m + \mu^2 K(\alpha,\mu)\big) \|g_i(t-1)\|_2^2 > 0$$

and constitutes a sufficient condition for (22). This is a polynomial of degree 2 in μ. From direct algebraic calculations, it is then straightforward to see that this last condition holds if and only if (15) holds with $\mu > \mu_0 + \delta$.

Part (ii). We have just shown that (17) holds at time T under the condition (15) if $\mu > \mu_0 + \delta$. By now exploiting Lemma 2, the error at time T takes the form $e(T) = A_{\sigma(T-1)} e(T-1)$ where $\sigma(\cdot)$ is the true switching function. Note that for any $i \in Q$ and any $x \in \mathbb{R}^n$, $\|A_i x\|_2 \leq \rho \|x\|_2$ where ρ is defined in (3). It follows that if

$$\|e(T-1)\|_2 \leq \frac{R}{\mu} \text{ with } \mu > (\mu_0 + \delta)\max(1,\rho),$$

as assumed in the statement of the theorem, then

$$\|e(T)\|_2 = \left\|A_{\sigma(T-1)} e(T-1)\right\|_2 \leq \frac{R}{(\mu/\rho)}$$

with $\mu/\rho > \mu_0 + \delta$. This in turn implies that

$$\|e(T+1)\|_2 = \left\|A_{\sigma(T)} A_{\sigma(T-1)} e(T-1)\right\|_2$$
$$\leq \frac{R}{(\mu/\rho)} \text{ with } \mu/\rho > \mu_0 + \delta,$$

which implies ... etc. ... In fact, from the definition of ρ, Condition (15) is satisfied for any $t \geq T$ so that $e(t) = A_{\sigma(t-1)} e(t-1)$ for any $t \geq T$. By Assumption 1, it can be concluded that the estimation error $e(t)$ converges asymptotically towards 0. ■

What Theorem 1 says is that, if the estimation error happens to lie inside the disk $D(\mu) = \{x \in \mathbb{R}^n : \|x\|_2 < R/\mu\}$ for a large enough μ and if the design parameter ε is chosen sufficiently small, then the estimation error will definitively converge asymptotically to zero. For this situation to occur, the modes must be strongly distinguishable within the data in the sense that the number R defined in (4) must be large.

As it turns out, the derived sufficient convergence conditions may be somewhat conservative. In practice we have noticed that convergence is generically achieved, thus suggesting that further research should be devoted to relaxing the condition of Theorem 1.

Remark 1: A requirement for an estimation problem to be well-posed is that the considered system is observable. Observability is a property of the system that refers to the possibility to infer uniquely the continuous state (and sometimes, also the discrete mode) from input and output measurements. Note that here, Eq. (15) constitutes a local observability condition. In effect, this equation says that if the estimated state comes to be close enough to the true state, then by Theorem 1, we are able to recover uniquely the true state. Since convergence is only locally proved, no absolute observability condition in the usual sense is explicitly invoked.

C. An approximate closed-form implementation

Note that the designed state observer is based on solving an ℓ_2 optimization problem at each time. However this may be a little computationally expensive in practice. In the objective of alleviating the computational requirement, we remark in this section that the previous observer admits an approximate analytic implementation at a much cheaper cost. The idea is based on replacing the ℓ_2 norm in the criterion appearing in (7) by a squared-norm while taking the square of the weights in (9). Hence we redefine the weights by

$$w_i^o(t-1) = \Big[\|y(t-1) - C_i\hat{x}(t-1) - D_iu(t-1)\|_2 + \varepsilon \Big]^{-2}, \tag{23}$$

that is, as the square of the ones defined in (9). As in (8), the weights are then normalized according to

$$w_i(t-1) = \frac{w_i^o(t-1)}{\sum_{i=1}^s w_i^o(t-1)}. \tag{24}$$

The prediction and the update equations for the estimated state become respectively

$$\hat{x}(t|t-1) = \operatorname*{arg\,min}_{\eta\in\mathbb{R}^n} \sum_{i=1}^s w_i(t-1) \|\eta - A_i\hat{x}(t-1) - B_iu(t-1)\|_2^2 \tag{25}$$

and

$$\hat{x}(t) = \operatorname*{arg\,min}_{\eta\in\mathbb{R}^n} \Big[\|\eta - \hat{x}(t|t-1)\|_2^2 + \gamma \sum_{i=s}^s w_i'(t) \|y(t) - C_i\eta - D_iu(t)\|_2^2 \Big]. \tag{26}$$

with $w_i'(t)$ defined as in (23)-(24) by replacing $\big(u(t-1), \hat{x}(t-1), y(t-1)\big)$ with $\big(u(t), \hat{x}(t|t-1), y(t)\big)$ instead. Note that the central point in the above least squares implementation of the non-smooth optimization problem is the squaring of the weights. By squaring the norms as in (25)-(26), we have also squared the weights so that (25)-(26) and (7)-(10) are roughly the same. Simple calculations then lead to the following analytic formulas for the state observer

$$\hat{x}(t|t-1) = \sum_{i=1}^s w_i(t-1)\Big(A_i\hat{x}(t-1) + B_iu(t-1)\Big)$$

$$\hat{x}(t) = \Big(I_n + \gamma \sum_{i=1}^s w_i'(t) C_i^\top C_i\Big)^{-1} \Big[\hat{x}(t|t-1) + \gamma \sum_{i=1}^s w_i'(t) C_i^\top \big(y(t) - D_iu(t)\big)\Big].$$

Here I_n is the identity matrix of dimension n. These closed-form formulas approximately implement the non-smooth optimization-based observer (7)-(10).

IV. A MORE GENERAL CASE

The observer structure described so far relies on the assumption that for any t, there exists only one index $\sigma(t) \in Q$ satisfying $y(t) = C_{\sigma(t)}x(t) + D_{\sigma(t)}u(t)$ and that the number R defined in (4) is relatively large. This may not be true in general as it may happen for example that more than one subsystem have the same (C, D)-matrices. In this type of situations, we need to seek mode discernibility not from a single output vector $y(t) \in \mathbb{R}^{n_y}$ but from a set of output vectors constructed over a time window of a certain size.

In this section we extend the proposed hybrid state observer to more general situations. The key idea to this extension is to construct a lifted switched system with inputs and outputs living in a higher dimensional space. The new output is formed by vertically concatenating the outputs of system (1) over a certain time window $[t-\tau,\ t]$. The new input is constructed similarly. We obtain the following model

$$\begin{cases} x(t) &= \mathcal{A}(\sigma_\tau(t-\tau))x(t-\tau) \\ & \quad +\mathcal{B}(\sigma_\tau(t-\tau))u_\tau(t-\tau) \\ y_\tau(t-\tau) &= \mathcal{C}(\sigma_\tau(t-\tau))x(t-\tau) \\ & \quad +\mathcal{D}(\sigma_\tau(t-\tau))u_\tau(t-\tau) \end{cases} \tag{27}$$

where the notations $u_\tau(t)$, $y_\tau(t)$, $\sigma_\tau(t)$ are defined as

$$\begin{aligned} y_\tau(t) &= \begin{bmatrix} y(t)^\top & \cdots & y(t+\tau-1)^\top \end{bmatrix}^\top \in \mathbb{R}^{\tau n_y}, \\ u_\tau(t) &= \begin{bmatrix} u(t)^\top & \cdots & u(t+\tau-1)^\top \end{bmatrix}^\top \in \mathbb{R}^{\tau n_u}, \\ \sigma_\tau(t) &= \sigma(t)\cdots\sigma(t+\tau-1) \in Q^\tau. \end{aligned} \tag{28}$$

The matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are defined for any word $\bar{\sigma} = \sigma_1 \ldots \sigma_\tau \in Q^\tau$, as

$$\begin{aligned} \mathcal{A}(\bar{\sigma}) &= A_{\sigma_\tau} \cdots A_{\sigma_1}, \\ \mathcal{B}(\bar{\sigma}) &= \begin{bmatrix} A_{\sigma_\tau} \ldots A_{\sigma_2} B_{\sigma_1} & A_{\sigma_\tau} \ldots A_{\sigma_3} B_{\sigma_2} & \cdots & A_{\sigma_\tau} B_{\sigma_{\tau-1}} & B_{\sigma_\tau} \end{bmatrix}, \end{aligned}$$

$$\mathcal{C}(\bar{\sigma}) = \Big[(C_{\sigma_1})^\top \quad (C_{\sigma_2}A_{\sigma_1})^\top \quad \cdots \quad (C_{\sigma_\tau}A_{\sigma_{\tau-1}} \cdots A_{\sigma_1})^\top \Big]^\top,$$

$$\mathcal{D}(\bar{\sigma}) = \begin{bmatrix} D_{\sigma_1} & 0 & \dots & 0 \\ C_{\sigma_2}B_{\sigma_1} & D_{\sigma_2} & \dots & 0 \\ C_{\sigma_3}A_{\sigma_2}B_{\sigma_1} & C_{\sigma_3}B_{\sigma_2} & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ C_{\sigma_\tau}A_{\sigma_{\tau-1}}\dots A_{\sigma_2}B_{\sigma_1} & (*) & \dots & D_{\sigma_\tau} \end{bmatrix},$$

where $(*)$ stands for $C_{\sigma_\tau}A_{\sigma_{\tau-1}}\dots A_{\sigma_3}B_{\sigma_2}$. We can view the lifted system (27) as a switched system in a similar way as (1). The set of discrete states of the system (27) is the finite set Q^τ, i.e., it is assumed that all switching paths are admissible. The state estimator discussed in the preceding sections can be applied to (27) with a few changes : At time t, we construct a state smoother to obtain

$$\hat{x}(t-\tau|t) = \arg\min_{\eta\in\mathbb{R}^n} \Big[\gamma_1 \|\eta - \hat{x}(t-\tau)\|_2^2 + \gamma \sum_{\bar{\sigma}\in Q^{\tau+1}} w_{\bar{\sigma}}(t) \times \|y_{\tau+1}(t-\tau) - \mathcal{C}(\bar{\sigma})\eta - \mathcal{D}(\bar{\sigma})u_{\tau+1}(t-\tau)\|_2\Big]$$

and from $\hat{x}(t-\tau|t)$, we predict the state at time t as

$$\hat{x}(t) = \arg\min_{\eta\in\mathbb{R}^n} \sum_{\bar{\sigma}\in Q^\tau} w'_{\bar{\sigma}}(t)\times \|\eta - \mathcal{A}(\bar{\sigma})\hat{x}(t-\tau|t) - \mathcal{B}(\bar{\sigma})u_\tau(t-\tau)\|_2 .$$

$w_{\bar{\sigma}}(t)$ and $w'_{\bar{\sigma}}(t)$ are defined respectively from $\big(u_{\tau+1}(t-\tau), \hat{x}(t-\tau), y_{\tau+1}(t-\tau)\big)$ and $\big(u_\tau(t-\tau), \hat{x}(t-\tau|t), y_\tau(t-\tau)\big)$ similarly as in (9). Note that the smoothing step can be removed by setting both γ and γ_1 to zero.

V. Numerical experiments

We now illustrate the performance of the new state observer on a three-modes switched linear system having state dimension equal to 3, 2 inputs and 2 outputs. The dynamics of the three modes are described by an equation of the form (1) with the following matrices.

$$A_1 = \begin{bmatrix} 0.83 & 0 & 0.07 \\ 0 & 0.80 & -0.10 \\ 0.07 & -0.10 & 0.72 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.17 & -1.40 \\ -2 & -0.65 \\ 0 & 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} -1 & 0.90 & -1 \\ -0.93 & 0 & 1 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & -1.50 \\ 0.07 & -1.33 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0.33 & 0.58 & -0.09 \\ -0.40 & 0.15 & -0.50 \\ -0.43 & 0.30 & 0.43 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -0.33 & 0 \\ 0.68 & 0.88 \\ -0.81 & -0.90 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 0.02 & -0.35 & 0.33 \\ -0.37 & 0 & -0.79 \end{bmatrix}, \quad D_2 = \begin{bmatrix} -0.97 & 0 \\ -0.02 & 0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -0.15 & -0.17 & 0.48 \\ -0.28 & -0.40 & -0.23 \\ -0.42 & 0.32 & -0.02 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 & -0.85 \\ -0.80 & -0.63 \\ -0.21 & -1.16 \end{bmatrix}$$

$$C_3 = \begin{bmatrix} 0 & 0 & 0.38 \\ 0 & 0.64 & -0.46 \end{bmatrix}, \quad D_3 = \begin{bmatrix} 0 & 0.88 \\ -1.11 & 0.27 \end{bmatrix}.$$

We select the input sequence to be the realization of a random vector process $\boldsymbol{u} \sim \mathcal{N}(0, I_2)$. The initial state is also generated at random from a centered normal distribution. The design parameters ε and γ are set to 10^{-5} and 1 respectively. Running first the observer on noise-free data, a perfect reconstruction of the continuous state is obtained despite a quite small value for R in Eq. (4): $R = 0.17$ and $R/\mu = 0.085$. Second, to make the simulation more realistic, the output is disturbed by a certain amount of noise ; we also let the state be corrupted by a process noise. In both cases, the signal to noise ratio is about 20 dB. Note that the covariances of these noises are unknown by the algorithm. That is the structure of the noise is not explicitly handled by the observer. The results depicted in Figure 1 show that the proposed estimate scheme is able to overcome this challenge. For all the three forms of the observer discussed in the paper, convergence occurs quite quickly after the initial time.

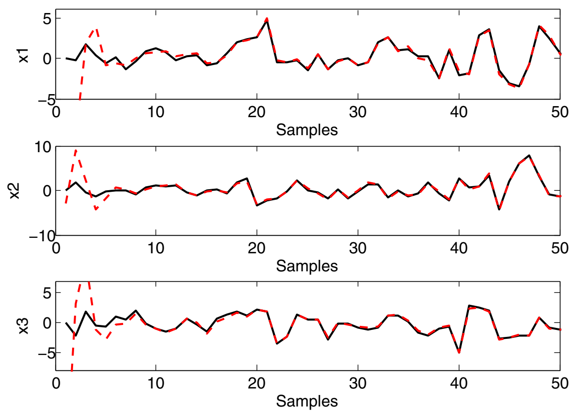

Fig. 1. True state (black solid) and estimated state (red dashed) trajectories obtained using the observer of Section III-B.

VI. Conclusion

In this paper we have presented a new hybrid observer for switched linear systems. While the majority of existing techniques rely on a prior explicit determination of the discrete state, the method of this paper is a non-smooth yet continuous optimization-based estimation approach. To recursively obtain an estimate of the state, we optimize at each time a weighted cost function over the entire set of subsystems. The considered cost function has the attractive property of being tolerant to gross fitting errors, thus allowing for an optimization over all the subsystems without prior identification of the discrete mode. On a non-trivial numerical example, the developed method has proved to be viable. Relaxing the condition of Theorem 1 is still an open problem to be considered in future research.

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