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Structural Controllability of Multi-Agent Systems subject to Simultaneous Failure of Links and Agents

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***Abstract*—In this paper, structural controllability of a leader-follower multi-agent system with multiple leaders is studied from a graph theoretic point of view. The preservation of structural controllability under simultaneous failure in both the communication links and the agents is investigated. The effects of the loss of agents or communication links on the controllability of an information flow graph have been the subject of two previous studies. This work expands the corresponding results by considering the effects of losses in both links and agents at the same time. To this end, the concepts of joint $(r, s)-$controllability and joint $t-$controllability are introduced as quantitative measures of reliability for a multi-agent system. A method is subsequently presented for investigating the problem of joint $t-$controllability in a directed information flow graph, using polynomial-time algorithms. The proposed methods are applied to a number of well-known directed graphs to clarify the results.**

I. Introduction

The past decade has seen a growing interest in the control of multi-agent networks. This type of system consists of a group of dynamic agents which interact according to a given information flow network. Distributed and cooperative control of these networked dynamic systems has found applications in emerging areas such as formation control of satellite clusters and motion coordination of robots [1], [2]. An important class of multi-agent systems which is commonly used in the literature is the one with leader-follower architecture. Various problems related to the control of leader-follower multi-agent systems include connectivity, containment, consensus, and flocking [3], [4].

Controllability of a single-leader multi-agent system is studied in [5], where a necessary and sufficient condition for controllability is derived using the Laplacian matrix of the interconnection graph. Controllability under switching topologies is investigated in [6], and it is shown that switching between fixed uncontrollable topologies can lead to a controllable system. The notion of equitable partitions is exploited in [7] and [8] to introduce a necessary condition for controllability of a leader-follower multi-agent system and provide a graph-theoretical interpretation of the controllability subspace. The papers [9] and [10] approach the problem using the notion of the controllability of a structured system, and derive graphical conditions for the controllability of the corresponding information flow digraph. The preservation of controllability in the face of failure in communication links and agents is then investigated, where the concepts of link and agent controllability degrees are introduced. While existing results on the controllability of multi-agent systems provide an important measure of reliability of network to faults, they cannot handle the important problem of simultaneous failure of communication links and agents.

The chief aim of this paper is to expand on the results of [9] and [10] by considering the case when both communication links and agents in the network are prone to faults. It is known that in real-world multi-agent systems, some faults can affect part of the network, containing a number of links and agents. This type of failure in multi-agent systems, where terrain properties or hardware faults disable a number of agents and limit the ability of others to communicate, motivates the study of controllability under simultaneous failure of links and agents.

The remainder of this paper is organized as follows. Section II gives some preliminaries on sets and graph theory, and also reviews some results from [9] and [10]. The tools and concepts introduced in this section are then used in Section III to investigate the notion of joint controllability in a directed information flow graph. In Section IV, the results are illustrated and discussed using sample graphs, and finally the paper is concluded in Section V.

II. Preliminaries and Notation

Throughout the paper, $\mathbb{N}$ denotes the set of all natural numbers, and $\mathbb{N}_k$ the set of integers $\{1, 2, \ldots, k\}$. Furthermore, $\mathbb{W} = \mathbb{N} \cup \{0\}$, $\mathbb{R}$ denotes the set of all real numbers, and any other set is represented by a curved capital letter. The cardinality of a set $\mathscr{X}$ (which is the number of its elements) is denoted by $|\mathscr{X}|$. The difference of two sets $\mathscr{X}$ and $\mathscr{Y}$ is denoted by $\mathscr{X} \backslash \mathscr{Y}$, and is defined as $\{x | x \in \mathscr{X} \wedge x \notin \mathscr{Y}\}$.

A. Directed Information Flow Graph of a Multi-Agent System and its Controllability

A directed graph or digraph is defined as an ordered pair of two sets $(\mathscr{V}, \mathscr{E})$, where $\mathscr{V} = \{\nu_1, \ldots, \nu_n\}$ is the set of vertices and $\mathscr{E} \subseteq \mathscr{V} \times \mathscr{V}$ is the set of directed edges. In the graphical representation, each edge $\epsilon := (\tau, \nu) \in \mathscr{E}$ is denoted by a directed arc from the vertex $\tau \in \mathscr{V}$ to vertex $\nu \in \mathscr{V}$. Vertices ν and τ are referred to as the head and tail of the edge ϵ, respectively. Notice that the definition of $\mathscr{E}$ does not allow for the existence of parallel arcs in the graphical representation of digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, i.e., two edges that

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share the same pair of head and tail are identical. Given a set of vertices $\mathscr{X} \subset \mathscr{V}$, the set of all edges for which the tails belong to $\mathscr{X}$ but the heads do not, is termed the outcut of $\mathscr{X}$, and is denoted by $\partial^+_{\mathscr{G}}\mathscr{X} \subset \mathscr{E}$. The cardinality of $\partial^+_{\mathscr{G}}\mathscr{X}$ is called the out-degree of $\mathscr{X}$, and is denoted by $d^+_{\mathscr{G}}\mathscr{X} = |\partial^+_{\mathscr{G}}\mathscr{X}|$. For $k \in \mathbb{N}_{n-2}$, a sequence of distinct edges in the form $(\tau, \nu_1), (\nu_1, \nu_2), \dots, (\nu_{k-1}, \nu_k)(\nu_k, \nu)$ is called a $\tau\nu$ path if no two edges share a common head or tail. For $\mathscr{R} \subset \mathscr{V}$ a $\tau\nu$ path is called $\mathscr{R}-$rooted if $\tau \in \mathscr{R}$. The set $\mathscr{R}$ associated with an $\mathscr{R}-$rooted $\tau\nu$ path is referred to as the root-set, and a vertex $\nu \in \mathscr{V}\backslash\mathscr{R}$ is called reachable from the root-set $\mathscr{R}$ if there exists an $\mathscr{R}-$rooted $\tau\nu$ path, for some $\tau \in \mathscr{R}$. Two distinct $\tau\nu$ paths are called edge-disjoint if they do not share any edges. Two edge-disjoint $\tau\nu$ paths are called disjoint if τ and ν are the only vertices that are common to both of them.

Consider a team of n single integrator agents given by:

$$\dot{x}_i(t) = u_i(t),\ i \in \mathbb{N}_n, \tag{1}$$

where the first $n - m$ agents are followers, and the last m agents are leaders, with the following control inputs

$$u_i(t) = \begin{cases} u^i_{ext}(t), & i \in \mathbb{N}_n\backslash\mathbb{N}_{n-m} \quad \text{(2a)} \\ \displaystyle\sum_{j \in \mathbb{N}_i} \alpha_{ij} x_j(t), & i \in \mathbb{N}_{n-m} \quad \text{(2b)} \end{cases}$$

where $\alpha_{ij} \in \mathbb{R}$ and $\alpha_{ii} \neq 0$ in (2b). Note that the leaders are influenced by external control inputs, whereas the followers are governed by a control law which is the linear combination of the states of neighboring agents as given by (2b). The interaction structure between the agents in (1) can be described by a directed information flow graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, where each vertex represents an agent, and a directed edge from vertex ν_j to vertex ν_i indicates that $x_j(t)$ is transmitted to agent i and $\alpha_{ij} \neq 0$ in (2b). Moreover, the condition $\alpha_{ii} \neq 0$ in (2b) implies the existence of a self-loop on each follower vertex of $\mathscr{G}$; however, the self-loops are omitted to simplify the graphical representations. In a digraph representing a leader-follower multi-agent system, the root-set $\mathscr{R}$ consists of the leading agents; note that $|\mathscr{V}| = n$ and $|\mathscr{R}| = m$. The state of each agent $x_i(t)$ is set to be its absolute position w.r.t. an inertial reference frame, and the agent dynamics is assumed to be decoupled along each axis of the frame.

Remark 1. *Consider a leader-follower multi-agent system represented by the information flow digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with the root-set $\mathscr{R}$. The control laws in* (2) *imply that no edges enter the root-set, i.e. $\forall \tau, \nu \in \mathscr{V},\ \nu \in \mathscr{R} \longrightarrow (\tau, \nu) \notin \mathscr{E}$.*

Definition 1. *The information flow digraph $\mathscr{G}$ corresponding to the leader-follower multi-agent system* (1)*, is called controllable if the coefficients α_{ij} in* (2b) *can be chosen such that by properly moving the leaders, the followers would assume any desired configuration in an arbitrary time.*

The following theorem from [10] provides a necessary and sufficient condition for the controllability of an information flow digraph as defined above.

Theorem 1. *The information flow digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with the root-set $\mathscr{R} \subset \mathscr{V}$ is controllable if and only if every vertex $\nu \in \mathscr{V}\backslash\mathscr{R}$ is reachable from the root-set $\mathscr{R}$.*

The next subsection summarizes the main results of [9] and [10], upon which Section III expands.

B. Link and Agent Controllability Degrees

Link and agent controllability degrees provide quantitative insight into the reliability of a leader-follower multi-agent system in the face of agent and link failure, as investigated in [9] and [10] for a single leader and multiple leaders, respectively. A conceptually related issue is the fault tolerance of networks and connectivity of their interconnection digraphs, as discussed in Section 1.5 of [11].

In Section 1.7 of [12], the results obtained by using the max-flow min-cut theorem for a single source and a single sink, are extended to the flow networks with multiple sources and sinks by adding two new nodes [12]. The work [10] exploits a similar technique to extend the results of [9] to a digraph $\mathscr{G}$ with multiple leaders designated in a root-set $\mathscr{R}$, by using the expansion of $\mathscr{G}$ w.r.t. $\mathscr{R}$, defined bellow.

Definition 2. *Given an information flow digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with the root-set $\mathscr{R} \subset \mathscr{V}$, the expansion of $\mathscr{G}$ w.r.t. $\mathscr{R}$ is denoted by $\mathscr{G}'$, and is defined as $\mathscr{G}' = (\mathscr{V}', \mathscr{E}')$, where for a given vertex $r \notin \mathscr{V}$, $\mathscr{V}' = \{r\} \cup \mathscr{V}$ and $\mathscr{E}' = \{(r, \nu) | \nu \in \mathscr{R}\} \cup \mathscr{E}$.*

The link controllability degree of an information flow digraph is defined as follows [10].

Definition 3. *An information flow digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with the root-set $\mathscr{R} \subset \mathscr{V}$ is said to be $p-$*link controllable *if p is the largest number such that the controllability of the digraph is preserved after removing any group of at most $p - 1$ edges. Moreover, a minimal set of p edges, whose removal makes $\mathscr{G}$ uncontrollable is referred to as a* critical link set *and is denoted by $\mathscr{C}_p \subset \mathscr{E}$. The number p is referred to as the* link controllability degree *of the digraph $\mathscr{G}$ w.r.t. the root-set $\mathscr{R}$, and is denoted by $lc(\mathscr{G}; \mathscr{R})$. Similarly, given a vertex $\nu \in \mathscr{V}\backslash\mathscr{R}$, the minimum number of edges of $\mathscr{G}$ whose removal makes the vertex ν unreachable from the set $\mathscr{R}$ is denoted by $lc(\mathscr{G}, \nu; \mathscr{R})$.*

The following theorem from [10] provides a necessary and sufficient condition for the $p-$link controllability of an information flow digraph.

Theorem 2. *The information flow digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with the root-set $\mathscr{R} \subset \mathscr{V}$ is $p-$link controllable if and only if*

$$\min_{\mathscr{R} \subseteq \mathscr{X} \subset \mathscr{V}} d^+_{\mathscr{G}}\mathscr{X} = p. \tag{3}$$

The agent controllability degree of an information flow digraph is defined as follows [10].

Definition 4. *An information flow digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with the root-set $\mathscr{R} \subset \mathscr{V}$ is said to be $q-$*agent controllable *if q is the largest number such that the controllability of the digraph* *is preserved after removing any group of at most $q-1$ non-root vertices. Moreover, a minimal set of q non-root vertices whose removal makes $\mathscr{G}$ uncontrollable is referred to as a* critical agent set*, and is denoted by $\mathscr{C}_q \subset \mathscr{V}\backslash\mathscr{R}$. The number q is referred to as the* agent controllability degree *of the digraph $\mathscr{G}$ w.r.t. the root-set $\mathscr{R}$ and is denoted by $ac(\mathscr{G};\mathscr{R})$. Similarly, given a vertex $\nu \in \mathscr{V}\backslash\mathscr{R}$, the minimum number of vertices of $\mathscr{G}$ whose removal makes the vertex ν unreachable from the set $\mathscr{R}$ is denoted by $ac(\mathscr{G},\nu;\mathscr{R})$.*

In Section 1.11 of [12], a technique involving the duplication of the nodes in the digraph is used to extend the results of the problem of finding a maximal flow from one set of nodes to another, to the information flow digraphs subject to both arc and node capacity bounds. The corresponding technique employs the max-flow min-cut theorem with constraints on maximum arc flow. The work [9] exploits a similar technique termed node-duplication to relate the link and agent controllability degrees of a given information flow digraph. The node-duplicated version of an information flow digraph $\mathscr{G}$ is defined as follows.

Definition 5. *Given an information flow digraph $\mathscr{G} = (\mathscr{V},\mathscr{E})$ with the root-set $\mathscr{R} \subset \mathscr{V}$, replace every non-root vertex $\nu \in \mathscr{V}\backslash\mathscr{R}$ with two vertices $\tilde{\nu}_1$ and $\tilde{\nu}_2$, which are connected together by an intermediate edge $\tilde{\epsilon}_\nu = (\tilde{\nu}_1,\tilde{\nu}_2)$. The resulting digraph $\tilde{\mathscr{G}} = (\tilde{\mathscr{V}},\tilde{\mathscr{E}})$ is called the* node-duplication *of $\mathscr{G}$.*

Remark 2. *Given an information flow digraph $\mathscr{G} = (\mathscr{V},\mathscr{E})$, its node duplication $\tilde{\mathscr{G}} = (\tilde{\mathscr{V}},\tilde{\mathscr{E}})$ does not possess any anti-parallel edges, i.e. $\forall \tilde{\tau},\tilde{\nu} \in \tilde{\mathscr{V}},\ (\tilde{\tau},\tilde{\nu}) \in \tilde{\mathscr{E}} \longrightarrow (\tilde{\nu},\tilde{\tau}) \notin \tilde{\mathscr{E}}$.*

The following lemma from [9] describes the relationship between the agent controllability degree of a digraph $\mathscr{G}$ and the link controllability degree of its node-duplication $\tilde{\mathscr{G}}$.

Lemma 1. *Given an information flow digraph $\mathscr{G} = (\mathscr{V},\mathscr{E})$ with the root-set $\mathscr{R} \subset \mathscr{V}$, let its node-duplication be denoted by $\tilde{\mathscr{G}}$. The following relation holds:*

$$\forall \nu \in \mathscr{V}\backslash\mathscr{R},\ ac(\mathscr{G},\nu;\mathscr{R}) = lc(\tilde{\mathscr{G}},\tilde{\nu}_1;\mathscr{R}). \tag{4}$$

The following theorems from [9] provide lower bounds on the number of edges in a $p-$link or $q-$agent controllable digraph.

Theorem 3. *If an information flow digraph $\mathscr{G} = (\mathscr{V},\mathscr{E})$ is $p-$link controllable, then $|\mathscr{E}| \geqslant (|\mathscr{V}|-1)p$. Moreover, there exists a $p-$link controllable digraph, for which the equality holds.*

Theorem 4. *If an information flow digraph $\mathscr{G} = (\mathscr{V},\mathscr{E})$ is $q-$agent controllable, then $|\mathscr{E}| \geqslant |\mathscr{V}|+q-2$. Moreover, there exists a $q-$agent controllable digraph, for which the equality holds.*

III. Joint Controllability

The following definitions and the subsequent lemmas and theorems provide useful tools for investigating the effects of simultaneous link and agent failure on the controllability of an information flow digraph.

Definition 6. *An information flow digraph $\mathscr{G} = (\mathscr{V},\mathscr{E})$ with the root-set $\mathscr{R} \subset \mathscr{V}$ is said to be* joint $(r,s)-$controllable *if it remains controllable, in case of simultaneous failure of any set of links of size $u \leqslant r$ and set of non-root vertices of size $v \leqslant s$, where $u+v < r+s$ (note the strict inequality in the last expression).*

The next lemma follows immediately from Definitions 3, 4 and 6.

Lemma 2. *The following statements hold:*

a) *If $\mathscr{G}$ is joint $(r,s)-$controllable, then for all $u \leqslant r$ and $v \leqslant s$, $\mathscr{G}$ is joint $(u,v)-$controllable.*
b) *If $\mathscr{G}$ is joint $(r,s)-$controllable, then $r \leqslant lc(\mathscr{G};\mathscr{R})$ and $s \leqslant ac(\mathscr{G};\mathscr{R})$.*
c) *If $\mathscr{G}$ is joint $(r,s)-$controllable and $lc(\mathscr{G};\mathscr{R}) = r$, then $s = 0$.*
d) *If $\mathscr{G}$ is joint $(r,s)-$controllable and $ac(\mathscr{G};\mathscr{R}) = s$, then $r = 0$.*

Definition 7. *An information flow digraph $\mathscr{G} = (\mathscr{V},\mathscr{E})$ with the root-set $\mathscr{R} \subset \mathscr{V}$ is said to be* joint $t-$controllable *if t is the largest number such that $\mathscr{G}$ is joint $(u,v)-$controllable for all $u+v \leqslant t$. Moreover, a minimal set of r vertices and $s = t-r$ edges whose removal makes $\mathscr{G}$ uncontrollable is referred to as a* critical agent-link set*, and is denoted by $\mathscr{C}_{rs} \subset \mathscr{V} \cup \mathscr{E}\backslash\mathscr{R}$. The number t is called the* joint controllability degree *of the digraph $\mathscr{G}$ w.r.t. root-set $\mathscr{R}$, and is denoted by $jc(\mathscr{G};\mathscr{R})$.*

From Definitions 6 and 7, it follows that a sufficient condition for the preservation of controllability in the face of simultaneous failure in links and agents is that the total number of link and agent faults is not more than the joint controllability degree of the underlying information flow digraph. The next remark follows immediately from Definitions 3, 4, 6 and 7.

Remark 3. *For an information flow digraph $\mathscr{G} = (\mathscr{V},\mathscr{E})$ with the root-set $\mathscr{R}$, the following inequality holds:*

$$jc(\mathscr{G};\mathscr{R}) \leqslant \min\{lc(\mathscr{G};\mathscr{R}), ac(\mathscr{G};\mathscr{R})\}. \tag{5}$$

The next lemma is a direct consequence of Remark 3, and Theorems 3, 4.

Lemma 3. *If an information flow digraph $\mathscr{G} = (\mathscr{V},\mathscr{E})$ is joint $t-$controllable, then $|\mathscr{E}| \geqslant (|\mathscr{V}|-1)t$ and $|\mathscr{E}| \geqslant |\mathscr{V}|+t-2$.*

In the next definition and the theorems which follow, the class of *jointly uncritical* digraphs are introduced and some of their characteristics are pointed out.

Definition 8. *Given an information flow digraph $\mathscr{G} = (\mathscr{V},\mathscr{E})$ with the root-set $\mathscr{R}$, let $\mathscr{R} \subseteq \mathscr{X} \subset \mathscr{V}$ be any solution to the minimization problem in* (3)*. Let also $\mathscr{F} \subset \mathscr{V}\backslash\mathscr{R}$ be the set of all vertices in the digraph $\mathscr{G}_\mathscr{X} = (\mathscr{V},\mathscr{E}\backslash\partial^+_\mathscr{G}\mathscr{X})$, which are not reachable from the root-set $\mathscr{R}$, and define $\mathscr{H} = \mathscr{X} \cup \{\nu \in \mathscr{F} | (\tau,\nu) \in \partial^+_\mathscr{G}\mathscr{X} \wedge \tau \in \mathscr{R}\} \subset \mathscr{V}$; if $\partial^+_\mathscr{G}\mathscr{H} = \varnothing$, then $\mathscr{G}$ is* jointly uncritical*. Moreover, if $\partial^+_\mathscr{G}\mathscr{H} = \varnothing$, then every vertex $\nu \in \mathscr{H}\backslash\mathscr{X}$ is termed an* uncritical agent*.*

The following three remarks are direct results of Definition 8.

Remark 4. *An information flow digraph $\mathscr{G}$ is jointly uncritical if and only if $\mathscr{G}$ has an uncritical agent.*

Remark 5. *Given a jointly uncritical information flow digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with the root-set $\mathscr{R}$, if $\nu \in \mathscr{V}$ is an uncritical agent, then $\exists\, \tau \in \mathscr{R}$, such that $(\tau, \nu) \in \mathscr{E}$.*

Remark 6. *In a jointly uncritical information flow digraph $\mathscr{G}$, the removal of any sets of uncritical agents does not affect the controllability of $\mathscr{G}$.*

Theorem 5. *If an information flow digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with the root-set $\mathscr{R} \subset \mathscr{V}$ is not jointly uncritical, then $ac(\mathscr{G};\mathscr{R}) \leqslant lc(\mathscr{G};\mathscr{R})$.*

Proof: It suffices to introduce a set $\mathscr{L} \subset \mathscr{V}\backslash\mathscr{R}$ with the property $|\mathscr{L}| \leqslant lc(\mathscr{G};\mathscr{R})$, whose removal makes $\mathscr{G}$ uncontrollable. To this end, consider a solution $\mathscr{R} \subseteq \mathscr{X} \subset \mathscr{V}$ to the minimization problem in (3), which means that $|\partial^+_{\mathscr{G}}\mathscr{X}| = p$. The following routine utilizes $\partial^+_{\mathscr{G}}\mathscr{X}$ to generate one such set $\mathscr{L}$ with the desired characteristics. ∎

Routine A:

$\mathscr{L} = \varnothing$
for all $(\tau, \nu) \in \partial^+_{\mathscr{G}}\mathscr{X}$ **do**
  if $\tau \notin \mathscr{R}$ **then**
    $\mathscr{L} = \mathscr{L} \cup \{\tau\}$
  else
    $\mathscr{L} = \mathscr{L} \cup \{\nu\}$
  end if
end for

The next corollary follows upon combining the results of Remark 3 and Theorem 5.

Corollary 1. *If an information flow digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with the root-set $\mathscr{R} \subset \mathscr{V}$ is joint $t-$controllable and not jointly uncritical, then $t \leqslant ac(\mathscr{G};\mathscr{R})$.*

Theorem 6. *If an information flow digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with the root-set $\mathscr{R} \subset \mathscr{V}$ is joint $(r,s)-$controllable and not jointly uncritical, then $r + s \leqslant lc(\mathscr{G};\mathscr{R})$.*

Proof: Consider a solution $\mathscr{R} \subseteq \mathscr{X} \subset \mathscr{V}$ to the minimization problem in (3). According to Theorem 2, the link controllability degree of $\mathscr{G}$ is equal to the out-degree of $\mathscr{X}$, i.e. $|\partial^+_{\mathscr{G}}\mathscr{X}| = lc(\mathscr{G};\mathscr{R})$, and it follows from Lemma 2 that $r \leqslant lc(\mathscr{G};\mathscr{R})$. If $r = lc(\mathscr{G};\mathscr{R})$, then Lemma 2(c) requires that $s = 0$ and hence the statement of the above theorem holds. If on the other hand $r < lc(\mathscr{G};\mathscr{R})$, then choose a set of edges $\mathscr{Z}_r \subset \mathscr{E}$, such that $\mathscr{Z}_r \subset \partial^+_{\mathscr{G}}\mathscr{X}$ and $|\mathscr{Z}_r| = r$. Use Routine A after replacing $\partial^+_{\mathscr{G}}\mathscr{X}$ with $\partial^+_{\mathscr{G}}\mathscr{X}\backslash\mathscr{Z}_r$ to generate a set $\mathscr{L} \subset \mathscr{V}$. $\mathscr{L} \cup \mathscr{Z}_r$ is a set of $|\mathscr{L}|$ vertices and $|\mathscr{Z}_r| = r$ edges, for which $\mathscr{G}_{\mathscr{L},\mathscr{Z}_r} = (\mathscr{V}\backslash\mathscr{L}, \mathscr{E}\backslash\mathscr{Z}_r)$ is uncontrollable. However, $\mathscr{G}$ is joint $(r,s)-$controllable, which implies that $s \leqslant |\mathscr{L}|$. On the other hand, it follows from Routine A that $|\mathscr{L}| \leqslant |\partial^+_{\mathscr{G}}\mathscr{X}\backslash\mathscr{Z}_r|$. Hence $s \leqslant |\partial^+_{\mathscr{G}}\mathscr{X}\backslash\mathscr{Z}_r|$ or $s \leqslant lc(\mathscr{G};\mathscr{R}) - r$, which completes the proof. ∎

Corollary 2. *If an information flow digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ is joint $(r,s)-$controllable and not jointly uncritical, then $|\mathscr{E}| \geqslant (|\mathscr{V}| - 1)(r + s)$.*

Proof: The proof is straightforward and follows directly from Theorems 3, 6. ∎

Corollary 2 provides a necessary condition on the number of edges in a jointly uncritical digraph for joint $(r,s)-$controllability, and can be useful in the design of reliable multi-agent control systems. The next definition provides a mechanism to transform the problem of joint $t-$controllability in a given digraph into $q-$agent controllability of another digraph. This will, in turn, enable the multi-agent control system designer to take advantage of the polynomial-time algorithms developed in [9] and [10] for the latter problem.

Definition 9. *Given a digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, replace every edge $\epsilon \in \mathscr{E}$ with two edges $\hat{\epsilon}_1$ and $\hat{\epsilon}_2$ in the same direction as ϵ, and connect them through an intermediate vertex $\hat{\nu}_\epsilon$, termed a* black vertex. *The resulting digraph $\hat{\mathscr{G}} = (\hat{\mathscr{V}}, \hat{\mathscr{E}})$ is called the* edge-duplication *of $\mathscr{G}$. Every vertex of $\hat{\mathscr{G}}$ that is not a black vertex is referred to as a* white vertex.

Remark 7. *The following equalities hold for the number of vertices and edges in a given digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ and its edge-duplication $\hat{\mathscr{G}} = (\hat{\mathscr{V}}, \hat{\mathscr{E}})$:*

$$|\hat{\mathscr{V}}| = |\mathscr{V}| + |\mathscr{E}|, \tag{6a}$$

$$|\hat{\mathscr{E}}| = 2|\mathscr{E}|. \tag{6b}$$

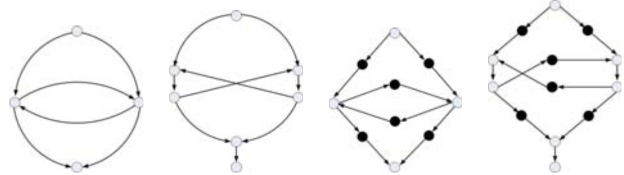
Remark 8. *Given a digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ and its edge-duplication $\hat{\mathscr{G}} = (\hat{\mathscr{V}}, \hat{\mathscr{E}})$, every white vertex $\hat{\nu}_\nu \in \hat{\mathscr{V}}$ corresponds to one vertex $\nu \in \mathscr{V}$ and every black vertex $\hat{\nu}_\epsilon \in \hat{\mathscr{V}}$ corresponds to one edge $\epsilon \in \mathscr{E}$. There exists a one-to-one correspondence between the sets $\mathscr{V} \cup \mathscr{E}$ and $\hat{\mathscr{V}}$.*

Theorem 7. *Consider a digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with the root-set $\mathscr{R} \subset \mathscr{V}$ and its edge-duplication $\hat{\mathscr{G}} = (\hat{\mathscr{V}}, \hat{\mathscr{E}})$. The digraph $\mathscr{G}$ is joint $t-$controllable if and only if $\hat{\mathscr{G}}$ is $t-$agent controllable.*

Proof: The proof follows from the fact that Definition 9 specifies a bijection between the sets $\mathscr{V} \cup \mathscr{E}$ and $\hat{\mathscr{V}}$. Using this bijection, any critical agent set of $\hat{\mathscr{G}}$ can be transformed into a critical agent-link set of $\mathscr{G}$ and vice versa. ∎

Remark 9. *For a joint $t-$controllable digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, it was shown in Lemma 3 that $|\mathscr{E}| \geqslant |\mathscr{V}| + t - 2$. Now, using* (6) *along with the results of Theorems 4 and 7, one can conclude that there exists a joint $t-$controllable digraph, for which $|\mathscr{E}| = |\mathscr{V}| + t - 2$.*

A digraph $\mathscr{G}$, as well as its node-duplication $\tilde{\mathscr{G}}$ and edge-duplication $\hat{\mathscr{G}}$, constructed according to Definitions 5 and 9, are depicted in Figs. 1(a)$-$(c). The digraph in 1(a), with the upper-most vertex as the root, is $2-$link and $2-$agent controllable. The digraph is also joint $(1,1)-$controllable. It does not have any uncritical agents, and is joint $2-$controllable. According to Theorem 7, the latter is tantamount to $2-$agent controllability of the digraph in Fig. 1(c). If the node-duplication process of Definition 5 is applied to every white vertex in an edge-duplicated digraph $\hat{\mathscr{G}}$, a new digraph is generated, which can be used to investigate the agent controllability of $\hat{\mathscr{G}}$ efficiently. The new digraph is termed the *node-edge-duplication* of $\mathscr{G}$, and is denoted by $\breve{\mathscr{G}}$. The node-edge-duplication of the digraph in Fig. 1(a) is depicted in Fig. 1(d). Lemma 1 provides a method for converting the problem of agent controllability in $\hat{\mathscr{G}}$ to link controllability in $\breve{\mathscr{G}}$. Polynomial-time algorithms presented in [9], [10] for investigating the link controllability of a given digraph can now be applied to $\breve{\mathscr{G}}$, which, in fact, gives the joint controllability degree of the original digraph $\mathscr{G}$.



(a) $\mathscr{G}$ (b) $\tilde{\mathscr{G}}$ (c) $\hat{\mathscr{G}}$ (d) $\breve{\mathscr{G}}$

Fig. 1. A digraph $\mathscr{G}$, its node-duplication $\tilde{\mathscr{G}}$, its edge-duplication $\hat{\mathscr{G}}$, and its node-edge-duplication $\breve{\mathscr{G}}$ are depicted in (a), (b), (c) and (d), respectively. The upper most vertex in all four digraphs is assumed to be a root.

Two special cases of interest are addressed in the sequel.

Proposition 1. *Given a complete digraph $\mathscr{G}_{cn} = (\mathscr{V}_{cn}, \mathscr{E}_{cn})$, where $|\mathscr{V}_{cn}| = n$ and $\mathscr{E}_{cn} = \mathscr{V}_{cn} \times \mathscr{V}_{cn}$, choose a vertex r as the root and remove the $n-1$ edges, which are headed by r. The resulting information flow digraph is joint $(n-1)-$controllable.*

Proof: The proof follows from the fact that $\mathscr{G}_{cn}$ has exactly $n-1$ disjoint $r\nu$ paths for every $\nu \in \mathscr{V}_{cn} \backslash \{r\}$. ■

Proposition 2. *Consider a Kautz digraph $\mathscr{G}_k = (\mathscr{V}_k, \mathscr{E}_k)$, where $\mathscr{V}_k$ and $\mathscr{E}_k$ are given by (See Section 3.3 of [11]):*

$$\mathscr{V}_k = \{\nu_1, \ldots, \nu_n\}, n = |\mathscr{V}_k|,$$
$$\mathscr{E}_k = \{(\nu_i, \nu_j) | i, j \in \mathbb{N}_n \wedge j \equiv (-id - \tau) \bmod n,\ \tau \in \mathbb{N}_d\},$$

for some $d \in \mathbb{N} \backslash \{1\}$ and $\kappa \in \mathbb{N}$, where $n = d^\kappa + d^{\kappa-1}$. Choose a vertex r as the root and remove all of the edges which are headed by r. The resulting information flow digraph is joint $d-$controllable.

Proof: The proof follows from the fact that $\mathscr{G}_k$ has exactly d disjoint $r\nu$ paths for every $\nu \in \mathscr{V}_k \backslash \{r\}$. ■

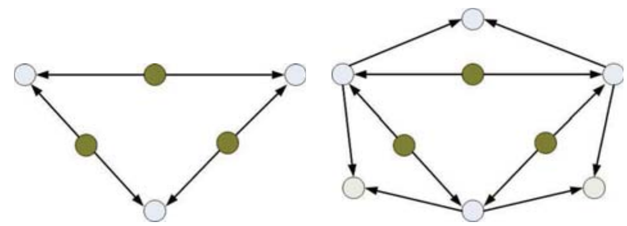
IV. Examples and Discussion

The following examples will illustrate and elaborate upon the ideas discussed in Section III.

Example 1. *A jointly uncritical digraph.*

Consider the digraph in Fig. 2(a) with the colored vertices as the roots. Every non-root vertex in this digraph is an uncritical agent whose removal will not affect the controllability of the digraph. This digraph is $2-$link and $3-$agent controllable, and is also joint $(2, k)-$controllable for any $k \in \mathbb{N}_3$. Remark 3 states that the joint controllability degree of this digraph cannot exceed its link controllability degree and this digraph is in effect joint $2-$controllable. The joint controllability degree, however, does not provide much information about this digraph because it is jointly uncritical.

On the other hand, the digraph in Fig. 2(b) with the colored vertices as the roots, is not jointly uncritical and is $2-$link and $2-$agent controllable. It is also joint $(1,1)-$controllable, and its joint controllability degree is 2.



(a) (b)

Fig. 2. (a) An example of a graph which is jointly uncritical (with three uncritical agents). (b) An example of a graph which is not jointly uncritical (with no uncritical agents).

Example 2. *Circulant digraphs.*

Circulant digraphs are introduced and discussed in Section 3.4.5 of [11]. Accordingly, a circulant digraph $\mathscr{G}_c = (\mathscr{V}_c, \mathscr{E}_c)$ with $|\mathscr{V}_c| = n$ is given by:

$$\mathscr{V}_c = \{\nu_1, \ldots, \nu_n\},$$
$$\mathscr{E}_c = \{(\nu_i, \nu_j) | i, j \in \mathbb{N}_n \wedge j - i \equiv b \bmod n,\ b \in \mathscr{B}\},$$

for some $\mathscr{B} \subseteq \mathbb{N}_{n-1}$. Choose a vertex $r \in \mathscr{V}_c$ as the root, and remove every edge whose head is r. Then in the resulting information flow digraph $\mathscr{G}_c$, $lc(\mathscr{G}_c; \{r\}) = |\mathscr{B}|$ and $ac(\mathscr{G}_c; \{r\}) \leqslant |\mathscr{B}|$. The former equality follows upon noting that $\mathscr{G}_c$ has exactly $|\mathscr{B}|$ edge-disjoint $r\nu$ paths for every $\nu \in \mathscr{V}_c \backslash \{r\}$. The latter inequality, on the other hand, follows from Theorem 5 and the fact that $\mathscr{G}_c$ is not jointly uncritical. The choices of $\mathscr{B}_1 = \{1\}$, $\mathscr{B}_2 = \{1, n-1\}$, and $\mathscr{B}_3 = \{1, n-2\}$ correspond to a simple loop $\mathscr{G}_1 = (\mathscr{V}_1, \mathscr{E}_1)$, a distributed double-loop $\mathscr{G}_2 = (\mathscr{V}_2, \mathscr{E}_2)$, and a daisy chain loop $\mathscr{G}_3 = (\mathscr{V}_3, \mathscr{E}_3)$, respectively, and they are introduced in Section 3.4.1 of [11]. For a simple loop $lc(\mathscr{G}_1; \{r\}) = ac(\mathscr{G}_1; \{r\}) = jc(\mathscr{G}_1; \{r\}) = 1$, while for the other two cases $lc(\mathscr{G}_i; \{r\}) = ac(\mathscr{G}_i; \{r\}) = jc(\mathscr{G}_i; \{r\}) = 2,\ i = 2, 3$. These three have the additional property that for every $r+s > jc(\mathscr{G}_i; \{r\}),\ i \in \mathbb{N}_3, \mathscr{G}_i$ is not joint $(r,s)-$controllable. Accordingly, the joint controllability degree alone provides a complete characterization of the controllability preservation properties for $\mathscr{G}_i,\ i \in \mathbb{N}_3$. The three digraphs are depicted in Fig. 3(a)$-$(c) for $|\mathscr{V}_i| = 5,\ i \in \mathbb{N}_3$, with the colored vertices selected as the roots.

On the other hand, with $\mathscr{B}_4 = \{2, 3, 5\}$, $|\mathscr{V}_4| = 6$ and the uppermost vertex selected as the root, the resulting information flow digraph $\mathscr{G}_4$, shown in Fig. 3(d), is $3-$link and $2-$agent controllable [9]. The joint controllability degree for $\mathscr{G}_4$ is 2, and unlike $\mathscr{G}_i,\ i \in \mathbb{N}_3$, $jc(\mathscr{G}_4; \{r\}) = 2$ does not proffer a full characterization of the controllability preservation properties in $\mathscr{G}_4$. Accordingly, $\mathscr{G}_4$ is joint $(r,s)-$controllable for $(r,s) \in \{(2,1),(3,0)\}$, whereas $r+s > jc(\mathscr{G}_4;\{r\})$.

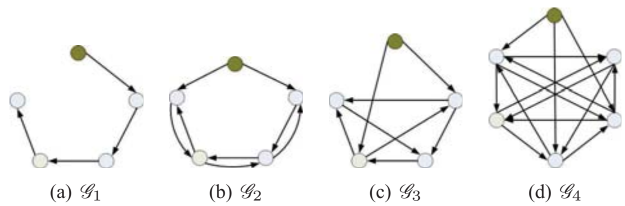


Fig. 3. The digraphs in Example 2.

Let the values of $(r,s) \in \mathbb{W} \times \mathbb{W}$ for which $\mathscr{G}_i$ is joint $(r,s)-$controllable be shown as discrete points in the plane. For $i \in \mathbb{N}_3$, the line $r+s = jc(\mathscr{G}_i;\{r\})$ divides the plane into two regions, where for $r+s \leqslant jc(\mathscr{G}_i;\{r\})$, $\mathscr{G}_i$ is joint $(r,s)-$controllable and otherwise it is not. This is depicted in Fig. 4 for $\mathscr{G}_2$ and $\mathscr{G}_3$, where the closed shaded region contains all pairs of integers belonging to the joint controllability set (these pairs are shown by black circles). This property, however, does not hold for $\mathscr{G}_4$; it is evident from Fig. 5 that there exist two points above the $r+s = jc(\mathscr{G}_4;\{r\})$ line for which $\mathscr{G}_4$ is joint $(r,s)-$controllable.

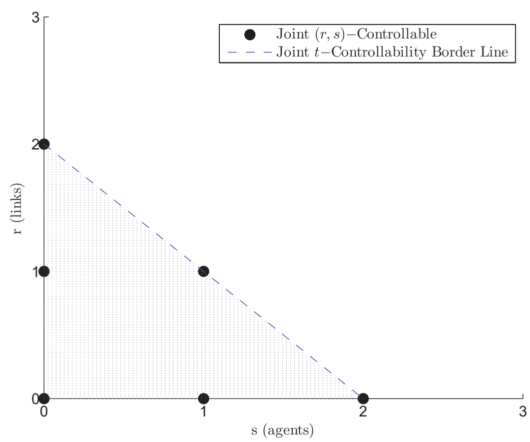


Fig. 4. Joint controllability of $\mathscr{G}_2$ and $\mathscr{G}_3$ in Example 2. The filled circles in the shaded area represent the pair of integers belonging to the jointly controllable set.

V. Conclusion

Structural controllability of a network of single-integrator agents with leader-follower architecture was investigated, and the notions of joint $(r,s)-$controllability and joint $t-$controllability were introduced. These notions provide quantitative reliability measures in a multi-agent system subject to simultaneous failure of communication links and agents. Graphical conditions to determine one or more so-called "uncritical agents" in the network were derived, and examples of digraphs containing such agents were presented. It was further stated that the failure of the uncritical agents would

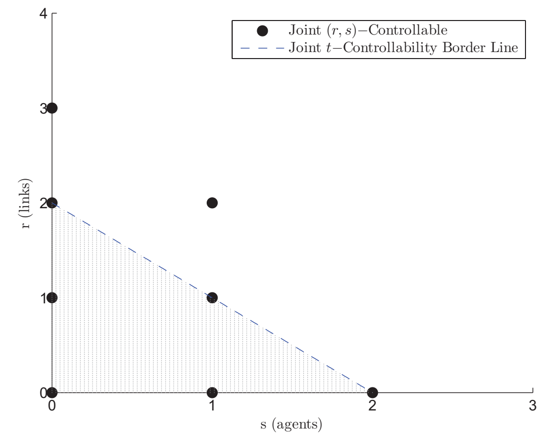


Fig. 5. Joint controllability of $\mathscr{G}_4$ in Example 2. The filled circles represent the pair of integers which belong to the jointly controllable set.

not affect the controllability of a jointly uncritical digraph. Moreover, a method was presented to transform the problem of joint $t-$controllability of a given digraph into $t-$agent controllability of another digraph. The latter problem could then be handled using existing polynomial-time algorithms.

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