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An optimization approach to resolve the competing aims of active fault detection and control

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Abstract— **The paper deals with the problem of active fault detection and control for multiple models. It is assumed that a fault detector is given and the goal is to design an input signal generator such that detection and control aims are achieved. Since these two aim are conflicting, it is necessary to express a desired compromise between them. The paper investigates three formulations that allow for respecting both competing aims. In the first formulation both aims are combined into a single criterion. In other two formulations, one aim is reflected in the criterion and the other aim is enforced as a constraint.**

I. Introduction

Ever increasing requirements on safety, availability, and low costs require to use adequate methods for detecting an undesirable behavior of a monitored system. Although passive fault detection methods [1] perform quite well in many applications, there are situations in which faults can remain undetected because of an operating regime. It motivates the development of active fault detection methods that can deal with such situations by introducing an auxiliary input signal.

An active fault detection method based on the sequential probability ratio test was considered in [2], [3]. Another approach to active fault detection for systems with energy bounded noises was treated in [4]. Since the inputs of a system are usually utilized for control, it is necessary to tackle the problem of simultaneous control and active detection. Early works [5], [6] studied the relationships and limitations of the integrated fault detection and control, and more detailed treatement was given in [7], [8], [9].

A unifying general formulation of the active fault detection and control problem for stochastic systems accompanied by a formal solution based on closed loop information processing strategy (IPS) was introduced in [10]. The importance of the unified formulation consists in the ability to derive and study individual special cases in a coherent way. Although the formal solution is not computationally tractable, it provides a basis for proposing suitable approximate solutions.

The goal of the paper is to present three problem formulations to active fault detection and control for a given detector. Since the optimal solution based on the closed loop IPS is intractable, a design procedure based on the open loop IPS and the upper bound on the probability of misclassification introduced in [9] is proposed.

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II. Problem Formulation

Consider a system that can be described at each time step $k \in \mathcal{T} = \{0, 1, \ldots, F\}$ of the finite horizon $F < \infty$ by one of two linear stochastic models of the structure

$$\mathbf{x}_{k+1} = \mathbf{A}_{\mu_k}\mathbf{x}_k + \mathbf{B}_{\mu_k}\mathbf{u}_k + \mathbf{G}_{\mu_k}\mathbf{w}_k, \tag{1}$$

$$\mathbf{y}_k = \mathbf{C}_{\mu_k}\mathbf{x}_k + \mathbf{H}_{\mu_k}\mathbf{v}_k, \tag{2}$$

where the couple $\mathbf{x}_k$, μ_k represents the unmeasured state of the system. The state variable $\mu_k \in \mathcal{M} = \{1, 2\}$ indicates the model in effect at time step k, and the vector $\mathbf{x}_k \in \mathbb{R}^{n_x}$ collects remaining state variables. The input and output are denoted as $\mathbf{u}_k \in \mathcal{U}_k$ and $\mathbf{y}_k \in \mathbb{R}^{n_y}$ respectively. The set of admissible inputs $\mathcal{U}_k \subseteq \mathbb{R}^{n_u}$ represents a priori known constraints that usually stem from physical or logical limitations imposed by the system. The state noise $\mathbf{w}_k \in \mathbb{R}^{n_w}$ and measurement noise $\mathbf{v}_k \in \mathbb{R}^{n_v}$ are mutually independent white noises with a Gaussian distribution with zero means and unit covariance matrices. They are also independent of the initial state $[\mathbf{x}_0, \mu_0]^T$. The initial condition $\mathbf{x}_0$ has the Gaussian distribution with a known mean value $\hat{\mathbf{x}}_{0|-1}$ and a covariance matrix $\mathbf{P}_{0|-1}$. The probability function $P(\mu_0)$ is also known. Finally, it is assumed that the state variable μ_k is constant over the whole horizon F, and the system matrices $\mathbf{A}_{\mu_k}$, $\mathbf{B}_{\mu_k}$, $\mathbf{G}_{\mu_k}$, $\mathbf{C}_{\mu_k}$, and $\mathbf{H}_{\mu_k}$ are given.

When the system matrices with $\mu_k = 1$ describe the normal operation of the system and the other set of matrices represents the faulty behavior, then the estimation of the state variable μ_k can be regarded as a fault detection problem. In the most general case, the aim is to design a system that generates decisions about the fault in the system and also an input signal that should improve the quality of detection and simultaneously control the system. A general solution to this problem and particular special cases were discussed in [10]. In this paper a special case in which a detector is given in advance is considered.

The problem of active detection and control with a given detector is depicted in Fig. 1. A given detector $\mathbf{D}$ analyzes the system $\mathbf{S}$ by means of the input-output data and generates the decision d_k about a potential fault at each time step. The goal is to design an input signal generator $\mathbf{C}$ that generates input $\mathbf{u}_k$ such that the detection and control aims are pursued and the given detector $\mathbf{D}$ is taken into account.

The given detector is fixed in advance and described at

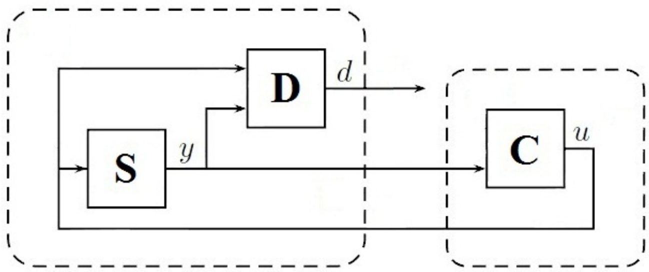

Fig. 1. The schema of active detection and control with a given detector.

each time step $k \in \mathcal{T}$ by the relation

$$d_k = \sigma_k\left(\mathbf{y}_0^k, \mathbf{u}_0^{k-1}\right), \tag{3}$$

where $\sigma_k(\mathbf{y}_0^k, \mathbf{u}_0^{k-1}),\ k \in \mathcal{T}$ are given functions. The notation $\mathbf{y}_i^j$ represents a sequence of the variables $\mathbf{y}_k$ from the time step i up to the time step j. If $i > j$ then the sequence $\mathbf{y}_i^j$ is empty and the corresponding variable is simply left out from an expression. The designed input signal generator **C** can generally be described as

$$\mathbf{u}_k = \boldsymbol{\gamma}_k\left(\mathbf{y}_0^k, \mathbf{u}_0^{k-1}\right), \tag{4}$$

where $\boldsymbol{\gamma}_k(\mathbf{y}_0^k, \mathbf{u}_0^{k-1}),\ k \in \mathcal{T}$ is a sequence of functions to be designed.

In order to evaluate the quality of the input signal generator **C** in terms of detection and control aims, a criterion is needed. It is assumed that the detection and control aims have already been expressed by a designer for each time step $k \in \mathcal{T}$ using the cost functions $L_k^{\mathrm{d}}(\mu_k, d_k)$ and $L_k^{\mathrm{c}}(\mathbf{x}_k, \mathbf{u}_k)$, respectively. The cost function $L_k^{\mathrm{c}}(\mathbf{x}_k, \mathbf{u}_k)$ is convex, and the cost function $L_k^{\mathrm{d}}(\mu_k, d_k)$ satisfies the inequality $L_k^{\mathrm{d}}(\mu_k, \mu_k) \leq L_k^{\mathrm{d}}(\mu_k, d_k)$ for all $\mu_k \in \mathcal{M}$, $d_k \in \mathcal{M}$, $d_k \neq \mu_k$ at each time step $k \in T$, and the strict inequality holds at least at one time step. Since the values of these cost functions are random variables, it is common to define detection and control criteria as

$$J^{\mathrm{d}}\left(\boldsymbol{\gamma}_0^F\right) = \mathrm{E}\left\{\sum_{k=0}^{F} L_k^{\mathrm{d}}(\mu_k, d_k)\right\}, \tag{5}$$

$$J^{\mathrm{c}}\left(\boldsymbol{\gamma}_0^F\right) = \mathrm{E}\left\{\sum_{k=0}^{F} L_k^{\mathrm{c}}(\mathbf{x}_k, \mathbf{u}_k)\right\}, \tag{6}$$

where $\mathrm{E}\{\cdot\}$ is the expectation operator. The input signal generator **C** that is optimal according to the first criterion usually differs from the system **C** obtained by minimizing the second criterion. Therefore it is necessary to define a desired compromise between the detection and control aims and solve a multi-objective optimization problem [11].

This paper investigates three different problem formulations that allow for incorporation of both criteria into the design procedure. Although a problem formulation consisting in combining both criteria into a single criterion is the simplest way, it does not always yield a desirable result. In many situations the aim is not to minimize both criteria at the same time, but rather to maintain one criterion below a specified threshold while minimizing the other one. The following three problem formulations are considered.

- Fault detection and control problem I (FDC I). The detection and control aims are weighted and combined into a single criterion. This problem formulation could be useful in situations when both cost functions express the same quantity (e.g. time, money, energy). The goal is to minimize the criterion

$$J\left(\boldsymbol{\gamma}_0^F\right) = \mathrm{E}\left\{\sum_{k=0}^{F} L_k(\mu_k, d_k, \mathbf{x}_k, \mathbf{u}_k)\right\}, \tag{7}$$

subject to

$$\mathbf{u}_0^F \in \mathcal{U},$$

where $L_k(\cdot) = \alpha_k L_k^{\mathrm{d}}(\mu_k, d_k) + (1-\alpha_k) L_k^{\mathrm{c}}(\mathbf{x}_k, \mathbf{u}_k)$ is an overall cost function, $\alpha_k \in [0,1]$ is the weighting factor chosen by a designer, and $\mathcal{U} = \mathcal{U}_0 \times \mathcal{U}_1 \cdots \times \mathcal{U}_F$ is the set of admissible input sequences.

- Fault detection and control problem II (FDC II). The control aim is placed into the criterion, and the detection aim is used as a constraint. This problem formulation should be preferred when a specified level of the detection criterion (expressing e.g. the maximum probability of misclassification) has to be guaranteed. The goal is to minimize the criterion

$$J^{\mathrm{c}}\left(\boldsymbol{\gamma}_0^F\right) = \mathrm{E}\left\{\sum_{k=0}^{F} L_k^{\mathrm{c}}(\mathbf{x}_k, \mathbf{u}_k)\right\}, \tag{8}$$

subject to

$$\mathrm{E}\left\{\sum_{k=0}^{F} L_k^{\mathrm{d}}(\mu_k, d_k)\right\} \leq L_{\max}^{\mathrm{d}},\ \mathbf{u}_0^F \in \mathcal{U} \tag{9}$$

where $L_{\max}^{\mathrm{d}}$ is the maximum acceptable level of the detection criterion.

- Fault detection and control problem III (FDC III). The detection aim is placed into the criterion, and the control aim is enforced as a constraint. This problem formulation could be utilized for example in a situation where the input signal generator should excite the system to confirm or reject the suspicion that a sensor is failing, but the control criterion (e.g. the maximum allowable energy of the inputs) must not exceed a certain value. The goal is to minimize the criterion

$$J^{\mathrm{d}}\left(\boldsymbol{\gamma}_0^F\right) = \mathrm{E}\left\{\sum_{k=0}^{F} L_k^{\mathrm{d}}(\mu_k, d_k)\right\}, \tag{10}$$

subject to

$$\mathrm{E}\left\{\sum_{k=0}^{F} L_k^{\mathrm{c}}(\mathbf{x}_k, \mathbf{u}_k)\right\} \leq L_{\max}^{\mathrm{c}},\ \mathbf{u}_0^F \in \mathcal{U}, \tag{11}$$

where $L_{\max}^{\mathrm{c}}$ is the maximum acceptable value of the control criterion.

III. Optimal Solution

The solutions to the posed problem formulations FDC I, FDC II, and FDC III can be obtained using different assumptions on the availability of the outputs at individual time steps. There are three basic information processing strategies (IPS): open loop (OL), open loop feedback (OLF) and closed loop (CL) [12]. The mutual relationships between all three IPS's and the benefit of the CL IPS in the context of the active change detection problem were discussed in [13]. Although the design procedure presented in this paper is based on the OL IPS, a general solution utilizing the CL IPS is briefly discussed.

In the case of FDC I, the optimal solution based on the CL IPS can be obtained by solving the backward recursive equation

$$V_k^*\left(\mathbf{y}_0^k,\mathbf{u}_0^{k-1}\right)=\min_{\mathbf{u}_k\in\mathcal{U}_k}\mathrm{E}\left\{L_k(\mu_k,\sigma_k(\mathbf{y}_0^k,\mathbf{u}_0^{k-1}),\mathbf{x}_k,\mathbf{u}_k)+ V_{k+1}^*\left(\mathbf{y}_0^{k+1},\mathbf{u}_0^k\right)\middle|\,\mathbf{y}_0^k,\mathbf{u}_0^k\right\}, \quad (12)$$

where $\mathrm{E}\{\cdot|\cdot\}$ is the conditional expectation operator, and the cost-to-go function $V_k^*(\mathbf{y}_0^k,\mathbf{u}_0^{k-1})$ (also called Bellman function) expresses the minimum expected costs incurred from time step k to time step F given the input-output data $\mathbf{y}_0^k,\mathbf{u}_0^{k-1}$. The initial condition of the backward recursive equation is $V_{F+1}^*=0$. Since an analytical solution to this backward recursive equation seldom exists and a numerical solution is usually computationally prohibitive, it is necessary to resort to an approximate solution [14].

As one could anticipate, the introduction of the expectation constraints in formulations FDC II and FDC III makes the problem even harder to solve using the CL IPS. The problem with expectation constraint was extensively studied for controlled Markov decision processes with discrete variables [15]. In general, the increase in complexity originates from restricting the set of all feasible functions $\boldsymbol{\gamma}_k(\mathbf{y}_0^k,\mathbf{u}_0^{k-1})$ to a nontrivial subset $\boldsymbol{\Gamma}_k$ of which an explicit description is difficult to obtain [16]. To avoid these issues a simplistic design procedure based on the OL IPS will be considered in what follows.

IV. Suboptimal Solution

Since the OL IPS assumes that the actual outputs are not utilized, the input signal generator $\mathbf{C}$ can be described as

$$\mathbf{u}_k=\boldsymbol{\gamma}_k, \quad (13)$$

where $\boldsymbol{\gamma}_k$ is a function of time step k but not the input-output data $\mathbf{y}_0^k,\mathbf{u}_0^{k-1}$. It means that the input sequence $\mathbf{u}_0^F$ can be designed off-line.

A. Application of the open loop strategy

In all three problem formulations, it is possible to proceed in the following way. Besides providing the optimal solution in the case of FDC I, the backward recursive equation (12) motivates us to write both criteria (5) and (6) for an arbitrary function $\boldsymbol{\gamma}_k(\mathbf{y}_0^k,\mathbf{u}_0^{k-1})$ in the recursive form

$$V_k^\bullet\left(\mathbf{y}_0^k,\mathbf{u}_0^{k-1}\right)=\mathrm{E}\left\{L_k^\bullet+V_{k+1}^\bullet\left(\mathbf{y}_0^{k+1},\mathbf{u}_0^k\right)\right\}, \quad (14)$$

where $\bullet$ is used as a wildcard. Using this recursive form and the assumption (13) of the OL IPS, the detection criterion can be expressed as

$$J^{\mathrm{d}}\left(\mathbf{u}_0^{F-1}\right)=\sum_{k=0}^{F}\int_{\mathbf{y}_0^k}\mathrm{E}\left\{L_k^{\mathrm{d}}\left(\mu_k,\sigma_k\left(\mathbf{y}_0^k,\mathbf{u}_0^{k-1}\right)\right)\middle|\,\mathbf{y}_0^k,\mathbf{u}_0^{k-1}\right\} \prod_{i=0}^{k}p\left(\mathbf{y}_i|\mathbf{y}_0^{i-1},\mathbf{u}_0^{i-1}\right)d\mathbf{y}_0^k, \quad (15)$$

the control criterion reduces to

$$J^{\mathrm{c}}\left(\mathbf{u}_0^F\right)=\sum_{k=0}^{F}\int_{\mathbf{y}_0^k}\mathrm{E}\left\{L_k^{\mathrm{c}}\left(\mathbf{x}_k,\mathbf{u}_k\right)\,\middle|\,\mathbf{y}_0^k,\mathbf{u}_0^k\right\} \prod_{i=0}^{k}p\left(\mathbf{y}_i|\mathbf{y}_0^{i-1},\mathbf{u}_0^{i-1}\right)d\mathbf{y}_0^k, \quad (16)$$

and the combined criterion becomes

$$J\left(\mathbf{u}_0^F\right)=\sum_{k=0}^{F}\int_{\mathbf{y}_0^k}\mathrm{E}\left\{L_k\left(\mu_k,\sigma_k(\mathbf{y}_0^k,\mathbf{u}_0^{k-1}),\mathbf{x}_k,\mathbf{u}_k\right)\middle|\,\mathbf{y}_0^k,\mathbf{u}_0^k\right\} \prod_{i=0}^{k}p\left(\mathbf{y}_i|\mathbf{y}_0^{i-1},\mathbf{u}_0^{i-1}\right)d\mathbf{y}_0^k, \quad (17)$$

where $p(\mathbf{y}_i|\mathbf{y}_0^{i-1},\mathbf{u}_0^{i-1})$ is the conditional predictive pdf of the output $\mathbf{y}_i$.

B. Detector, Detection and Control Objectives Specification

A general given detector and general cost functions expressing detection and control objectives have been considered so far. To simplify the presentation of what follows, a particular problem is presented in this subsection and consequently illustrated through the numerical example in the next section.

It is assumed that the given detector is time invariant and generates decisions d_k in the maximum a posteriori probability sense

$$d_k=\sigma\left(\mathbf{y}_0^k,\mathbf{u}_0^{k-1}\right)=\arg\max_{\mu_k}P\left(\mu_k|\mathbf{y}_0^k,\mathbf{u}_0^{k-1}\right), \quad (18)$$

which is quite common in the multiple model change detection area [17].

The aim of detection is to minimize the probability of making the wrong decision at the final time step $k=F$, which can be expressed by the cost function

$$L_k^{\mathrm{d}}\left(\mu_k,d_k\right)=\begin{cases}1 & k=F\wedge d_k\neq\mu_k,\\ 0 & \text{otherwise.}\end{cases} \quad (19)$$

It was shown in [18] that given the cost function (19) and detector (18), the detection criterion (15) expresses the probability of making a wrong decision at the final time step $k=F$. Since the value of this criterion cannot be computed analytically, it was proposed in [9] to use an upper bound $\mathcal{B}(\mathbf{u}_0^{F-1})$ instead. This upper bound is usually referred to as the Bhattacharyya bound and the relation is the following

$$J^{\mathrm{d}}\left(\mathbf{u}_0^{F-1}\right) \leq \mathcal{B}\left(\mathbf{u}_0^{F-1}\right) = \sqrt{P(\mu_0=1)P(\mu_0=2)}e^{-K}, \quad (20)$$

$$K = \frac{1}{2}\ln\frac{|\mathbf{\Sigma}|}{\sqrt{|\mathbf{\Sigma}^1||\mathbf{\Sigma}^2|}} + \frac{1}{8}\left(\hat{\mathbf{y}}^1 - \hat{\mathbf{y}}^2\right)^T \mathbf{\Sigma}^{-1}\left(\hat{\mathbf{y}}^1 - \hat{\mathbf{y}}^2\right),$$

where $\mathbf{\Sigma} = (\mathbf{\Sigma}^1 + \mathbf{\Sigma}^2)/2$, $\hat{\mathbf{y}}^i = \mathrm{E}\{\mathbf{y}_0^F|\mathbf{u}_0^{F-1}, \mu_0 = i\}$, $\mathbf{\Sigma}^i = \mathrm{E}\{(\mathbf{y}_0^F - \hat{\mathbf{y}}_i)(\mathbf{y}_0^F - \hat{\mathbf{y}}_i)^T|\mathbf{u}_0^{F-1}, \mu_0 = i\}$, $i = 0, 1$. It can simply be shown that $\hat{\mathbf{y}}^i$ are affine functions of $\mathbf{u}_0^{F-1}$, and $\mathbf{\Sigma}^i$ are independent of $\mathbf{u}_0^{F-1}$.

Finally, the aim of control consists in minimizing the energy of the input signal, which is given by the quadratic cost function

$$L_k^{\mathrm{c}}\left(\mathbf{x}_k, \mathbf{u}_k\right) = \mathbf{u}_k^T\mathbf{u}_k, \quad (21)$$

and by substituting this particular cost function into (16), it follows that

$$J^{\mathrm{c}}\left(\mathbf{u}_0^F\right) = \sum_{k=0}^{F}\mathbf{u}_k^T\mathbf{u}_k. \quad (22)$$

C. Numerical solution discussion

In order to get concrete optimization problem formulations out of the problems presented in Section II the assumptions presented in Section IV can be used. The expression in (20) is used to upper bound J^{d} and this expression is substituted into the general formulations in Section II. The resulting optimization problems are in general non-convex and they are therefore not possible to solve using “standard tools” as LP, QP or SDP solvers, which are commonly used in areas like Model Predictive Control for linear systems, [19]. In this section it is described how these non-convex optimization problems can be solved numerically. The approach chosen in this work is to use a global optimization routine `bmibnb` in the freely available MATLAB toolbox YALMIP, [20]. This routine implements a spatial branch and bound routine similar to the one introduced in [21] for bilinear non-convex optimization problems. The main idea in the algorithm is to compute convex envelopes that work as a convex outer approximations of the nonlinear functions. During the branch and bound process, better and better outer approximations are computed and these are used to compute lower bounds on the optimal objective function value. In the spirit of branch and bound, also upper bounds on the optimal objective function value are computed, and these are used to prune the branch and bound search tree. In this work, the lower bounds are computed using CPLEX, [22], and the upper bounds using SNOPT, [23].

Since this solution strategy is based on non-convex global optimization, the computational performance cannot in general be expected to be tractable. Hence, the choice of this algorithm is motivated by that we would like to see how good the rest of the concept can possibly get and the YALMIP solver provides us with a tool which is very suitable for the experiments performed in this work. In a practical implementation, some relaxed version of the problems are more tractable to solve, especially if the procedure is to be performed in real-time.

The three different approaches are now studied separately.

1) FDC I:

$$\min_{\mathbf{u}_0^F} \alpha\mathcal{B}\left(\mathbf{u}_0^{F-1}\right) + (1-\alpha)\sum_{k=0}^{F}\mathbf{u}_k^T\mathbf{u}_k \quad (23)$$

subject to (1), (2), and $\mathbf{u}_0^F \in \mathcal{U}$.

The objective function which is a weighted sum of the Bhattacharyya bound and the control cost is a non-convex function for $\alpha \in (0, 1]$. Note that, for $\alpha = 0$, the objective function is simply a convex quadratic function. In order to solve this non-convex (when $\alpha \in (0, 1]$) problem to global optimality, or at least to a user defined suboptimality level, the `bmibnb` solver in YALMIP is used.

2) FDC II:

$$\min_{\mathbf{u}_0^F}\sum_{k=0}^{F}\mathbf{u}_k^T\mathbf{u}_k \quad (24)$$

subject to (1), (2) and

$$\mathcal{B}(\mathbf{u}_0^{F-1}) \leq L_{\max}^{\mathrm{d}},\ \mathbf{u}_0^F \in \mathcal{U}. \quad (25)$$

Using the definition of $\mathcal{B}$, the constraint in (25) can be rewritten as

$$\begin{aligned} &-\left(\hat{\mathbf{y}}^1 - \hat{\mathbf{y}}^2\right)^T\mathbf{\Sigma}^{-1}\left(\hat{\mathbf{y}}^1 - \hat{\mathbf{y}}^2\right) \leq \\ &4\ln\left(\frac{|\Sigma|}{\sqrt{|\Sigma^1||\Sigma^2|}}\right) + 8\ln\left(L_{\max}^{\mathrm{d}}\right) \quad (26) \\ &-4\ln\left(P(\mu_0 = 1)P(\mu_0 = 2)\right) \triangleq \kappa, \end{aligned}$$

where κ is a constant. Hence, the constraint in (25) can be rewritten as a non-convex quadratic constraint. The optimization problem to minimize (24) subject to (26) can be solved using YALMIP's `bmibnb` solver.

3) FDC III: Since $\hat{\mathbf{y}}^1$ and $\hat{\mathbf{y}}^2$ are the only optimization variables in the expression in (20), $\mathcal{B}$ can from an optimization point of view be simplified to an expression in the form

$$\gamma\exp\left(-\frac{1}{8}\left(\hat{\mathbf{y}}^1 - \hat{\mathbf{y}}^2\right)^T\mathbf{\Sigma}^{-1}\left(\hat{\mathbf{y}}^1 - \hat{\mathbf{y}}^2\right)\right), \quad (27)$$

where γ is considered as a constant. Since optimization problems with scaled objective functions are equivalent, the constant γ can be neglected during the optimization. Moreover, the logarithm is a monotonically increasing function. Therefore, it is equivalent to minimize $-(\hat{\mathbf{y}}^1 - \hat{\mathbf{y}}^2)^T\mathbf{\Sigma}^{-1}(\hat{\mathbf{y}}^1 - \hat{\mathbf{y}}^2)$ instead of the exponential function of this expression. As a conclusion, the problem can be formulated as minimization of a concave quadratic function subject to a convex constraint set, i.e.,

$$\min_{\mathbf{u}_0^{F-1}} -\left(\hat{\mathbf{y}}^1 - \hat{\mathbf{y}}^2\right)^T\mathbf{\Sigma}^{-1}\left(\hat{\mathbf{y}}^1 - \hat{\mathbf{y}}^2\right) \quad (28)$$

subject to (1), (2) and $\sum_{k=0}^{F}\mathbf{u}_k^T\mathbf{u}_k \leq L_{\max}^{\mathrm{c}},\ \mathbf{u}_0^F \in \mathcal{U}$. Also this problem can be solved using `bmibnb` in YALMIP.

V. SUBOPTIMAL SOLUTION - NUMERICAL EXAMPLES

Two numerical examples will be presented in this section. The aim of the first example is to graphically illustrate the constraint sets and the criteria for all three problem formulations. The second example focuses on the designing the OL optimal input sequences for a longer horizon.

A. First numerical example

In the first numerical example, just one-step prediction (i.e. $F = 1$) is considered, and the system matrices are the following

$$\mathbf{A}_1 = \mathbf{A}_2 = \mathbf{C}_1 = \mathbf{C}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$\mathbf{G}_1 = \mathbf{G}_2 = \mathbf{H}_1 = \mathbf{H}_2 = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix},$$
$$\mathbf{B}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}, \ \mathbf{B}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \quad (29)$$

Constraint sets $\mathcal{U}_k$ are defined for each $k \in \mathcal{T}$ as $\mathcal{U}_k := \{\mathbf{u}_k \in \mathbb{R}^2 | \|\mathbf{u}_k\|_\infty \leq 5\}$. The initial condition $\mathbf{x}_0$ is given by the mean value $\hat{\mathbf{x}}_{0|-1} = [0, \ 0]^T$ and the covariance matrix $\mathbf{P}_{0|-1} = 0.1\mathbf{I}_2$, where $\mathbf{I}_n$ is the identity matrix of order n. The a priori probabilities are $P(\mu_0 = 1) = P(\mu_0 = 2) = 0.5$.

1) FDC I: The combined criterion $J(\mathbf{u}_0^F)$ expresses a trade-off between the detection aim and the control aim, and the solution may lie anywhere within the constraint set $\mathcal{U}$ depending on the parameter α_0. As an example, the optimal solution for $\alpha_0 = 0.995$ is depicted in Fig. 2.

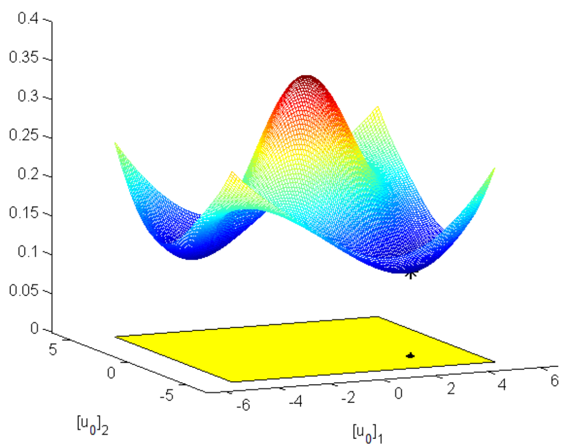

Fig. 2. Constraint set and criterion for FDC I. The black cross denotes the optimal input $\mathbf{u}_0$ and yellow area represents the constraint set.

2) FDC II: The control criterion $J^c(\mathbf{u}_0^F)$ is a convex function and its unconstrained minimum is attained at $\mathbf{u}_k = \mathbf{0} \ \forall k \in \mathcal{T}$. However, such an input signal provides poor information for detection because it may not sufficiently excite the system. The Bhattacharyya bound determines a non-convex set that excludes the input signals close to origin and prevents insufficient excitation. The optimal solution lies on the boundary of set given by the Bhattacharyya bound, as it is demonstrated in Fig. 3 for the maximum allowable probability of misclassification $L^d_{\max} = 0.3$.

3) FDC III: Since the logarithm of the Bhattacharyya bound is a concave function, the solution to the unconstrained minimization would result into an input signal with the infinite amplitude. Nevertheless, there are two constraint sets $\mathbf{u}_0^F \in \mathcal{U}$ and $\sum_{k=0}^{F} \mathbf{u}_k^T \mathbf{u}_k \leq L^c_{\max}$. The maximum

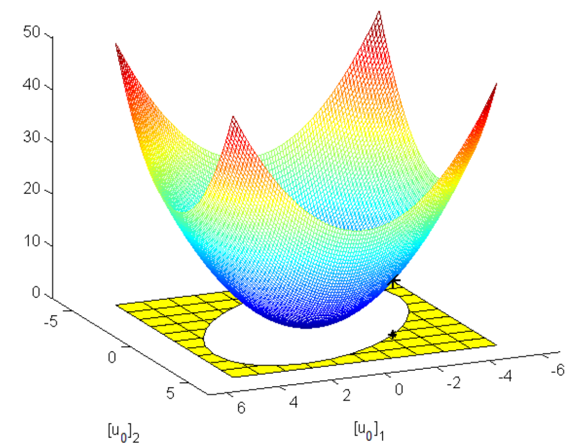

Fig. 3. Constraint set and criterion for FDC II. The black cross denotes the optimal input $\mathbf{u}_0$ and yellow area represents the constraint set.

allowed value for control criterion $L^c_{\max} = 20$ defines a disk $\{\mathbf{u}_0 \in \mathbb{R}^2 | \mathbf{u}_0^T \mathbf{u}_0 \leq 20\}$ with the center at the origin. Since this disk lies entirely in the set $\mathcal{U}$, the optimal solution lies on the boundary of the disk (i.e. $\mathbf{u}_0^T \mathbf{u}_0 = 20$), see Fig. 4.

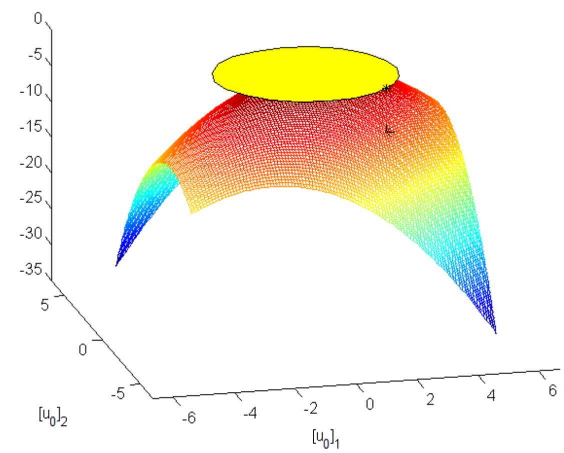

Fig. 4. Constraint set and criterion for FDC III. The black cross denotes the optimal input $\mathbf{u}_0$ and yellow area represents the constraint set.

B. Second numerical example

The second numerical example focuses on comparison of the optimal input sequences using four different experiments. The experiments are defined as follows

- Experiment 1: FDC I, $\alpha = 0$
- Experiment 2: FDC III, $L^c_{\max} = 40$
- Experiment 3: FDC II, $L^d_{\max} = 0.01$
- Experiment 4: FDC I, $\alpha = 1$.

The system matrices are the following

$$\mathbf{A}_1 = 0.9, \ \mathbf{A}_2 = 0.6, \ \mathbf{B}_1 = 0.1, \ \mathbf{B}_2 = 1,$$
$$\mathbf{C}_1 = \mathbf{C}_2 = 1, \ \mathbf{G}_1 = \mathbf{G}_2 = \mathbf{H}_1 = \mathbf{H}_2 = \sqrt{2}. \quad (30)$$

Constraint sets $\mathcal{U}_k$ are defined for each $k \in \mathcal{T}$ as $\mathcal{U}_k := \{\mathbf{u}_k \in \mathbb{R} | \|\mathbf{u}_k\| \leq 5\}$. The initial condition $\mathbf{x}_0$ is given by the mean value $\hat{\mathbf{x}}_{0|-1} = 0$ and the covariance $\mathbf{P}_{0|-1} = 0.1$. The a priori probabilities are $P(\mu_0 = 1) = P(\mu_0 = 2) = 0.5$ and prediction horizon is $F = 40$.

The optimal input signals of the selected experiments are depicted in Fig. 5. The first experiment shows the optimal input signal, when the detection cost is not considered. This setup results into the zero input signal. The second experiment illustrates the control cost constrained problem, while the the third experiment represents the detection constrained problem. In both cases, the optimal input signal resembles a harmonic signal with a variable amplitude. Despite a difference in problem formulations, the form of the optimal OL input sequence is similar to the form of input for some cases presented in [24]. The fourth experiment shows the active detector, when the control cost is not considered. The input signal oscillates between minimum and maximum admissible input value and provides the most useful information to the detector. This problem formulation was discussed in details in [9].

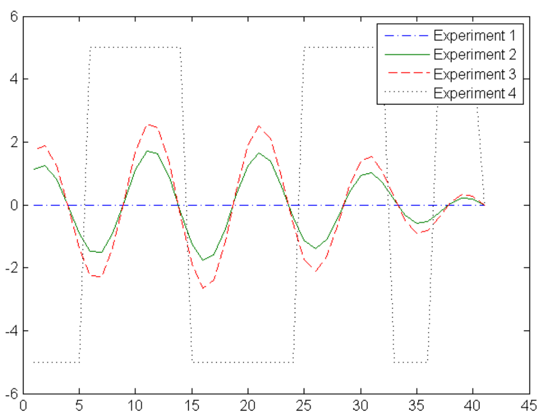

Fig. 5. Optimal input sequences.

VI. Conclusion

Three alternative formulations of active fault detection and control were presented in the paper. The first formulation aims at minimization of a cost function given as a trade-off between control and detection. The second formulation aims at minimization of a control objective, while respecting a detection constraint and the third formulation aims at minimization of a detection objective, while respecting a control constraint. The optimal closed loop solution to all three problems is numerically intractable. Therefore a suboptimal solution based on the open loop information processing strategy, quadratic control cost criterion, and the Bhattacharyya bound as the detection criterion was presented. The presented framework of three problem formulations provides a useful tool for designing active detection and control systems. The designer can choose the formulation that suits the best the current application in the sense of the design requirements.

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