2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Robust Parametric Identification of Sinusoidal Signals: an Input-to-State Stability Approach

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Abstract— **In this work, a robust method to estimate sinusoidal signals of unknown frequency, amplitude and phase is described. The stability properties of the devised estimation method under perturbed condition are studied by Input-to-State Stability (ISS) analysis. Compared to averaging approaches, the ISS-Lyapunov theory allows to study the stability for any value of adaptation parameters.**

Index Terms— **Frequency estimation, phase locking loops, adaptive systems, input-to-state stability.**

I. Introduction

Algorithms which are capable of extracting sinusoids from periodic signals and to estimate their parameters in real-time are a very active area of research in the fields of signal processing, power quality assessment (see [1], [2], [3], [4]), active noise cancellation (see [5] and the references therein), and sinusoidal disturbance rejection (i.e., vibration control), (see [6], [7], [8]).

Many techniques have been proposed in the literature to provide robust estimates of amplitude, frequency and phase of sinusoidal signals, among which the Kalman Filter, [9], and the Extended Kalman Filter (EKF) represent the most used tools for their ease of implementation and the popularity gained among signal processing practitioners (see [10]). The main disadvantage of using Kalman Filtering consists in that the models of both the process and the noise must be assumed for the design of the filter. Indeed, this technique is strongly related to the class of Internal Model (IM)-based methods, in which a lumped model of the signal generator is assumed and its parameters are adapted by using real-time measurements, (see [11] and [12]). Nonetheless, both the the EKF and the IM are known to be very sensitive with respect to their design parameters. In this regard, adaptive notch filtering represents a valid alternative when an accurate model of the process is not available (see [13], [14] and [15]). It consists of a of a very sharp notch whose base frequency adaptively tracks that of the input signal.

Recently, with the aim of providing simple algorithms suited for the digital implementation, orthogonal finite impulse response (FIR) discrete-time adaptive filters have been proposed to perform real-time frequency estimation (see e.g., [16] and [17]).

Also nonlinear methods have been recently proposed to obtain more robust estimates in presence of noisy signals and for Amplitude-Frequency-Phase (AFP) reconstruction of non-stationary sinusoids (see [18], [19], [20], [21] and the references therein). These techniques have been successfully used to monitor the quality of electrical power delivery and have been shown to provide accurate and reliable real-time frequency estimates [2]. On the other side, the stability results available for the existing nonlinear AFP algorithms can provide only local stability guarantees, or, when averaging analysis is used, global results are valid only for small adaptation gains (see [22]).

In this framework, we are going to present a new AFP method with semi-global stability guarantees for any value of the adaptation parameters. Moreover, the stability of the method against bounded perturbations (noise or limited-amplitude disturbnce signals) will be proven by ISS analysis. The ISS-Lyapunov dissipation inequalities can be also used to assess the transitory performance of the frequency-estimator.

II. Notation and Basic Definitions

Let $\mathbb{R}$, $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$ denote the real, the non-negative real and the strict positive real sets of numbers, respectively. Given a vector $\mathbf{x} \in \mathbb{R}^n$, we will denote as $|\mathbf{x}|$ the Euclidean norm of $\mathbf{x}$. Moreover, given a time-varying vector $\mathbf{x}(t) \in \mathbb{R}^n$, $t \in \mathbb{R}_{\geq 0}$ we will denote as $\|\mathbf{x}\|_\infty$ the quantity

$$\|\mathbf{x}\|_\infty = \sup_{t\geq 0} |\mathbf{x}(t)|.$$

Moreover, let L^n_∞ denote the set of piece-wise continuous signals $\mathbf{u}(t), \mathbf{u} : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ with finite $\|\mathbf{u}\|_\infty$ norm.

The notions of functions of class $\mathcal{K}$, class $\mathcal{K}_\infty$, and class $\mathcal{KL}$ are used to characterize stability properties. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to the class $\mathcal{K}$ if it is continuous, stricly increasing and $\alpha(0) = 0$. If, in addition $\lim_{s\to\infty} = \infty$ then it belongs to the class $\mathcal{K}_\infty$. A continuos function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to the class $\mathcal{KL}$ if, for any fixed $t \in \mathbb{R}_{\geq 0}$, the function $\beta(\cdot, t)$ is a $\mathcal{K}$-function with respect to the first argument and if, for any fixed $s \in \mathbb{R}_{\geq 0}$, the function $\beta(s, t)$ is monotonically decreasing with respect to t and $\lim_{t\to\infty} \beta(s, t) = 0$.

Consider now the following system

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}) \tag{1}$$

with $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^m$, $f(0, 0) = 0$ and $f(\mathbf{x}, \mathbf{u})$ locally Lipschitz in $\mathbb{R}^n \times \mathbb{R}^m$.

Definition 2.1 (ISS): The system (1) is ISS (Input to State Stable) if there exist a $\mathcal{KL}$-function $\beta(\cdot, \cdot)$ and a class $\mathcal{K}$-function such that, for any input $\mathbf{u} \in L^m_\infty$ and any initial

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condition $\mathbf{x}_0 \in \mathbb{R}^n$, the trajecotory of the system verifies

$$|\mathbf{x}(t)| \leq \beta(|\mathbf{x}_0|, t) + \gamma(\|\mathbf{u}\|_\infty) \tag{2}$$

□

Definition 2.2 (ISS-Lyapunov Function): A function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ of class $\mathcal{C}^1$ is an ISS-Lyapunov function for (1) if there exist three $\mathcal{K}_\infty$-functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$ and a $\mathcal{K}$-function $\mathcal{X}(\cdot)$ such that

$$\underline{\alpha}(|\mathbf{x}|) \leq V(\mathbf{x}) \leq \overline{\alpha}(|\mathbf{x}|), \;\; \forall \mathbf{x} \in \mathbb{R}^n \tag{3}$$

and

$$|\mathbf{x}| \geq \mathcal{X}(|\mathbf{u}|) \Rightarrow \frac{\partial V}{\partial \mathbf{x}} f(\mathbf{x}, u) \leq -\alpha(|\mathbf{x}|), \; \forall \mathbf{x} \in \mathbb{R}^n, \; \forall \mathbf{u} \in \mathbb{R}^m \tag{4}$$

□

Theorem 2.1 ([23]): The system (1) is ISS if and only if it admits an ISS-Lyapunov function. □

III. Problem Statement

Consider the task of detecting the frequency $\omega^* \in \mathbb{R}$, the phase $\vartheta_0 \in \mathbb{R}$ and the amplitude $A \in \mathbb{R}$ of the sinusoidal signal:

$$\begin{array}{l} y(t) = A\cos(\vartheta(t)), \;\; t \in \mathbb{R}_{\geq 0} \\ \left\{ \begin{array}{lcl} \dot{\vartheta} & = & \omega^*, \;\; t \in \mathbb{R}_{\geq 0} \\ \vartheta(0) & = & \vartheta_0, \end{array} \right. \end{array} \tag{5}$$

Let us introduce the auxiliary filtered signals $x_1(t)$, $x_2(t)$ and $x_3(t)$ obtained as:

$$\left\{ \begin{array}{l} \dot{x}_1(t) = \lambda\left(y(t) - x_1(t)\right), \;\; t \in \mathbb{R}_{\geq 0} \\ x_1(0) = x_{1_0}. \end{array} \right. \tag{6}$$

$$\left\{ \begin{array}{l} \dot{x}_2(t) = \lambda\left(x_1(t) - x_2(t)\right), \;\; t \in \mathbb{R}_{\geq 0} \\ x_2(0) = x_{2_0}. \end{array} \right. \tag{7}$$

and

$$\left\{ \begin{array}{l} \dot{x}_3(t) = \lambda\left(x_2(t) - x_3(t)\right), \;\; t \in \mathbb{R}_{\geq 0} \\ x_3(0) = x_{3_0}. \end{array} \right. \tag{8}$$

where $\lambda \in \mathbb{R}_{>0}$ is an arbitrary positive constants. In the sequel we will denote as $H_3(s)$ the transfer function in the Laplace domain such that

$$[x_3](s) = H_3(s)[y](s). \tag{9}$$

with

$$H_3(s) = \frac{\lambda^3}{(\lambda + s)^3}. \tag{10}$$

The time-derivatives of $x_3(t)$ are available up to the 3-rd order, that is:

$$\begin{array}{rcl} \ddot{x}_3(t) & = & \lambda\left(\dot{x}_2(t) - \dot{x}_3(t)\right) \\ & = & \lambda^2\left(x_3(t) - 2x_2(t) + x_1(t)\right) \\ \dddot{x}_3(t) & = & \lambda^2\left(\dot{x}_3(t) - 2\dot{x}_2(t) + \dot{x}_1(t)\right) \\ & = & \lambda^3(x_2(t) - x_3(t) - 2(x_1(t) - x_2(t)) + (y(t) - x_1(t))) \\ & = & \lambda^3\left(-x_3(t) + 3x_2(t) - 3x_1(t) + y(t)\right) \end{array} \tag{11}$$

Now, let us define the following time-dependent variables obtained as linear combinations of variables $x_1(t)$, $x_2(t)$, and $x_3(t)$:

$$\begin{array}{l} z_0(t) \triangleq x_3(t) \\ z_1(t) \triangleq -\dot{x}_3(t) = \lambda\left(x_3(t) - x_2(t)\right) \\ z_2(t) \triangleq -\ddot{x}_3(t) = -\lambda^2\left(x_3(t) - 2x_2(t) + x_1(t)\right) \\ z_3(t) \triangleq \dddot{x}_3(t) = \lambda^3\left(-x_3(t) + 3x_2(t) - 3x_1(t) + y(t)\right) \end{array} \tag{12}$$

Let us consider the sinusoidal equilibrium reached asymptotically for $t \to +\infty$ (or for a suitable specific choice of the initial states $x_1(0) = \overline{x}_{10}, x_2(0) = \overline{x}_{20}^*, \overline{x}_2(0) = \overline{x}_{20}^*, \overline{x}_3(0) = \overline{x}_{30}^*$), denoting with $\overline{x}_0(t), \overline{x}_1(t)$, $\overline{x}_2(t)$, $\overline{x}_3(t)$, the stationary filtered sinusoidal signals and $\overline{z}_0(t), \overline{z}_1(t)$, $\overline{z}_2(t)$, $\overline{z}_3(t)$ the correspondent stationary auxiliary signals, whose amplitude depends on the actual frequency ω^* of the measured variable $y(t)$:

$$\begin{array}{l} \overline{z}_0(t) = A_z \cos(\vartheta_z(t)) \\ \overline{z}_1(t) = A_z \omega^* \sin(\vartheta_z(t)) \\ \overline{z}_2(t) = A_z {\omega^*}^2 \cos(\vartheta_z(t)) \\ \overline{z}_3(t) = A_z {\omega^*}^3 \sin(\vartheta_z(t)) \end{array} \tag{13}$$

where

$$\begin{array}{rcl} A_z & = & A|H_3(j\omega^*)| \\ \vartheta_z(t) & = & \vartheta(t) + \angle H_3(j\omega^*) \end{array} \tag{14}$$

Now, it is convenient to solve the problem in the unknowns $A_z, \vartheta_z(t)$ and ω^* from the set of equations (13) and then to infer the original parameters by (14). At any time instant $t : \vartheta_z(t) \neq \pi/2 + k\pi$, the problem of detecting the squared frequency $\Omega^* = {\omega^*}^2$ can be solved in algebraic way as

$$\Omega^* = \overline{z}_2(t)(\overline{z}_0(t))^{-1}. \tag{15}$$

Moreover, for any time instant $t : \vartheta_z(t) \neq k\pi$, it holds that

$$\Omega^* = \overline{z}_3(t)(\overline{z}_1(t))^{-1}. \tag{16}$$

As the actual phase $\vartheta_z(t)$ is not known, it is not possible to choose a priori which of the two expression has to be used. A possible solution consists in minimizing a mixed objective:

$$\begin{array}{rcl} \Omega^* & = & \arg\min\limits_{\Omega \in \mathbb{R}_{>0}} \left(\Omega\overline{z}_0(t) - \overline{z}_2(t)\right)^2 + \left(\Omega\overline{z}_1(t) - \overline{z}_3(t)\right)^2 \\ & = & \dfrac{\overline{z}_0(t)\overline{z}_2(t) + \overline{z}_1(t)\overline{z}_3(t)}{(\overline{z}_0(t))^2 + (\overline{z}_1(t))^2} \end{array} \tag{17}$$

Note that the previous expression holds for any t at the sinusoidal equilibrium, due to the orthogonality of $\overline{z}_0(t)$ and $\overline{z}_1(t)$. Moreover, from the first two equation in (13), it follows that

$${\omega^*}^2(\overline{z}_0(t))^2 + (\overline{z}_1(t))^2 \;\; = \;\; A_z^2 {\omega^*}^2. \tag{18}$$

Then, the parameter A_z can be computed by

$$A_z = \sqrt{\frac{\Omega^*(\overline{z}_0(t))^2 + (\overline{z}_1(t))^2}{\Omega^*}} \tag{19}$$

and, finally, the phase of the signal under concern can be determined as

$$\vartheta_z(t) = \operatorname{atan}\left(\frac{\overline{z}_1(t)}{A_z\omega^*} \Big/ \frac{\overline{z}_0(t)}{A_z}\right). \tag{20}$$

Now, considering that $H_3(s) = (H_1(s))^3$, with

$$H_1(s) \triangleq \frac{\lambda}{\lambda + s}\,, \tag{21}$$

it follows that

$$\begin{aligned} |H_3(j\omega^*)| &= \frac{\lambda^3}{\left(\lambda^2+\omega^{*2}\right)^{3/2}}\,, \\ \angle H_3(j\omega^*) &= 3\,\mathrm{atan}\left(\frac{-\omega*}{\lambda}\right). \end{aligned} \tag{22}$$

Finally, in view of (14) and (22), the original parameters can be retrieved as:

$$\begin{aligned} A &= A_z\left(\frac{\sqrt{\lambda^2+\omega^{*2}}}{\lambda}\right)^3, \\ \vartheta(t) &= \vartheta_z(t) - 3\,\mathrm{atan}\left(\frac{-\omega^*}{\lambda}\right), \\ &= \vartheta_z(t) + 3\,\mathrm{atan}\left(\frac{\omega^*}{\lambda}\right). \end{aligned} \tag{23}$$

Summing up, the problem of estimating three unknown parameters has been solved by introducing three auxiliary signals and by solving, at each time instant, a scalar algebraic equation in the unknown ω^*. At this point, we have solved the problem assuming that the auxiliary signals are considered during the sinusoidal equilibrium, but during the transient modes of behavior or in the likely situation where external disturbances affect the measured signal $y(t)$, the stationary signals $\overline{z}_0(t), \overline{z}_1(t), \overline{z}_2(t), \overline{z}_3(t)$ are not available. Then, we seek an estimation method depending only on measurable quantities. Note that the instantaneous values of the signals $z_0(t), z_1(t), z_2(t), z_3(t)$, evolving from arbitrary initial conditions, cannot be used in (17) in place of stationary ones, because the denominator may eventually assume a 0-value, thus a singularity-free estimation method is needed. Instead of directly computing a minimizer for (17), it is preferable to set up a dynamic optimization scheme with guaranteed asymptotic convergence properties.

In the next section, by using Lyapunov arguments, we will propose an adaptation law capable to ensure the semi-global input-to-state stability (i.e. for any compact set of initial conditions) of the estimtor dynamics in nominal and in perturbed (noisy) conditions.

IV. Input-to-State Stability of the Frequency Estimation System

Given the perturbed sinusoidal signal

$$\begin{aligned} &\hat{y}(t) = A\cos(\vartheta(t)) + d(t), \\ &\begin{cases} \dot{\vartheta}(t) = \omega^*, \;\; t \in \mathbb{R}_{\geq 0} \\ \vartheta(0) = \vartheta_0, \end{cases} \end{aligned} \tag{24}$$

where $d(t) \in L^1_\infty$ is an unmeasureable additive disturbance with $\|d\|_\infty \leq \overline{d}$, $\overline{d} \in \mathbb{R}_{\geq 0}$ finite, let us denote as $\hat{\mathbf{x}}(t) = [\hat{x}_1(t)\ \hat{x}_2(t)\ \hat{x}_3(t)]^\top$ the filter's states evolving from an arbitrary initial condition $\hat{\mathbf{x}}_0 = [\hat{x}_{1_0}\ \hat{x}_{2_0}\ \hat{x}_{3_0}]^\top$ according to:

$$\begin{cases} \dot{\hat{x}}_1(t) = \lambda\left(\hat{y}(t) - \hat{x}_1(t)\right), \;\; t \in \mathbb{R}_{\geq 0} \\ \hat{x}_1(0) = \hat{x}_{1_0}. \end{cases} \tag{25}$$

$$\begin{cases} \dot{\hat{x}}_2(t) = \lambda\left(\hat{x}_1(t) - \hat{x}_2(t)\right), \;\; t \in \mathbb{R}_{\geq 0} \\ \hat{x}_2(0) = \hat{x}_{2_0}. \end{cases} \tag{26}$$

$$\begin{cases} \dot{\hat{x}}_3(t) = \lambda\left(\hat{x}_2(t) - \hat{x}_3(t)\right), \;\; t \in \mathbb{R}_{\geq 0} \\ \hat{x}_3(0) = \hat{x}_{3_0}. \end{cases} \tag{27}$$

and let $\hat{\mathbf{z}} = [\hat{z}_0(t)\ \hat{z}_1(t)\ \hat{z}_2(t)\ \hat{z}_3(t)]^\top$ be the vector of the corresponding auxiliary signals obtained by (see (13) for the nominal case)

$$\begin{aligned} &\hat{z}_0(t) = \hat{x}_3(t) \\ &\hat{z}_1(t) = \lambda\left(\hat{x}_3(t) - \hat{x}_2(t)\right)) \\ &\hat{z}_2(t) = -\lambda^2\left(\hat{x}_3(t) - 2\hat{x}_2(t) + \hat{x}_1(t)\right) \\ &\hat{z}_3(t) = \lambda^3\left(-\hat{x}_3(t) + 3\hat{x}_2(t) - 3\hat{x}_1(t) + \hat{y}(t)\right) \end{aligned} \tag{28}$$

We propose the following frequency adaptation law, using perturbed signals:

$$\begin{aligned} \dot{\Omega}(t) = -\mu\Big[&\left(\hat{z}_0(t)\hat{z}_2(t) + \hat{z}_1(t)\hat{z}_3(t)\right)\left(\Omega(t)\hat{z}_0(t) - \hat{z}_2(t)\right)\hat{z}_0(t) \\ &+ \left((\hat{z}_0(t))^2 + (\hat{z}_1(t))^2\right)\left(\Omega(t)\hat{z}_1(t) - \hat{z}_3(t)\right)\hat{z}_1(t)\Big] \end{aligned} \tag{29}$$

with $\mu \in \mathbb{R}_{>0}$ arbitrary.

To analyze the stability properties of the frequency estimation system (25), (26), (27), (28) and (29), let us consider the dynamics of the error vector $\tilde{\mathbf{x}}(t) \triangleq [\hat{x}_1(t) - \overline{x}_1(t)\ \ \hat{x}_2(t) - \overline{x}_2(t)\ \ \hat{x}_3(t) - \overline{x}_3(t)]^\top$, that can be expressed in state-space form as follows:

$$\begin{cases} \dot{\tilde{\mathbf{x}}}(t) = \mathbf{A}\tilde{\mathbf{x}}(t) + \mathbf{b}\,d(t), \;\; t \in \mathbb{R}_{\geq 0} \\ \tilde{\mathbf{x}}(0) = \hat{\mathbf{x}}_0 - \overline{\mathbf{x}}_0 \end{cases} \tag{30}$$

with

$$\mathbf{A} = \begin{bmatrix} -\lambda & 0 & 0 \\ \lambda & -\lambda & 0 \\ 0 & \lambda & -\lambda \end{bmatrix}.$$

and $\mathbf{b} = [\lambda\ 0\ 0]^\top$. $\mathbf{A}$ being Hurwitz, there exists a positive definite matrix $\mathbf{P}:\ \mathbf{P}\mathbf{A} + \mathbf{A}^\top\mathbf{P} = -\mathbf{I}$. Let $W(\tilde{\mathbf{x}}) = \tilde{\mathbf{x}}^\top\mathbf{P}\tilde{\mathbf{x}}$, then there exist two positive scalars $a_1, a_2 \in \mathbb{R}_{>0}$ such that

$$a_1|\tilde{\mathbf{x}}|^2 \leq W(\tilde{\mathbf{x}}) \leq a_2|\tilde{\mathbf{x}}|^2\,, \quad \forall\tilde{\mathbf{x}}\,.$$

The derivative of W along the system's state trajectory satisfies the inequality

$$\frac{\partial W}{\partial\tilde{\mathbf{x}}}\left(\mathbf{A}\tilde{\mathbf{x}} + \mathbf{b}d\right) \quad\leq\quad -|\tilde{\mathbf{x}}|^2 + 2\,\|\mathbf{P}\|\,|\mathbf{b}|\,|d|\,|\tilde{\mathbf{x}}|.$$

For any $0 < \epsilon < 1$, let

$$\mathcal{X}(s) = \frac{2\,\|\mathbf{P}\|\,|\mathbf{b}|}{1-\epsilon}s$$

with $s \in \mathbb{R}_{\geq 0}$. It is easy to show that

$$|\tilde{\mathbf{x}}| \geq \mathcal{X}(|d|) \;\;\Rightarrow\;\; \frac{\partial W}{\partial\tilde{\mathbf{x}}}\left(\mathbf{A}\tilde{\mathbf{x}} + \mathbf{b}d\right) \leq -|\tilde{\mathbf{x}}|^2,$$

and that the system is ISS with asymptotic gain

$$\gamma_x(s) = a_1^{-1}a_2\,\mathcal{X}(s).$$

In view of the ISS property of the linear auxiliary filter (30), the error vector $\tilde{\mathbf{x}}(t)$ will enter in a closed ball of radius $\gamma_x(\|d\|_\infty) + \nu \leq \gamma_x(\overline{d}) + \nu$ in finite time T_ν, for any $\nu \in$

$\mathbb{R}_{>0}$. As a consequence, the auxiliary error vector $\tilde{\mathbf{z}}(t) = [\hat{z}_0(t) - \overline{z}_0(t), \tilde{z}_1(t) - \overline{z}_1(t), \tilde{z}_2(t) - \overline{z}_2(t), \tilde{z}_3(t) - \overline{z}_3(t)]^\top$ will enter in finite-time $T_\delta = T_\nu$ in a closed ball of radius $\gamma_z(\overline{d}) + \delta$ centered at the origin, with

$$\delta = \sum_{i=0}^{3} \{2^i \lambda^i\} \nu, \quad \gamma_z(s) = \sum_{i=0}^{3} \{2^i \lambda^i\} \gamma_x(s), \forall s \in \mathbb{R}_{\geq 0}. \quad (31)$$

Let us now write

$$\begin{aligned}
\dot{\Omega}(t) &= -\mu\Big\{ \Big[(\overline{z}_0(t)+\tilde{z}_0(t))(\overline{z}_2(t)+\tilde{z}_2(t))+(\overline{z}_1(t)+\tilde{z}_1(t)) \\
&\times (\overline{z}_3(t) + \tilde{z}_3(t))\Big] \Big[\Omega(t)(\overline{z}_0(t) + \tilde{z}_0(t)) - (\overline{z}_2(t) + \tilde{z}_2(t))\Big] \\
&\times (\overline{z}_0(t) + \tilde{z}_0(t)) + \Big[(\overline{z}_0(t) + \tilde{z}_0(t))^2 + (\overline{z}_1(t) + \tilde{z}_1(t))^2\Big] \\
&\times \Big[\Omega(t)(\overline{z}_1(t) + \tilde{z}_1(t)) - (\overline{z}_3(t) + \tilde{z}_3(t))\Big](\overline{z}_1(t) + \tilde{z}_1(t))\Big\} \\
&= -\mu\Big\{[\overline{z}_0(t)\overline{z}_2(t) + \overline{z}_1(t)\overline{z}_3(t)][\Omega(t)\overline{z}_0(t) - \overline{z}_2(t)]\overline{z}_0(t) \\
&\quad + [(\overline{z}_0(t))^2 + (\overline{z}_1(t))^2][\Omega(t)\overline{z}_1(t) - \overline{z}_3(t)]\overline{z}_1(t)\Big\} \\
&\quad + \mu\tilde{f}_z(t, \tilde{\mathbf{z}}) + \mu\tilde{f}_\Omega(t, \tilde{\mathbf{z}})\Omega(t) \quad (32)
\end{aligned}$$

where

$$\begin{aligned}
\tilde{f}_z(t, \tilde{\mathbf{z}}) &\triangleq -\Big[(\overline{z}_0(t) + \tilde{z}_0(t))(\overline{z}_2(t) + \tilde{z}_2(t)) \\
&+ (\overline{z}_1(t)+ \tilde{z}_1(t))(\overline{z}_3(t) + \tilde{z}_3(t))\Big] [-(\overline{z}_2(t) + \tilde{z}_2(t))]\, \tilde{z}_0(t) \\
&+\Big[(\overline{z}_0(t)+\tilde{z}_0(t))^2+(\overline{z}_1(t)+\tilde{z}_1(t))^2\Big] [-(\overline{z}_3(t) + \tilde{z}_3(t))]\, \tilde{z}_1(t) \\
&+ \Big[\tilde{z}_0(t)\overline{z}_2(t) + \tilde{z}_0(t)\tilde{z}_2(t) + \tilde{z}_2(t)z_0(t) + \tilde{z}_1(t)\overline{z}_3(t) \\
&+ \tilde{z}_1(t)\tilde{z}_3(t) + \tilde{z}_3(t)z_1(t)\Big] \Big[- (\overline{z}_2(t) + \tilde{z}_2(t))\Big] z_0(t) \quad (33)
\end{aligned}$$

and

$$\begin{aligned}
\tilde{f}_\Omega(t, \tilde{\mathbf{z}}) &\triangleq -\Big[(\overline{z}_0(t) + \tilde{z}_0(t))(\overline{z}_2(t) + \tilde{z}_2(t)) \\
&+ (\overline{z}_1(t) + \tilde{z}_1(t))(\overline{z}_3(t) + \tilde{z}_3(t))\Big] (\overline{z}_0(t) + \tilde{z}_0(t))\, \tilde{z}_0(t) \\
&+ \Big[(\overline{z}_0(t) + \tilde{z}_0(t))^2 + (\overline{z}_1(t) + \tilde{z}_1(t))^2\Big] (\overline{z}_1(t) + \tilde{z}_1(t))\, \tilde{z}_1(t) \\
&+ \Big[\tilde{z}_0(t)\overline{z}_2(t) + \tilde{z}_0(t)\tilde{z}_2(t) + \tilde{z}_2(t)z_0(t) + \tilde{z}_1(t)\overline{z}_3(t) \\
&+ \tilde{z}_1(t)\tilde{z}_3(t) + \tilde{z}_3(t)z_1(t)\Big] (\overline{z}_0(t) + \tilde{z}_0(t))\, z_0(t) \\
&+ \Big[(2\overline{z}_0(t) + \tilde{z}_0(t))\tilde{z}_0(t) + (2\overline{z}_1(t) + \tilde{z}_1(t))\tilde{z}_1(t)\Big] \\
&\times (\overline{z}_1(t) + \tilde{z}_1(t))\, z_1(t) \quad (34)
\end{aligned}$$

The adaptation law (29), rewritten in terms of nominal auxiliary z signals and relative errors, is described by (32). Note that, the functions $\tilde{f}_z(t, \tilde{\mathbf{z}})$ and $\tilde{f}_\Omega(t, \tilde{\mathbf{z}})$ introduced in (32) and defined in (33) and (34), verify

$$\tilde{f}_z(t, 0) = 0, \quad \tilde{f}_\Omega(t, 0) = 0;$$

for all $t \in \mathbb{R}_{\geq 0}$. Moreover, being $y(t)$ bounded and $\overline{z}_0(t), \ldots, \overline{z}_3(t)$ bounded, there exist two $\mathcal{K}_\infty$-functions $\sigma_z(\cdot)$ and $\sigma_\Omega(\cdot)$ such that:

$$\begin{aligned}
|\tilde{f}_z(t, \tilde{\mathbf{z}}(t))| &\leq \sigma_z(|\tilde{\mathbf{z}}(t)|), \\
|\tilde{f}_\Omega(t, \tilde{\mathbf{z}}(t))| &\leq \sigma_\Omega(|\tilde{\mathbf{z}}(t)|),
\end{aligned} \quad (35)$$

In order to characterize the stability properties of the estimator, let us consider the following candidate cost-function in Ω:

$$\begin{aligned}
J(\Omega) &\triangleq \frac{\overline{z}_0(t)\overline{z}_2(t) + \overline{z}_1(t)\overline{z}_3(t)}{(\overline{z}_0(t))^2 + (\overline{z}_1(t))^2} (\Omega\overline{z}_0(t) - \overline{z}_2(t))^2 \\
&\quad + (\Omega\overline{z}_1(t) - \overline{z}_3(t))^2 \\
&= \Omega^*[\Omega^2(\overline{z}_0(t))^2 + (\overline{z}_2(t))^2 - 2\Omega\overline{z}_0(t)\overline{z}_2(t)] \\
&\quad +\Omega^2(\overline{z}_1(t))^2 + (\overline{z}_3(t))^2 - 2\Omega(\overline{z}_1(t))(\overline{z}_3(t)) \\
&= \Omega^* A_z^2 \Big[\Omega^2 + \Omega^{*2} - 2\Omega\Omega^*\Big] (\cos(\vartheta(t)))^2 \\
&\quad +\Omega^* A_z^2 \Big[\Omega^2 + \Omega^{*2} - 2\Omega\Omega^*\Big] (\sin(\vartheta(t)))^2 \\
&= \Omega^* A_z^2 (\Omega - \Omega^*)^2 (\cos(\vartheta(t)))^2 \\
&\quad +\Omega^* A_z^2 (\Omega - \Omega^*)^2 (\sin(\vartheta(t)))^2 \\
&= \Omega^* A_z^2 (\Omega - \Omega^*)^2,
\end{aligned} \quad (36)$$

then, J is a *time-invariant* function of Ω, derived from the stationary sinusoidal $\overline{z}$ signals. Introducing the error variable $\tilde{\Omega} = \Omega - \Omega^*$, let us denote with $V(\tilde{\Omega}) \triangleq J(\Omega)$ a candidate ISS-Lyapunov function for the estimator's dynamics.

The derivative of the candidate ISS-Lyapunov function V along the system's trajectory verifies the following inequalities:

$$\begin{aligned}
\frac{\partial V}{\partial \tilde{\Omega}} \dot{\Omega} &= \Big[2\Omega^*\Big(\Omega(t)\overline{z}_0(t) - \overline{z}_2(t)\Big)\overline{z}_0(t) \\
&\quad +2\Big(\Omega(t)\overline{z}_1(t) - \overline{z}_3(t)\Big)\overline{z}_1(t)\Big]\dot{\Omega}(t) \\
&= -2\mu\left((\overline{z}_0(t))^2+(\overline{z}_1(t))^2\right) \times \\
&\Big[\Omega^*(\Omega(t)\overline{z}_0(t)-\overline{z}_2(t))\overline{z}_0(t)+(\Omega(t)\overline{z}_1(t)-\overline{z}_3(t))\,\overline{z}_1(t)\Big]^2 \\
&\quad + 2\mu\left(\tilde{f}_z(t, \tilde{\mathbf{z}})+\tilde{f}_\Omega(t, \tilde{\mathbf{z}})\Omega(t)\right)\Big((\overline{z}_0(t))^2+(\overline{z}_1(t))^2\Big)\times \\
&\Big[\Omega^*(\Omega(t)\overline{z}_0(t)-\overline{z}_2(t))\overline{z}_0(t)+(\Omega(t)\overline{z}_1(t)-\overline{z}_3(t))\,\overline{z}_1(t)\Big] \\
&\leq -2\mu A_z^4 \min\{1, \Omega^*\} \Omega^{*2}\, (\Omega(t)-\Omega^*)^2 + \\
&\quad 2\mu A_z^3\Omega^* |\Omega(t)-\Omega^*| \Big(\sigma_z(|\tilde{\mathbf{z}}(t)|) + \sigma_\Omega(|\tilde{\mathbf{z}}(t)|)\Omega(t)\Big) \\
&\leq -\mu\Big(\alpha^* - \sigma_2(|\tilde{\mathbf{z}}(t)|)\Big)|\tilde{\Omega}(t)|^2 + \mu\sigma_1(|\tilde{\mathbf{z}}(t)|)|\tilde{\Omega}(t)|
\end{aligned} \quad (37)$$

where

$$\begin{aligned}
&\alpha^* \triangleq 2A_z^4 \min\{1, \Omega^*\} \Omega^{*2} \\
&\sigma_1(|\tilde{\mathbf{z}}(t)|) \triangleq 2A_z^3\Omega^* \Big(\sigma_z(|\tilde{\mathbf{z}}(t)|)+\Omega^*\sigma_\Omega(|\tilde{\mathbf{z}}(t)|)\Big) \\
&\sigma_2(|\tilde{\mathbf{z}}(t)|) \triangleq 2A_z^3\Omega^*\, \sigma_\Omega(|\tilde{\mathbf{z}}(t)|)|\tilde{\Omega}(t)|^2
\end{aligned} \quad (38)$$

Now, the following result characterizes the ISS stability properties of the frequency estimator in presence of bounded disturbances.

Theorem 4.1 (ISS of frequency estimation system): Given a signal $y(t)$ generated by (24), with nominal fundamental frequency ω^*, the frequency estimation system given by (25), (26), (27), (28) and (29) is ISS with respect to any additive disturbance signal $d(t) \in L^1_\infty$ such that

$$\|d\|_\infty < \overline{d} < \gamma_z^{-1}\left(\sigma_2^{-1}(\alpha^*)\right) \quad (39)$$

where α^* is defined as in (38), $\Omega^* = \omega^{*2}$, with A_z given by (14), σ_2 by (38) and γ_z by (31). $\square$

Proof: Due to the the ISS property of the auxiliary filter (see (31)), for any positive $\delta \in \mathbb{R}_{>0}$ there exists a finite time-instant T_δ such that $|\tilde{\mathbf{z}}(t)| \leq \gamma_z(\overline{d}) + \delta, \forall t \geq T_\delta$, which implies

$$\sigma_2(|\tilde{\mathbf{z}}(t)|) \leq \sigma_2(\gamma_z(\overline{d}) + \delta), \ \ \forall t \geq T_\delta. \tag{40}$$

If the bound on disturbances $\overline{d}$ verifies

$$\alpha^* - \sigma_2(\gamma_z(\overline{d}) + \delta) > 0, \tag{41}$$

for some $\delta \in \mathbb{R}_{>0}$, then, for any $t > T_\delta$, the following bound on the derivative of V can be established

$$\begin{aligned}\frac{\partial V}{\partial \tilde{\Omega}}\dot{\Omega}(t) &\leq -\mu\Big(\alpha^* - \sigma_2(\gamma_z(\overline{d})+\delta)\Big)|\tilde{\Omega}(t)|^2 + \mu\sigma_1(|\tilde{\mathbf{z}}(t)|)|\tilde{\Omega}(t)| \\ &\leq -c\,|\tilde{\Omega}(t)|^2 + \mu\sigma_1(|\tilde{\mathbf{z}}(t)|)|\tilde{\Omega}(t)|, \ \ \forall t \geq T_\delta\end{aligned} \tag{42}$$

where $c \triangleq \mu\left[\alpha^* - \sigma_2(\gamma_z(\overline{d})+\delta)\right]$ is a positive constant. Finally, for any $0 < \epsilon < 1$, let

$$\mathcal{X}_\Omega(s) = \frac{1}{c(1-\epsilon)}\mu\sigma_1(s).$$

It is easy to prove that

$$|\tilde{\Omega}(t)| \geq \mathcal{X}_\Omega(|\tilde{\mathbf{z}}(t)|) \ \Rightarrow \ \frac{\partial V}{\partial \tilde{\Omega}}\dot{\Omega}(t) \leq -c|\tilde{\Omega}(t)|^2, \ \ \forall t \geq T_\delta. \tag{43}$$

Considering that, for any finite initial condition Ω_0, the derivative $\dot{\Omega}(t)$ is bounded in the interval $[0, T_\delta]$, then $\Omega(T_\delta)$ is finite and $\tilde{\Omega}(T_\delta)$ is, in turn, finite. Hence, thanks to (42) and (43), for any disturbance signal $d(t)$ bounded by (39), V is an ISS-Lyapunov function for the frequency estimator dynamics with respect to the $\tilde{\mathbf{z}}(t)$ input. Being the dynamics of $\tilde{\mathbf{z}}$ ISS with respect to the disturbance $d(t)$, is follows that the frequency estimation system is in turn ISS with respect to $d(t)$, that is, there exist a $\mathcal{KL}$-function $\beta(\cdot,\cdot)$ and a $\mathcal{K}$-function $\gamma_\Omega(\cdot)$ such that $|\tilde{\Omega}(t)| \leq \beta(\tilde{\Omega}(T_\delta), t - T_\delta) + \gamma_\Omega(\|d\|_\infty)$. In particular, the asymptotic ISS gain is given by:

$$\gamma_\Omega(s) = \mathcal{X}_\Omega\left(\gamma_z(s)\right), \ \ s \in [0, \overline{d}).$$

A graphical representation of the ISS transient and asymptotic bounding functions involved in the present proof is depicted in Figure 1. $\blacksquare$

V. Digital implementation and Example

A simulated experiment of frequency detection and tracking has been carried out by discretizing in time the filter's dynamics and the frequency adaptation law (Euler's implicit integration step), with sample time $T_s = 5^{-3}$ s. The simulated signal takes the form of (24), with $A = 1$ and $\omega^* = 5$ rad/s, $t \in [0, 10]$; $\omega^* = 15$ rad/s, $t > 10$, while $d(t)$ is a L^1_∞ random noise with uniform distribution in the interval $[-0.25, 0.25]$. As shown in Figures 2 and 3, the sudden frequency changed has been traked even in presence of noise. The effect of the filter's parameter λ on the noise sensitivity and on the convergence speed of the frequency estimator has been shown in Figure 3.

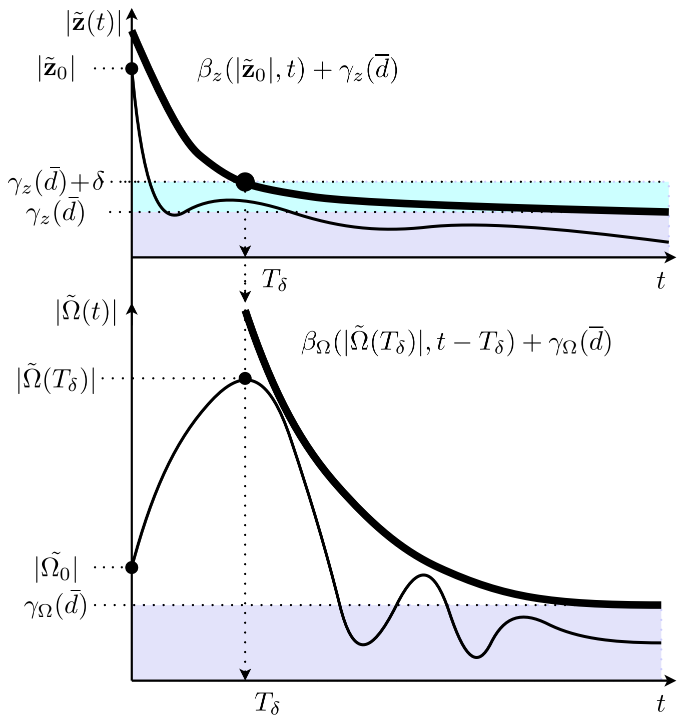

Fig. 1. Bounds on the norms of the auxiliary error vector $|\tilde{\mathbf{z}}|$ and of the frequency estimation error $|\tilde{\Omega}|$ established by ISS analysis. The bound on the norm of the frequency estimation error decreases starting from time T_δ, being $|\tilde{\mathbf{z}}(t)| \leq \gamma_z(\overline{d}) + \delta \leq \alpha^*, \ \forall t \geq T_\delta$ (see (40), (41) and (42)).

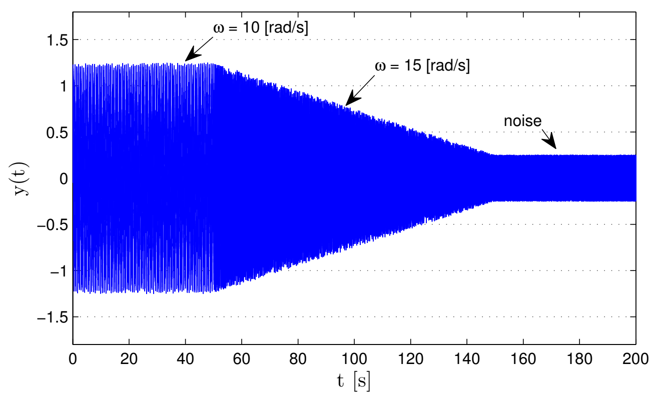

Fig. 2. Noisy input signal, with step-wise frequency change at time $t = 50$s and linear amplitude decrease.

It emerges clear from Figure 4 that, for constant $\lambda = 5$, the convergence speed of the estimator depends on the amplitude of the input sinusoid, as predicted by the Lyapunov stability analysis. Conversely, the sensitivity to the noise (which has not been scaled), does not change significatively.

Notably, when the filter is fed only by noise (starting from $t = 150$ s), the frequency estimate starts to diverge. Indeed, inequality (41) does not hold in this case, and the boundedness of the estimate cannot be guaranteed.

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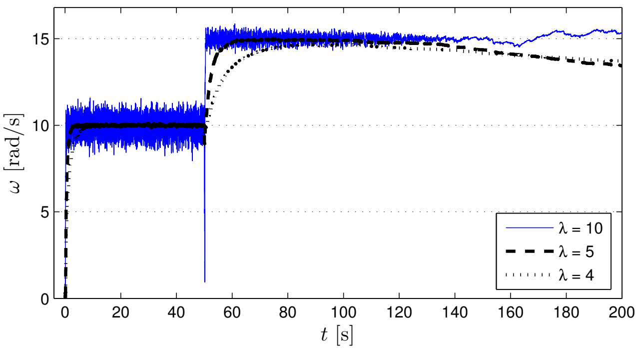

Fig. 3. Estimated frequency trend $\omega(t)$ during the experiment for different values of the λ parameter, with $A = 1$ constant. The time response of the estimator is faster for higher values of λ, at the cost of an increased sensitivity to the noise.

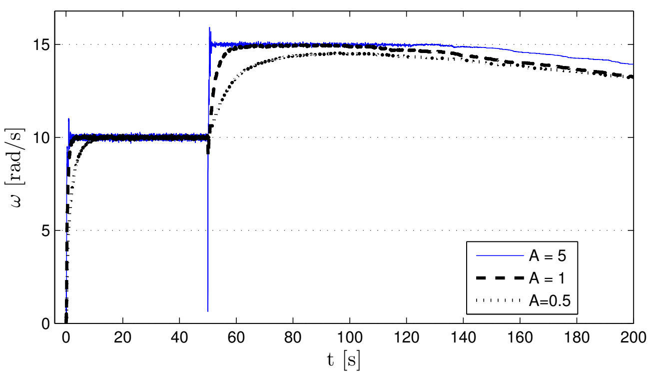

Fig. 4. Estimated frequency trend $\omega(t)$ during the experiment for different amplitudes of the incoming signal, for $\lambda = 5$ constant. According to the Lyapunov stability analysis, the convergence speed of the estimator is faster for higher signal amplitudes.

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