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Boundary second-order sliding-mode control of an uncertain heat process with spatially varying diffusivity

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Abstract— **The primary concern of the present paper is the regulation of an uncertain heat process with collocated boundary sensing and actuation. The underlying heat process is governed by an uncertain parabolic partial differential equation (PDE) with mixed boundary conditions. The process exhibits an unknown spatially varying diffusivity parameter, and is affected by a smooth uncertain boundary disturbance which is, possibly, unbounded in magnitude. The proposed robust synthesis is formed by the linear feedback design and by the "Twisting" second-order sliding-mode control algorithm, suitably combined and re-worked in the infinite-dimensional setting. A non-standard Lyapunov functional is invoked to prove the global asymptotic stability of the resulting closed-loop system in a suitable Sobolev space. The proof is accompanied by a set of simple tuning rules for the controller parameters. The effectiveness of the developed control scheme is supported by simulation results.**

Keywords: **Infinite-dimensional systems; Boundary Control; Heat equation; Uncertain systems; Second-order sliding modes.**

I. INTRODUCTION

The primary concern of the present paper is the regulation of an uncertain heat process with collocated boundary sensing and actuation. The boundary control problem for heat processes was studied, e.g., in [3], [6], [8] under more strict assumptions on the admitted uncertainties and perturbations compared to those made in the present work. In this paper we address the boundary control problem for an uncertain heat process, governed by a parabolic partial differential equation (PDE) with a scalar spatial variable $\xi \in [0, 1]$ and with Robin's boundary conditions. An appropriate extension of the "Twisting" second-order sliding mode (2-SM) control technique (see [9], [13] for details on the application of this algorithm in the finite-dimensional setting) allows us to address the following main features:

- The diffusivity parameter is admitted to be uncertain
- Only collocated boundary sensing and actuation are assumed to be available.
- The proposed controller is simple to implement, and rejects a class of non-vanishing matched perturbations of arbitrary shape, possibly unbounded in magnitude, requiring just the knowledge of a constant upper bound to the magnitude of the disturbance time derivative.
- The plant input is continuous, whereas its first-order time derivative is discontinuous.
- The global asymptotic stability of the error system is achieved in the Sobolev space $W^{2,2}(0, 1)$.

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In the closely related recent publication [4] a similar problem has been studied by combining an integral-type first-order sliding mode controller and a backstepping transformation (see [8]). A similar dynamics as that considered in the present paper, with Dirichlet (instead of Robin's) BCs, has been dealt with in the above work. However, the controller tuning inequalities resulting from the presented Lyapunov analysis depend on the spatiotemporal derivatives of the solution, which are, normally, not available for feedback in practice, thereby making the result presented in [4] of local nature.

In the present work the positive diffusivity parameter is admitted to have an uncertain spatially-varying profile, and a space varying reference is considered. In the resulting closed-loop system, the discontinuous 2-SM controller is connected to the plant input through a dynamical filter (an integrator) thereby augmenting the system state with its time derivative. While passing through the filter, the discontinuous signal is smoothed out, and the so-called chattering phenomenon, extremely undesired in practice, is thus attenuated. Due to such a dynamic input extension, the global asymptotic stabilization of the underlying uncertain heat process is achieved in a stronger norm of a Sobolev space, involving spatial state derivatives up to the second order. The stability proof is based on a non-smooth Lyapunov functional construction and it leads to a set of simple tuning rules for the controller parameters.

The rest of the paper is outlined as follows. In Section 2, the control problem is formulated. In Section 3, a stabilizing boundary controller is developed and the associated stability proof is presented. Simulation results are given in Section 4. Finally, Section 5 collects some concluding remarks.

A. Notation

The notation used throughout is fairly standard. $L_2(0,1)$ stands for the Hilbert space of square integrable functions $z(\zeta),\ \zeta \in (0,1)$, whose L_2-norm is given by

$$\|z(\cdot)\|_2 = \sqrt{\int_0^1 z^2(\zeta)d\zeta}. \quad (1)$$

$W^{0,2}(0,1)$denotes the Hilbert space $L_2(0,1)$. $W^{1,2}(0,1)$ denotes the Sobolev space of absolutely continuous scalar functions $z(\zeta)$ on $(0,1)$ with square integrable derivative $z_\zeta(\zeta)$ and the norm

$$\|z(\cdot)\|_{1,2} = \sqrt{\|z(\cdot)\|_2 + \|z_\zeta(\cdot)\|_2} \quad (2)$$

$W^{2,2}(0,1)$ denotes the Sobolev space of absolutely continuous scalar functions $z(\zeta)$ on $(0,1)$ with square integrable derivatives $z^{(i)}(\zeta)$ up to the order $i = 2$ and the weighted norm

$$\|z(\cdot)\|_{2,2,\varphi} = \sqrt{\|z(\cdot)\|_2 + \|z_\zeta(\cdot)\|_2 + \|[\varphi(\cdot)z_\zeta(\cdot)]_\zeta\|_2} \quad (3)$$

which depends on the weighting function $\varphi(\cdot) \in W^{1,2}(0,1)$.

II. Problem formulation

Consider the space- and time-varying scalar field $Q(\xi,t)$ with the monodimensional spatial variable $\xi \in [0,1]$ and time variable $t \geq 0$. Let it be governed by a perturbed version of the parabolic PDE which is commonly referred to as the "**Heat Equation**":

$$Q_t(\xi,t) = [\theta(\xi)Q_\xi(\xi,t)]_\xi, \quad (4)$$

where the subscripts t and ξ denote temporal and spatial derivatives, respectively, and $\theta(\cdot) \in C^1(0,1)$ is a positive-definite spatially-varying parameter called *thermal conductivity* (or, more generally, *diffusivity*). The initial condition (IC) is given by

$$Q(\xi,0) = Q^0(\xi) \in W^{2,2}(0,1). \quad (5)$$

Throughout, we assume controlled and perturbed Robin's (i.e., mixed) boundary conditions (BCs) of the form

$$\begin{aligned} Q_\xi(0,t) &= \alpha_0 Q(0,t) + \beta_0 && (6)\\ Q_\xi(1,t) &= -\alpha_1 Q(1,t) + \beta_1 + u(t) + \psi(t), && (7)\end{aligned}$$

where $\alpha_i \geq 0$ and β_i $(i = 0,1)$, are arbitrary constants (no restrictions on the sign of the β_i's are met), $u(t) \in \mathbb{R}$ is a modifiable source term (boundary control input) and $\psi(t) \in \mathbb{R}$ represents an **uncertain** sufficiently smooth disturbance.

We consider the time-independent and spatially varying reference $Q^r(\xi)$ which satisfies the boundary value problem

$$\begin{aligned} [\theta(\xi)Q^r_\xi(\xi)]_\xi &= 0 && (8)\\ Q^r_\xi(0) &= \alpha_0 Q^r(0) + \beta_0 && (9)\\ Q^r(1) &= Q^r_1 && (10)\end{aligned}$$

for an arbitrary, user-selectable, constant Q^r_1.

The class of admissible disturbances is specified by the following restriction on their time derivative.

Assumption 1: The disturbance $\psi(t)$ is twice continuously differentiable and there exists an *a priori* known constant M such that

$$|\psi_t(t)| \leq M \quad (11)$$

for almost all $t \geq 0$.

The spatially varying diffusivity is supposed to satisfy the next restriction

Assumption 2: There exist *a priori* known constants Θ_m, Θ_M such that

$$0 < \Theta_m \leq \theta(\xi) \leq \Theta_M, \quad \forall \xi \in [0,1]. \quad (12)$$

With the assumptions above the evolution of the considered heat process is studied in the Sobolev space $W^{2,2}(0,1)$ and the control objective is to steer the $W^{2,2}$-norm of the deviation

$$x(\xi,t) = Q(\xi,t) - Q^r(\xi) \quad (13)$$

of the scalar field $Q(\xi,t)$ from the *a priori* given reference to zero, despite the presence of an uncertain, arbitrarily shaped, smooth boundary disturbance $\psi(t)$ fulfilling the Assumption 1. Boundary sensing at $\xi = 1$ of the deviation $x(\xi,t)$ and of its time derivative $x_t(\xi,t)$ is assumed to be the only available information on the state of the system. The deviation variable $x(\xi,t)$ is governed by the heat equation

$$x_t(\xi,t) = [\theta(\xi)x_\xi(\xi,t)]_\xi \quad (14)$$

subject to the next Robin-type BCs

$$\begin{aligned} x_\xi(0,t) - \alpha_0 x(0,t) &= 0 && (15)\\ x_\xi(1,t) + \alpha_1 x(1,t) &= u(t) + \psi(t) + \gamma_1, && (16)\end{aligned}$$

with the constant

$$\gamma_1 = \beta_1 - Q^r_\xi(1) - \alpha_1 Q^r_1, \quad (17)$$

which can be derived by considering (13), and its spatial derivative $x_\xi(\xi,t) = Q_\xi(\xi,t) - Q^r_\xi(\xi)$, along with the conditions (7) and (10). The corresponding ICs are

$$x(\xi,0) = x^0(\xi), \quad x^0(\xi) = Q^0(\xi) - Q^r(\xi) \quad (18)$$

It is worth noticing that the disturbance-free system (14)-(18) in open-loop is only stable, rather than asymptotically stable. Thus, the modifiable control variable $u(t)$ should be designed in order to make the zero solution $x(\xi,t) = 0$ of the closed-loop system (14)-(18) globally asymptotically stable in the $W^{2,2}$-space despite the presence of an unknown disturbance $\psi(t)$ affecting the state of the system through its boundary. Since non-homogeneous boundary conditions are in force, the meaning of the boundary-value problem (14)-(18) is subsequently viewed in the mild sense.

The mild solutions coincide with those of the following PDE in distributions

$$x_t(\xi,t) = [\theta(\xi)x_\xi(\xi,t)]_\xi + \theta(1)[u(t) + \psi(t) + \gamma_1]\delta(\xi - 1) \quad (19)$$

subject to the homogeneous Robin BCs

$$x_\xi(0,t) - \alpha_0 x(0,t) = x_\xi(1,t) + \alpha_1 x(1,t) = 0 \quad (20)$$

and to the ICs (18). Indeed, (weak) solutions of the boundary-value problem (**??**)-(20) are defined by means of the corresponding Green function, yielding the same integral equation.

To this end, we note that according to [1, Theorem 3.3.3], the unforced system (14)-(18) with $u(t) \equiv 0$ possesses a unique classical solution in the state space $W^{2,2}(0,1)$ (cf. [1, Definition 3.2.9].

III. Main Result

To achieve the control goal, the system state is augmented through a *dynamic input extension* by inserting an integrator at the plant input. The control derivative $u_t(t)$ is then regarded as a fictitious control variable to be generated by a suitable feedback mechanism. The next dynamical controller

$$\begin{aligned} u_t(t) &= -\lambda_1 \text{sign}\, x(1,t) - \lambda_2 \text{sign}\, x_t(1,t) - W_1 x(1,t) \\ &- W_2 x_t(1,t), \qquad u(0) = 0, \end{aligned} \tag{21}$$

is currently under study, where the initial condition $u(0)$ is set to zero for certainty. In the above controller description, λ_1, λ_2, W_1 and W_2 are constant tuning parameters subject to the inequalities

$$\lambda_2 > M, \ \ \lambda_1 > \lambda_2 + M, \ \ W_1 > \frac{1}{2}\frac{\Theta_M}{\Theta_m}, \ \ W_2 > 0. \tag{22}$$

Remark 1: Since the dynamic control input is governed by the ordinary differential equation (21) with discontinuous (multi-valued) right-hand side, the precise meaning of the solutions of the distributed parameter system (14)-(18), driven by the discontinuous dynamic controller (21), is then specified in the sense of Filippov [5]. Extension of the Filippov concept towards the infinite-dimensional setting may be found in [10], [13]. As in the finite-dimensional case, a motion along the discontinuity manifold is referred to as a sliding mode.

Since the present paper focuses on the stabilizing synthesis we do not analyze the well-posedness of the closed-loop system because of space limitations and due to the parabolic character of the system that ensures it. Thus, in the remainder, we simply assume the following.

Assumption 3: The closed-loop system (**??**)-(21) possesses a unique mild solution $x(\cdot,t) \in W^{2,2}(0,1)$ whose time derivative $x_t(\cdot,t) \in W^{2,2}(0,1)$ constitutes a (weak) solution of the distribution boundary-value problem

$$x_{tt}(\xi,t) = [\theta(\xi)x_{t\xi}(\xi,t)]_\xi + \theta(1)\{u_t[y](t) + \psi_t(t)\}\delta(\xi-1) \tag{23}$$

$$\begin{aligned} &x_{t\xi}(0,t) - \alpha_0 x_t(0,t) = 0, \\ &x_{t\xi}(1,t) + \alpha_1 x_t(1,t) = 0. \end{aligned} \tag{24}$$

with respect to $x_t(\xi,t)$, which is formally obtained by differentiating (**??**)-(20) in the time variable.

The following relation

$$\|x_t\|_2 = \|[\theta(\xi)x_\xi(\xi,t)]_\xi\|_2 \tag{25}$$

is particularly concluded from (14). Along with the technical lemmas of the next subsection, relation (25) will be instrumental in our further derivation.

We are now in a position to state our main result.

Theorem 1 *Consider the perturbed heat process (4)-(7) subject to the dynamic control strategy (21), (22). Let Assumptions 1 and 2 be satisfied. Then the solutions (x, x_t) of the resulting error boundary-value problem (23)-(24) are globally asymptotically stable in the space $W^{2,2}(0,1) \times L_2(0,1)$.*

A. Instrumental Lemmas

We now present several technical lemmas that will be instrumental in the subsequent proof of Theorem 1.

Lemma 1: Let $z(\xi) \in W^{1,2}(0,1)$. Then, the following inequality holds:

$$\|z(\cdot)\|_2^2 \le 2(z^2(i) + \|z_\xi(\cdot)\|_2^2), \quad i = 0, 1. \tag{26}$$

Proof of Lemma 1: Given $z(\xi) \in W^{1,2}(0,1)$, it is absolutely continuous and therefore,

$$z(\xi) = z(0) + \int_0^\xi z_\xi(\eta)d\eta, \ for\ any\ \xi \in [0,1]. \tag{27}$$

which, considering the well known inequality $\|w\|_1 < \|w\|_2$ which is valid for all $w \in L_2(0,1)$, can be estimated as

$$\begin{aligned} |z(\xi)| &\le |z(0)| + \int_0^\xi |z_\xi(\eta)|d\eta \le |z(0)| + \int_0^1 |z_\xi(\eta)|d\eta \\ &= |z(0)| + \|z_\xi(\cdot)\|_1 \le |z(0)| + \|z_\xi(\cdot)\|_2 \end{aligned} \tag{28}$$

Now squaring both sides of (28), applying Young's inequality $2ab < a^2 + b^2$, and integrating both sides over the spatial domain $\xi \in [0,1]$, yield (26) with $i = 0$. The proof of (26) with $i = 1$ becomes identical under the change of coordinate $\zeta = 1 - \xi$. Lemma 1 is proved. □

Lemma 2: The functional

$$\begin{aligned} \tilde{V}(x, x_t) &= \lambda_1\theta(1)|x(1,t)| + \frac{1}{2}\theta(1)W_1 x^2(1,t) \\ &+ \frac{1}{2}\|x_t(\cdot,t)\|_2^2, \end{aligned} \tag{29}$$

being computed on the mild solutions (x, x_t) of the boundary-value problem (23)-(24), upper estimates the weighted $W^{2,2}(0,1) \times L_2(0,1)$-norm of these solutions in the sense that

$$\alpha(\|x(\cdot,t)\|_{2,2,\theta}^2 + \|x_t(\cdot,t)\|_2^2) \le \tilde{V}(x, x_t) \tag{30}$$

for any time instant $t \ge 0$ and for some positive constant α

Proof of Lemma 2: Successively applying relation (26) with $i = 1$ to a mild solution $z = x(\xi,t)$ and then to the term $z = \theta(\xi)x_\xi(\xi,t)$ yields

$$\|x(\cdot,t)\|_2^2 \le 2(x^2(1,t) + \|x_\xi(\cdot,t)\|_2^2), \tag{31}$$

$$\|\theta(\cdot)x_\xi(\cdot,t)\|_2^2 \le 2(\theta^2(1)x_\xi^2(1,t) + \|[\theta(\cdot)x_\xi(\cdot,t)]_\xi\|_2^2). \tag{32}$$

By exploiting the next relation

$$\Theta_m^2\|x_\xi(\cdot,t)\|_2^2 \le \|\theta(\cdot)x_\xi(\cdot,t)\|_2^2 \le \Theta_M^2\|x_\xi(\cdot,t)\|_2^2, \tag{33}$$

which is a trivial consequence of (12), it can be further manipulated (32) so as to obtain

$$\begin{aligned}\|x_\xi(\cdot,t)\|_2^2 &\le \frac{2}{\Theta_m^2}(\Theta_M^2 x_\xi^2(1,t)+\|[\theta(\cdot)x_\xi(\cdot,t)]_\xi\|_2^2)= \\ &= \rho_1 x_\xi^2(1,t)+\rho_2\|[\theta(\cdot)x_\xi(\cdot,t)]_\xi\|_2^2 \end{aligned} \quad (34)$$

with the positive constants ρ_1 and ρ_2 beng implicitly defined. By taking into account the BC (20), the above relations (31) and (34) can be rewritten in the form

$$\|x(\cdot,t)\|_2^2 \le 2x^2(1,t)+2\rho_1\alpha_1^2x^2(1,t) + 2\rho_2\|[\theta(\cdot)x_\xi(\cdot,t)]_\xi\|_2^2 \quad (35)$$

$$\|x_\xi(\cdot,t)\|_2^2 \le \rho_1\alpha_1^2x^2(1,t)+\rho_2\|[\theta(\cdot)x_\xi(\cdot,t)]_\xi\|_2^2. \quad (36)$$

Employing relation (25), it follows from (35)-(36) that

$$\begin{aligned}\|x(\cdot,t)\|_{2,2,\theta}^2 &= \|x(\cdot,t)\|_2^2+\|x_\xi(\cdot,t)\|_2^2+\|[\theta(\cdot)x_\xi(\cdot,t)]_\xi\|_2^2 \\ &\le (2+3\rho_1\alpha_1^2)x^2(1,t)+(3\rho_2+1)\|[\theta(\cdot)x_\xi(\cdot,t)]_\xi\|_2^2 \\ &= (2+3\rho_1\alpha_1^2)x^2(1,t)+(3\rho_2+1)\|x_t(\cdot,t)\|_2^2,\end{aligned}$$

and taking into account (29), the validity of (30) is thus concluded for all $t\ge 0$ and for some positive α. Lemma 2 is proved. □

Lemma 3: Let a set

$$\begin{aligned}\mathcal{D}_R^{\tilde V} &= \{(z(\xi),h(\xi))\in W^{2,2}(0,1)\times L_2(0,1): \\ &\qquad \tilde V(z,h)\le R\}\end{aligned} \quad (37)$$

be determined by means of functional (29) and be specified with some positive R. Then the following conditions

$$\int_0^1 z(1)h(\xi)\,d\xi \ge -\frac{1}{2}\left[\frac{R}{\lambda_1\Theta_m}|z(1)|+\|h\|_2^2\right], \quad (38)$$

$$\|h\|_2^2\le 2R,\quad \|h\|_2\le\sqrt{2R},\quad \|h\|_2^2\le\sqrt{2R}\|h\|_2, \quad (39)$$

hold for an arbitrary $(z(\xi),h(\xi))\in\mathcal{D}_R^{\tilde V}$.

Proof of Lemma 3: The following implications hold in light of the inequalities (12):

$$\begin{aligned}\tilde V(z,h) \le R &\Rightarrow \theta(1)\lambda_1|z(1)|\le R \\ &\Rightarrow |z(1)|\le\frac{R}{\lambda_1\theta(1)}\le\frac{R}{\lambda_1\Theta_m}.\end{aligned} \quad (40)$$

Furthermore, by the triangle inequality it yields

$$\begin{aligned}\int_0^1 z(1)h(\xi)\,d\xi &\ge -\frac{1}{2}\left[z^2(1)+\|h\|_2^2\right]= \\ &- \frac{1}{2}\left[|z(1)||z(1)|+\|h\|_2^2\right].\end{aligned} \quad (41)$$

Being coupled together, (40) and (41) immediately result in (38). In turn, the relations (39) follow from the trivial chain of implications (that consider the positive definiteness of $\theta(1)$):

$$\begin{aligned}\tilde V(z,h) \le R &\Rightarrow \frac{1}{2}\|h\|_2^2\le R \Rightarrow \|h\|_2\le\sqrt{2R} \Rightarrow \\ &\Rightarrow \|h\|_2^2\le\sqrt{2R}\|h\|_2.\end{aligned} \quad (42)$$

Lemma 3 is thus proved. □

B. Proof of Theorem 1

By Lemma 2, functional (29) is positive definite along the mild solutions (x,x_t) of the boundary-value problem (23)-(24). The time derivative of (29) along such solutions is

$$\begin{aligned}&\dot{\tilde V}(t)=\lambda_1\theta(1)x_t(1,t)sign(x(1,t)) \\ &+ W_1\theta(1)x(1,t)x_t(1,t)+\int_0^1 x_tx_{tt}d\xi \\ &= \lambda_1\theta(1)x_t(1,t)sign(x(1,t))+W_1\theta(1)x(1,t)x_t(1,t) \\ &+ \int_0^1 x_t[\theta(\xi)x_{t\xi}(\xi,t)]_\xi d\xi+\theta(1)x_t(1,t)[u_t(t)+\psi_t(t)].\end{aligned} \quad (43)$$

The integral term in the right hand side of (43), being integrated by parts by taking into account the homogeneous BC's (24), yields

$$\begin{aligned}\int_0^1 x_t[\theta(\xi)x_{t\xi}(\xi,t)]_\xi d\xi &= -\theta(1)\alpha_1x_t^2(1,t) \\ &-\theta(0)\alpha_0x_t^2(0,t)-\int_0^1\theta(\xi)x_{t\xi}^2d\xi.\end{aligned} \quad (44)$$

By substituting (21) into the last term of (43), and making simple manipulations, one obtains

$$\begin{aligned}\dot{\tilde V}(t) &= -\lambda_2\theta(1)|x_t(1,t)|-\theta(1)(W_2+\alpha_1)x_t^2(1,t) \\ &- \int_0^1\theta(\xi)x_{t\xi}^2d\xi-\theta(0)\alpha_0x_t^2(0,t) \\ &+ \theta(1)x_t(1,t)\psi_t(t).\end{aligned} \quad (45)$$

Due to the upper bound (11), one obtains

$$|\theta(1)x_t(1,t)\psi_t(t)|\le\theta(1)M|x_t(1,t)|, \quad (46)$$

By (46), and considering as well the inequality (12), relation (45) is further manipulated to

$$\begin{aligned}\dot{\tilde V}(t) &\le -\Theta_m(\lambda_2-M)|x_t(1,t)|-\Theta_m(W_2+\alpha_1)x_t^2(1,t) \\ &- \Theta_m\alpha_0x_t^2(0,t)-\Theta_m\|x_{t\xi}\|_2^2.\end{aligned} \quad (47)$$

Due to (22) and (47), the Lyapunov functional $\tilde V(t)$, being computed along the mild solutions of the closed-loop system, is a non-increasing function of time, and as a result the domain $\mathcal{D}_R^{\tilde V}$, given by (37) with an arbitrary $R\ge\tilde V(0)$, is **invariant** for the system trajectories. Thus, the subsequent analysis will take into account that the mild solutions (x,x_t) stay in the domain $\mathcal{D}_R^{\tilde V}$ forever.

Now consider the "augmented" functional

$$\begin{aligned}\tilde V_R(t) &= \tilde V(t)+\frac{1}{2}\kappa_R\theta(1)(W_2+\alpha_1)x^2(1,t) \\ &+ \kappa_R\int_0^1 x(1,t)x_t(\xi,t)\,d\xi\end{aligned} \quad (48)$$

where κ_R is a sufficiently small positive constant to subsequently be specified.

By Lemma 3 specified with $z=x$ and $h=x_t$, and considering the inequality (12), in the domain $\mathcal{D}_R^{\tilde V}$ function

$\tilde{V}_R$ can be estimated as

$$\begin{aligned}\tilde{V}_R(x,x_t) &\geq \lambda_1\Theta_m|x(1,t)| \\ &+\frac{1}{2}\Theta_m[W_1+\kappa_R(W_2+\alpha_1)]x^2(1,t) \\ &+\frac{1}{2}\|x_t\|_2^2-\frac{\kappa_R}{2}\left[\frac{R}{\lambda_1\Theta_m}|x(1,t)|+\|x_t\|_2^2\right] \\ &= \left(\lambda_1\Theta_m-\frac{\kappa_R R}{2\lambda_1\Theta_m}\right)|x(1,t)|+\frac{1}{2}(1-\kappa_R)\|x_t\|_2^2 \\ &+\frac{1}{2}\Theta_m(W_1+\kappa_R(W_2+\alpha_1))x^2(1,t)\end{aligned} \tag{49}$$

Let us specify $\kappa_R>0$ such that

$$\kappa_R<\min\left\{\frac{2\lambda_1^2\Theta_m^2}{R},1\right\}. \tag{50}$$

Then, it follows from (49), (50) that the augmented functional (48) is lower estimated by functional (29) as

$$\tilde{V}_R\quad(x,x_t)\geq\mu\tilde{V}(x,x_t) \tag{51}$$

$$\mu=\min\left\{1-\frac{\kappa_R R}{2\lambda_1^2\Theta_m^2},\frac{W_1+\kappa_R(W_2+\alpha_1)}{W_1},(1-\kappa_R)\right\} \tag{52}$$

It means that along with (29), the functional $\tilde{V}_R$ is positive definite on the mild solutions (x,x_t) of the boundary-value problem (23)-(24) within the invariant set $\mathcal{D}_R^{\tilde{V}}$.

Let us now evaluate the time derivative of $\tilde{V}_R(t)$:

$$\begin{aligned}\dot{\tilde{V}}_R&=\dot{\tilde{V}}+\kappa_R\theta(1)(W_2+\alpha_1)x(1,t)x_t(1,t)\\ &+\kappa_R\int_0^1 x_t(1,t)x_t(\xi,t)d\xi+\kappa_R\int_0^1 x(1,t)x_{tt}(\xi,t)\xi.\end{aligned} \tag{53}$$

By utilizing the first inequality of (39) specified with $h=x_t$ and applying the well known inequality $\|z\|_1\leq\|z\|_2$ (which is valid for any $z\in L_2(0,1)$) the magnitude of the first integral term in the right hand side of (53) is estimated by

$$\begin{aligned}\left|\kappa_R\int_0^1 x_t(1,t)x_t(\xi,t)d\xi\right|&\leq\kappa_R|x_t(1,t)|\int_0^1|x_t(\xi,t)|d\xi\\ &\leq\kappa_R|x_t(1,t)|\|x_t\|_2\leq\sqrt{2R}\kappa_R|x_t(1,t)|.\end{aligned} \tag{54}$$

By straightforward integration one finds that the last integral term in (53) can be manipulated as follows

$$\begin{aligned}&\kappa_R x(1,t)\int_0^1 x_{tt}(\xi,t)d\xi=\kappa_R x(1,t)\\ &\times\int_0^1\left([\theta(\xi)x_{t\xi}(\xi,t)]_\xi+\theta(1)[u_t(t)+\psi_t(t)]\delta(x-1)\right)d\xi\\ &=\kappa_R x(1,t)[\theta(1)x_{t\xi}(1,t)-\theta(0)x_{t\xi}(0,t)]\\ &+\kappa_R x(1,t)\theta(1)(u_t(t)+\psi_t(t))\end{aligned} \tag{55}$$

Considering the BCs (24), the terms in the right hand side of (55) can be further elaborated as

$$\begin{aligned}&\kappa_R x(1,t)[\theta(1)x_{t\xi}(1,t)-\theta(0)x_{t\xi}(0,t)]=\\ &-\kappa_R\theta(1)\alpha_1 x(1,t)x_t(1,t)-\kappa_R\theta(0)\alpha_0 x(1,t)x_t(0,t)\end{aligned} \tag{56}$$

$$\begin{aligned}&\kappa_R\theta(1)x(1,t)(u_t(t)+\psi_t(t))=-\kappa_R\theta(1)\lambda_1|x(1,t)|\\ &-\kappa_R\theta(1)\lambda_2 x(1,t)\text{sign }x_t(1,t)-\kappa_R\theta(1)W_1x^2(1,t)\\ &-\kappa_R\theta(1)W_2x(1,t)x_t(1,t)+\kappa_R\theta(1)x(1,t)\psi_t(t).\end{aligned} \tag{57}$$

The next relation follows by applying the Young's inequality

$$\begin{aligned}&|\kappa_R\theta(0)\alpha_0 x(1,t)x_t(0,t)|\leq\\ &\kappa_R\theta(0)\left(\frac{1}{2}x^2(1,t)+\frac{1}{2}\alpha_0^2x_t^2(0,t)\right),\end{aligned}$$

and the following estimates

$$|\kappa_R\theta(1)\lambda_2 x(1,t)\text{sign }x_t(1,t)|\leq\kappa_R\theta(1)\lambda_2|x(1,t)|, \tag{58}$$

$$|\kappa_R\theta(1)x(1,t)\psi_t(t)|\leq\kappa_R\theta(1)M|x(1,t)|, \tag{59}$$

hold for the corresponding terms in (57) by virtue of Assumption 1. Employing (45)-(47), (54)-(59), and the inequality (12), the time derivative (53) is finally manipulated to

$$\begin{aligned}\dot{\tilde{V}}_R(t)\quad&\leq\quad-\Theta_m\left(\lambda_2-M-\frac{\kappa_R\sqrt{2R}}{\Theta_m}\right)|x_t(1,t)|\\ &-\quad\Theta_m(W_2+\alpha_1)x_t^2(1,t)\\ &-\quad\frac{1}{2}\Theta_m\alpha_0\left(2-\kappa_R\alpha_0\right)x_t^2(0,t)\\ &-\quad\Theta_m\|x_{t\xi}\|_2^2-\kappa_R\Theta_m[(\lambda_1-\lambda_2)-M]|x(1,t)|\\ &-\quad\kappa_R\left(W_1\Theta_m-\frac{1}{2}\Theta_M\right)x^2(1,t).\end{aligned} \tag{60}$$

It is clear that all the terms appearing in the right-hand side of (60) are nonpositive provided that the tuning condition (22), imposed on the controller parameters, hold and, in place of (50), the next more restrictive condition on the coefficient κ_R is additionally satisfied:

$$\kappa_R<\min\left\{\frac{2\lambda_1^2\Theta_m^2}{R},1,\frac{\Theta_m(\lambda_2-M)}{\sqrt{2R}},\frac{2}{\alpha_0}\right\}. \tag{61}$$

By Lemma 1, specialized with $z=x$, the mild solutions $x(\xi,t)\in W^{2,2}(0,1)$ satisfy the estimate (26). Moreover, its spatial and temporal derivatives $x_\xi(\xi,t)\in W^{1,2}(0,1)$ and $x_t(\xi,t)\in W^{1,2}(0,1)$ satisfy the next estimates

$$\|z_\xi(\cdot,t)\|_2^2\leq 2(z_\xi^2(i,t)+\|z_{\xi\xi}(\cdot,t)\|_2^2) \tag{62}$$

$$\|z_t(\cdot,t)\|_2^2\leq 2(z_t^2(i,t)+\|z_{t\xi}(\cdot,t)\|_2^2) \tag{63}$$

for $i=0,1$ and for almost all $t\geq 0$, which result from (26) by substituting $x_\xi(\xi,t)$ and $x_t(\xi,t)$ for $z(\xi,t)$, respectively. By (63) with $i=1$ it yields the relation $x_t^2(1,t)+\|x_{t\xi}\|_2^2\geq\frac{1}{2}\|x_t\|_2^2$. In light of the above, the next estimate can be made

$$-\Theta_m(W_2+\alpha_1)x_t^2(1,t)-\Theta_m\|x_{t\xi}\|_2^2\leq-\Theta_m\gamma_1\|x_t\|_2^2 \tag{64}$$

where $\gamma_1=\frac{1}{2}\min\{W_2+\alpha_1,1\}$. Relation (60) can further be manipulated to

$$\begin{aligned}
\dot{\tilde{V}}_R(t) &\leq -\Theta_m\left(\lambda_2 - M - \frac{\kappa_R\sqrt{2R}}{\Theta_m}\right)|x_t(1,t)| \\
&- \Theta_m\gamma_1\|x_t\|_2^2 - \frac{1}{2}\Theta_m\alpha_0\left(2-\kappa_R\alpha_0\right)x_t^2(0,t) \\
&- \kappa_R\Theta_m[(\lambda_1-\lambda_2)-M]|x(1,t)| \\
&- \kappa_R\left(W_1\Theta_m - \frac{1}{2}\Theta_M\right)x^2(1,t) \\
&\leq -\gamma_2(|x(1,t)| + x^2(1,t) + \|x_t\|_2^2) \qquad (65)
\end{aligned}$$

$$\gamma_2 = \Theta_m \min\{\kappa_R[(\lambda_1-\lambda_2)-M], \kappa_R\left(W_1 - \frac{1}{2}\frac{\Theta_M}{\Theta_m}\right), \gamma_1\}. \qquad (66)$$

On the other hand, (49) is readily estimated as

$$\tilde{V}_R(t) \geq \gamma_3(|x(1,t)| + x^2(1,t) + \|x_t\|_2^2) \qquad (67)$$

with positive, implicitly defined, constant parameter γ_3.

Relations (65) and (67), coupled together, result in

$$\dot{\tilde{V}}_R(t) \leq -\frac{\gamma_2}{\gamma_3}\tilde{V}_R(t) \qquad (68)$$

that establishes the exponential convergence of $\tilde{V}_R(t)$, initialized within (37), to zero as $t \to \infty$.

To complete the proof it remains to note that due to the upper estimate (51) of the functional $\tilde{V}(t)$ by the functional $\tilde{V}_R(t)$, it follows that $\tilde{V}(t)$, being computed on the mild solutions (x, x_t) of the boundary-value problem (23)-(24), converges asymptotically to zero, too, and by virtue of Lemma 2, the local asymptotic stability of (23)-(24) with the augmented state (x, x_t) in the $W^{2,2}(0,1) \times L_2(0,1)$-space is established with the initial set (37). Since the initial set (37) can be specified with an arbitrarily large $R > 0$ the **global** asymptotic stability in the $W^{2,2}(0,1)\times L_2(0,1)$-space is then concluded. Theorem 1 is thus proved. □

IV. Simulations

Consider the perturbed heat equation (4) with constant diffusivity $\theta = 1$. The parameters of the uncontrolled Robin's BC (6) are set as $\alpha_0 = 1$ and $\beta_0 = -5$.

The boundary value problem (8)-(10) specialized for a constant diffusivity has a solution $Q^r(\xi) = Q_0^r + \xi(Q_1^r - Q_0^r)$ which linearly depends on the spatial variable, where the reference boundary value $Q^r(1) = Q_1^r$ is arbitrarily selected as $Q_1^r = 15$ and the resulting value for Q_0^r is derived from the other parameters according to $Q_0^r = \frac{Q_1^r - \beta_0}{1+\alpha_0} = 10$, which is obtained by imposing the BC (6) on the solution $Q^r(\xi)$. Parameter β_1 is arbitrarily set to the value $\beta_1 = 1$. The disturbance $\psi(t)$ is selected as $\psi(t) = 4\cos(0.5\pi t)$. The magnitude of the disturbance time derivative ψ_t can be easily upper-estimated as $M = 6.5$, as required by (11). The initial conditions have been set to $Q^0(\xi) = 3 + 2\sin(4\pi\xi)$.

Controller (21) has been implemented with the parameters $\lambda_1 = 15$, $\lambda_2 = 7$, $W_1 = W_2 = 1$ which are selected in accordance with (22). Figure 1 shows the solution $Q(\xi,t)$. It can be seen that the solution converges to the chosen linear reference $Q^r(\xi)$ along the entire solution domain, as expected.

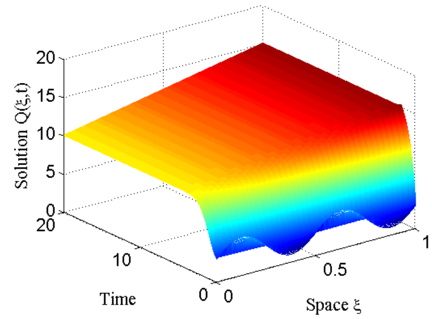

Fig. 1. The solution $Q(\xi, t)$.

V. Concluding Remarks

Using a dynamic version of a second-order sliding mode control algorithm, the problem of boundary global asymptotic stabilization of an uncertain heat process is solved in the presence of a persistent smooth disturbance by means of a continuous control. Finite-time convergence of the proposed algorithm, which would be the case if confined to a finite dimensional treatment, cannot be proved using the proposed Lyapunov functional, and it remains among other problems to be tackled in the future within the present framework.

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