2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Averaged Attractive Ellipsoid Technique Applied to Inventory Projectional Control with Uncertain Stochastic Demands

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Abstract— **This paper consider the inventory robust control problem with an uncertain demand and bounded control actions. The considered system deals with a nonlinear discrete-time stochastic model of a special structure. The robust control designing, consisting in the inventory product level minimization by the corresponding adjustment of the production rate, is shown to be converted into certain averaged attractive ellipsoid "minimization" problem under some specific constraints of BMI's (Bilinear Matrix Inequalities) type. The application of an adequate coordinate changing transforms these BMI's into a set of LMI's (Linear Matrix Inequalities) that permits to use directly the standard MATLAB - toolbox. The matrix generalization of the SLLN (Strong Law of Large Numbers) provides an instrument for the stochastic analyses of the considered process.**

I. Introduction

Inventory control and *revenue management* have become very active research topics and have been extensively studied in the academic literature in Economics, Operations Management, and Marketing [7], [8] and [10]. In these settings, suppliers maximize their profits over a time horizon (subject to some constraints) or minimizes their inventory product level by adjusting their prices and production allocations. The comprehensive reviews of modern inventory management techniques *under the complete information* on the system dynamics and demands can be found in [14] and [26].

The analogous problems in *the presence of uncertainties* or *incomplete information* on random demands are considered in [20] and [22]. The simplest case of such type of uncertainties is studied in [15] where the demand is treated as a discrete random variable that leads to the multi-model consideration and the control action is *a production rate*. This optimization problem is of the "min-max" type where the maximum is taken over the set of possible dynamic models and minimum - over admissible production rate referred below to as the *inventory control*. The solution is based on the *Robust Stochastic Maximum Principle* implementation [17].

Here we consider the more complicated case of uncertainties:

- the demands are random and generated by an exogenous (may be, non-stationary) linear system from a given class with a "wight noise" as an input,

- the demand (as a random signal) is unmeasurable on-line.

Since the model under consideration is nonlinear, but, in some sense, is "quasi-linear" (i.e., the regular part of the right-hand side of a corresponding stochastic differential equation belongs to a "cone" or a stripe containing an origin), then it seems to be natural to use a feedback controller which is linear on current state estimates generated by an observer of the Luenberger-like type. In the presence of unmodelled dynamics or non-decreasing perturbations, obviously, the minimization of a market demands can not be made as small as one wishes, but may be located (in some "average sense") within some bounded convex zone contained in an ellipsoid of the corresponding dimension, or in other word, in an "*averaged attracting ellipsoid*". Varying the gain matrices of both the linear feedback and an observer device one can "minimize" this ellipsoid providing more preferable robust dynamics for this controlled uncertain model.

The synthesis problem for a class of deterministic linear systems with bounded uncertainties and disturbances traced back to the pioneer works of Bertsekas [3], [9] and Chernous'ko [6] among others where the *Dynamic Programming Method* and *Ellipsoidal Calculus* were applied. For the perturbation of a bounded energy ($L_2(0,\infty)$-case) the explicit solution for the control designing was summarized in [24] using H_∞-approach and extended (to l_1-case) with *Linear Matrix Inequality* (LMI) application in [11]. The extension of the H_∞-technique for the class of stochastic continuos systems with completely measurable states and stochastic noises of the multiplicative-type was done in [23]. The stability of stochastic nonlinear models was studied also in [2] but under the assumption that all state are complete measurable. The closely related numerical procedure for the gain matrix optimization of a stochastic continuous time linear observer (filter) has been suggested and analyzed in [16], but, in our opinion, the joint stochastic "observation-control" problem still remains a great challenge for the control society.

The concept of an *Invariant Ellipsoid*, where all trajectory of a the market demands asymptotically arrive, in its more complete form were presented in [4]. In [13] the problem of synthesis of a static state-feedback controller for a linear time-invariant system, minimizing the size of the corresponding invariant ellipsoid, was reduced to optimization of a linear function under some set of LMI constraints. This method is very close in its philosophy with the, so-called, *Robust Attractive Ellipsoid Method* (RAEM) designed and applied here. Although many robust control problems can be formulated in terms of LMI and be solved with semi-definite programming [5], a significantly wider class of problems can be formulated in terms of *Bilinear Matrix Inequalities* (BMI) as in [18]. As for their numerical resolution, it can

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be mentioned that only in very few cases (such as static state feedback and dynamic output feedback) it is possible to convert the original problem to a convex one with the appropriate change of variables and obtain the equivalent LMI's.

In this paper we deal with demands of a *stochastic nature* and with a model dynamics given in a discrete time. The overall *goal* of this research is to introduce and study a inventory control problem. In particular, this paper considers an unmeasurable demand affecting a factory regime. We are interested in the adjustment of its production rate to minimize the inventory production level y under given class of unmeasured demands. Our "free" parameters to be adjusted are the gain matrices K (in the designed feedback) and L (in the designed state-observer)

The main *constraints* accepted in this paper are:

- the inventory production level y is assume to be bounded, i.e., $y \in [0, y^+]$, with a known upper bound;
- the admissible production rate u (or, the inventory control) constitutes the given interval $[0, u^+]$;
- the scalar control action u is considered as a nonlinear saturation function (Projectional Control) (see Fig.1) of the current estimate $\hat{x}$ of the state vector x which is not always available;

$$u = \pi(K\hat{x}) = \begin{cases} K\hat{x} & \text{if} \quad K\hat{x} \in [0, u^+] \\ 0 & \text{if} \quad K\hat{x} < 0 \\ u^+ & \text{if} \quad K\hat{x} > u^+ \end{cases} \tag{1}$$

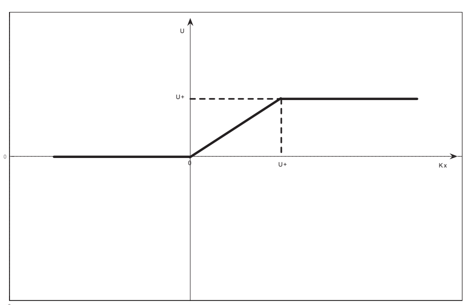


Fig. 1. The control action u.

of the current estimate $\hat{x}$ of the state vector x which is not always available;.

- the structure of the observer generating $\hat{x}$ is selected as a linear of a Luenberger-type form with a free gain-matrix L to be designed.

The principle *contributions* of this paper are as follows:

- the matrix form of the strong law of large numbers (SLLN) is designed here to analyze the

convergence to an "averaged" attractive ellipsoid with probability one;

- a specific form of BMI (after by a special transformation converted into an LMI) is obtained which provides this $a.s.$-convergence and, as a result, guaranteeing the robustness property of the designed controller for a class of "quasi-linear" exogenous models;
- the numerical matrix optimization procedure, based on the "Interior Point Method", which uses these LMI constraints, is suggested. It provides the optimal numerical values of the pair (K, L) guarantying the minimal attractive ellipsoid where almost all (a.s.) the demand trajectories arrive for any nonlinear controlled system from the considered class.

II. System Description and Problem Formulation

In this section we specify the basic model in discrete-time.

A. Discrete-time model

Following to [25], [21] and [17], consider the discrete-time stochastic process given by

$$\begin{gathered} y_{n+1} = [y_n + \tau u_n - \tau \bar{z}_n]_+ = [y_n + \tau u_n - \tau \bar{z}_n] + \Delta y_n \\ \Delta y_n = [y_n + \tau u_n - \tau \bar{z}_n]_+ - [y_n + \tau u_n - \tau \bar{z}_n] \\ \bar{z}_{n+1} = \bar{z}_n + \tau \Phi_n + \tau \sigma w_{n+1}, \ \tau, \sigma \text{ - const, } n = 0, 1, 2 \end{gathered} \tag{2}$$

where y_n is the inventory product level kept in the buffer of the capacity y^+, $u_n \in [0, u^+]$ is a production rate, τ is the discretization (sample-data) interval which is usually very small ($0 < \tau \ll 1$), Φ_n is a given deterministic sequence meaning the expected demand rate at the given environment conditions, w_{n+1} is a standard independent Gaussian random variable, σ is a positive constant number (which may be a priory unknown), $\bar{z}_n$ is 'the market demand process'.

The control processes u_n should be non-anticipative, i.e., it should be dependent on the current and past available information only. All random variables in (2) are assumed to be $\{\mathcal{F}_n\}_{n=0,1,2...}$-adapted. To meet the demand the factory, serving this market, should adjust its production rate $\{u_n\}$ to accommodate any possible changes in the current market situation.

Defining $x_{1,n} := y_n$, $x_{2,n} := \bar{z}_n$ and $x_n := \begin{bmatrix} x_{1,n} & x_{2,n} \end{bmatrix}^\intercal$, the system (2) can be represented in the vector form as

$$\begin{gathered} x_{n+1} = A x_n + B u_n + R_1 \Phi_n + R w_{n+1} + \Delta \tilde{y}_n \\ A = \begin{bmatrix} 1 & -\tau \\ 0 & 1 \end{bmatrix}, \ B = \begin{bmatrix} \tau \\ 0 \end{bmatrix} \\ R_1 := \begin{bmatrix} 0 \\ \tau \end{bmatrix}, \ R := \begin{bmatrix} 0 \\ \tau\sigma \end{bmatrix}, \ \Delta \tilde{y}_n := \begin{bmatrix} \Delta y_n \\ 0 \end{bmatrix} \end{gathered} \tag{3}$$

Assuming that only the inventory production level y_n is measurable, one may define the "output of the system" $\bar{y}_n$ as

$$\bar{y}_n = C x_n, \ C := \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{4}$$

The initial value x_0 is supposed to random having a of bounded second moment, i.e., $\mathsf{E}\left\{\|x_0\|^2\right\} < \infty$.

B. Problem formulation

So, *the formal formulation problem* looks now as follows: based on measurable (available) data $(y_0, y_1, ..., y_n)$ design an admissible output-feedback control $u_n = u_n(y_0, y_1, ..., y_n) \in \mathbb{R}^m$which provides "*a good*" (in some probabilistic sense) behavior of the given uncertain model (2) with a minimum possible inventory production level.

Below we restrict the class of all possible control actions $u_n = \pi(K\hat{x}_n)$ containing the term $\hat{x}_n$, referred to as the *estimate of the state* x_n at time n and generated by the recursive filter

$$\begin{gathered}\hat{x}_n = A\hat{x}_{n-1} + Bu_n + L(y_n - C\hat{x}_{n-1}) \\ \hat{x}_n \in \mathbb{R}^2,\ L \in \mathbb{R}^{2\times 1},\ \hat{x}_0 \text{ is fixed}\end{gathered} \tag{5}$$

The projection operator $\pi(\cdot)$ is defined in (1).

Define the state *estimation error* is given by $e_n := \hat{x}_n - x_n$, the *extended state vector* $z_n \in \mathbb{R}^4$ as $z_n = \begin{pmatrix} x_n^\intercal & e_n^\intercal \end{pmatrix}^\intercal$ as well as the quadratic function

$$V_n = z_n^T P z_n,\ P = P^\intercal > 0 \tag{6}$$

Definition 1: We say that $\{x_n\}_{n\geq 0}$ belongs asymptotically "**on average**" to the Robust Attractive Ellipsoid

$$\mathcal{E}(0, P_x) := \left\{x \in \mathbb{R}^n : x^T P_x x \leq 1\right\} \tag{7}$$

(with the center in the point $x = 0$ and the corresponding ellipsoidal matrix $P_x = P_x^\intercal > 0$) if for any initial conditions of the model (3) and an admissible control strategy $\{u_n\}_{n\geq 0}$, generated by (1), (5) and satisfying (6), the following property holds:

$$\limsup_{n\to\infty} \mathsf{E}\left\{x_n^T P_x x_n\right\} \leq 1 \tag{8}$$

Theorem 1 (on the attractive ellipsoid): If for the model (3)-(4), controlled by the feedback (1) using the state estimates generated by the observer (5), the following matrix inequalities hold

$$\begin{gathered}0 < \beta I_{4\times 4} \leq P = \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22}\end{bmatrix} \leq \alpha I_{4\times 4} \\ R^\intercal P_{11} R \leq \alpha_R I,\ \alpha_R > 0 \\ \tilde{W}_{\alpha,\lambda}(K, L, P) = \begin{bmatrix} \tilde{w}_{11} & \tilde{w}_{12} & \tilde{w}_{13} & \tilde{w}_{14} \\ \tilde{w}_{21} & \tilde{w}_{22} & \tilde{w}_{23} & \tilde{w}_{24} \\ \tilde{w}_{31} & \tilde{w}_{32} & \tilde{w}_{33} & \tilde{w}_{34} \\ \tilde{w}_{41} & \tilde{w}_{42} & \tilde{w}_{43} & \tilde{w}_{44}\end{bmatrix}\end{gathered} \tag{9}$$

with the sub-blocks

$$\begin{gathered}\tilde{w}_{11} = \begin{bmatrix} \bar{w}_{11} & \bar{w}_{12} \\ \bar{w}_{21} & \bar{w}_{22}\end{bmatrix} + 2G^\intercal K^\intercal K G + \\ \left(1+\tau^2\right)\tilde{G}^\intercal \tilde{G} \\ \bar{w}_{11} = [A+BK]^\intercal P_{11}[A+BK] - \lambda_{11} P_{11} \\ \bar{w}_{12} = [A+BK]^\intercal P_{11}[BK];\ \bar{w}_{12} = \bar{w}_{12} \\ \bar{w}_{22} = [BK]^\intercal P_{11}[BK] + [A-LC]^\intercal P_{22}[A-LC] \\ -\lambda_{22} P_{22}\end{gathered}$$

$$\begin{gathered}\tilde{w}_{21} = \begin{bmatrix} [A+BK]^\intercal P_{11} & 0_{2\times 2} \\ [BK]^\intercal P_{11} & [A-LC]^\intercal P_{22}\end{bmatrix}; \\ \tilde{w}_{12} = \tilde{w}_{21}^\intercal,\ \tilde{w}_{31} = \tilde{A}^T P \tilde{B}; \\ \tilde{w}_{13} = \tilde{w}_{31}^\intercal,\ \tilde{w}_{33} = \tilde{B}^\intercal P \tilde{B} - \varepsilon_2 I_{4\times 4} \\ \tilde{w}_{22} = \begin{bmatrix} P_{11} - \varepsilon_1 I_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & P_{22} - \varepsilon_1 I_{2\times 2}\end{bmatrix}, \\ \tilde{w}_{32} = P\tilde{B};\ \tilde{w}_{32} = \tilde{w}_{32}^\intercal \\ \tilde{w}_{41} = \tilde{A}^T P \tilde{C};\ \tilde{w}_{14} = \tilde{w}_{41}^\intercal,\ \tilde{w}_{42} = P\tilde{C},\ \tilde{w}_{24} = \tilde{w}_{42}^\intercal \\ \tilde{w}_{43} = \tilde{B}\tilde{C},\ \tilde{w}_{43} = \tilde{w}_{43}^\intercal,\ \tilde{w}_{44} = \tilde{C}^\intercal P \tilde{C} - \varepsilon_3 I_{4\times 4}\end{gathered}$$

$$\begin{gathered}0 < \Lambda = \operatorname{diag}[\lambda_{11}, \lambda_{22}],\ \ \|\Lambda\| := \lambda_{11} + \lambda_{22} < 1 \\ G = \begin{bmatrix} I_{2\times 2} & I_{2\times 2}\end{bmatrix}, \tilde{G} = \begin{bmatrix} I_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & 0_{2\times 2}\end{bmatrix}, \\ \tilde{B} = \begin{bmatrix} B_{2\times 1} & 0_{2\times 3} \\ 0_{2\times 1} & 0_{2\times 3}\end{bmatrix},\ \tilde{C} = \begin{bmatrix} I_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & 0_{2\times 2}\end{bmatrix} \\ \tilde{A} = \begin{pmatrix} A+BK & BK \\ 0 & A-LC\end{pmatrix},\ \varepsilon_1, \varepsilon_2, \varepsilon_2 > 0\end{gathered}$$

then $\{x_n\}_{n\geq 0}$ belongs asymptotically "on average" to the Robust Attractive Ellipsoid $\mathcal{E}(0, P_x)$ with

$$\begin{gathered}P_x = \varkappa^{-1}(\lambda, \alpha) P_{11} \\ \varkappa(\lambda, \alpha) := \frac{1}{1-\lambda}\left(Q_c + \lim_{n\to\infty}\sup \varepsilon_1 \|F_n\|^2 + \tilde{U}\right) \\ Q_c = (\alpha + \alpha_R) \\ \tilde{U} = (\tau\varepsilon_3 + 2\varepsilon_2)\left[1 + u^+\right]^2\end{gathered} \tag{10}$$

or, in other words,

$$\limsup_{n\to\infty} \mathsf{E}\left\{x_n^T \left[\frac{P_{11}}{\varkappa(\lambda, \alpha)}\right] x_n\right\} \leq 1 \tag{11}$$

The scheme of the proof. Using the properties of the random variables discussed above, we may conclude that fulfilling (9) implies

$$\begin{gathered}\mathsf{E}\{V_{n+1}/\mathcal{F}_n\} \overset{a.s}{\leq} Q_c + \|\Lambda\| V_n + \tilde{z}_n^\intercal \tilde{W}_{\alpha,\lambda}(K, L, P)\tilde{z}_n + \\ (\tau\varepsilon_3 + 2\varepsilon_2)\left[1 + \max(u_1^+, u_2^+)\right]^2 + \varepsilon_1 \|F_n\|^2\end{gathered}$$

$$\begin{gathered}\mathsf{E}\{V_{n+1}/\mathcal{F}_n\} \overset{a.s}{\leq} Q_c + \|\Lambda\| V_n + \\ (\tau\varepsilon_3 + 2\varepsilon_2)\left[1 + u^+\right]^2 + \varepsilon_1 \|F_n\|^2\end{gathered}$$

as well as $\limsup\limits_{n\to\infty} \mathsf{E}\{V_n\} \leq \varkappa(\lambda, \alpha)$, and, finally, (11) that completes the proof.

III. The averaged ellipsoid

Introduce the following *averaged quadratic form* or *the "averaged ellipsoid"* $\tilde{V}_n$ given by $\tilde{V}_n := n^{-1}\sum\limits_{t=1}^{n} V_t$ which can be also expressed as

$$\begin{gathered}\tilde{V}_n := n^{-1}\sum_{t=1}^{n} z_t^T P z_t = \operatorname{tr}\{\mathsf{E}\{Z_n\} P\} + \operatorname{tr}\{\Theta_n P\} \\ Z_n := n^{-1}\sum_{t=1}^{n} z_t z_t^T,\ \Delta_t := z_t z_t^T - E\left\{z_t z_t^T\right\} \\ \Theta_n = n^{-1}\sum_{t=1}^{n} z_t z_t^T - \mathsf{E}\{Z_n\} = n^{-1}\sum_{t=1}^{n} \Delta_t\end{gathered} \tag{12}$$

and $\{z_n\}_{n\geq 1}$ is generated by the recursion

$$\begin{array}{c} z_{n+1} = \tilde{A}(K,L)\, z_n + F_n + \\ \tilde{B}\left[\pi\left(K\hat{x}_n\right) - K\hat{x}_n\right] + \tilde{C}\Delta\tilde{y}_n + \tilde{w}_{n+1} \end{array} \tag{13}$$

Here $\tilde{A}$ are, in fact, the functions of K and L, namely, $\tilde{A} = \tilde{A}(K,L)$. Let us then show that under some conditions $\Theta_n \underset{n\to\infty}{\overset{a.s.}{\to}} 0$, or, in other words, demonstrate that the *Strong Law of Large Numbers* (SLLN) in its matrix form holds.

A. Matrix version of SLLN

The sufficient condition for the matrix form of the SLLN are given in the following theorem.

Theorem 2 (Matrix version of SLLN): Let the matrix Δ_n be defined as in (12) so that the following series converges:

$$\sum_{n=1}^{\infty}\left(\frac{\sigma_n}{n}\sqrt{R_{n-1}} + \frac{1}{n^2}\sigma_n^2\right) < \infty,\ R_0 := 0 \tag{14}$$

where

$$\begin{array}{c} \sigma_n := \operatorname{tr}\left\{\mathsf{E}\left\{\Delta_n\Delta_n^{\intercal}\right\}\right\} \\ R_n := n^{-2}\sum_{t=1}^{n}\sum_{s=1}^{n}\operatorname{tr}\left\{\mathsf{E}\left\{\Delta_t\Delta_s^{\intercal}\right\}\right\} \end{array} \tag{15}$$

Then

$$\Theta_n \underset{n\to\infty}{\overset{a.s.}{\to}} 0 \text{ and } \mathsf{E}\left\{\left\|\Theta_n\right\|^2\right\} \underset{n\to\infty}{\overset{a.s.}{\to}} 0 \tag{16}$$

The proof of this theorem can be found in [1] as the extension of Theorem 8.10 from [17].

B. Analysis of the double-averaged covariation R_n

The following intermediate results hold.

Theorem 3: If under the conditions of Theorem 1 additionally the 4-th conditional moments are bounded, i.e.,

$$\mathsf{E}\left\{\left\|w_n\right\|^4\right\} \leq m_{4,w} < \infty \tag{17}$$

there exist constants $\iota_1, \iota_2$and $\lambda \in [0,1)$ satisfying

$$0 < \boldsymbol{\tau}_1 < \lambda^{-2}\frac{1-\boldsymbol{\tau}_2}{1+\boldsymbol{\tau}_2^{-1}},\ \boldsymbol{\tau}_2 \in (0,1)$$

such that the following inequality holds

$$\begin{array}{c} \limsup\limits_{n\to\infty}\mathsf{E}\left\{V_n^2\right\} \leq \dfrac{c}{1-\tilde{\lambda}} \\ \tilde{\lambda} := \dfrac{1+\boldsymbol{\tau}_2^{-1}}{1-\boldsymbol{\tau}_2}\boldsymbol{\tau}_1\lambda^2,\ c = \left(1+\boldsymbol{\tau}_2^{-1}\right)\boldsymbol{\tau}_1^{-1}\bar{\beta}_n^2 \end{array} \tag{18}$$

Using the obtained estimate (18) we are ready to formulate the main result of this subsection.

Theorem 4: Under the conditions of Theorem 3 the following upper estimate is valid:

$$R_n \leq 4\left\|R\right\|^4\left(m_{4,w}\right)n^{-1} = O\left(n^{-1}\right) \tag{19}$$

So, based on this upper estimate one can conclude that SLLN holds for the considered class of random processes.

Proposition: Under the conditions of Theorem 3 the property (16) holds.

Indeed, by the upper estimate (19) and taking into account that

$$\limsup_{n\to\infty}\sigma_n^2 \leq \limsup_{n\to\infty}\mathsf{E}\left\{V_n^2\right\} \leq \frac{c}{1-\tilde{\lambda}} < \infty$$

it follows

$$\begin{array}{c} \sum\limits_{n=1}^{\infty}\left(\dfrac{\sigma_n}{n}\sqrt{R_{n-1}} + \frac{1}{n^2}\sigma_n^2\right) \leq \\ \text{Const}\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\sqrt{O\left(n^{-1}\right)} + \frac{1}{n^2}\right) < \infty \end{array}$$

This means that the conditions of Theorem 2 are fulfilled, and hence, the SLLN holds for the considered processes.

C. Analytical representation of the "averaged" attractive ellipsoid

Taking an upper limit of (6) and using the representation (12) as well as the property (16) we get

$$\begin{array}{c} \limsup\limits_{n\to\infty}\tilde{V}_n = \limsup\limits_{n\to\infty} n^{-1}\sum\limits_{t=1}^{n}V_t \\ \overset{a.s.}{=} \limsup\limits_{n\to\infty}\operatorname{tr}\left\{\mathsf{E}\left\{Z_n\right\}P\right\} = \\ \limsup\limits_{n\to\infty}\operatorname{tr}\left\{P^{1/2}\mathsf{E}\left\{Z_n\right\}P^{1/2}\right\} = \\ \operatorname{tr}\left\{P^{1/2}\mathcal{K}P^{1/2}\right\} = \operatorname{tr}\left\{\mathcal{K}P\right\} = \operatorname{tr}\left\{\mathcal{K}^{1/2}P\mathcal{K}^{1/2}\right\} \end{array}$$

where $\mathcal{K} := \limsup\limits_{n\to\infty}\mathsf{E}\left\{Z_n\right\}$ is the "upper limit -covariance" matrix of the extended error-vector z_n.

Theorem 5 (The main result): Under the conditions of Theorem 3 one can guarantee (with probability one) that

$$\begin{array}{c} \limsup\limits_{n\to\infty}\tilde{V}_n \overset{a.s.}{=} \limsup\limits_{n\to\infty}\mathsf{E}\left\{\tilde{V}_n\right\} = \\ \operatorname{tr}\left\{\mathcal{K}^{1/2}P\mathcal{K}^{1/2}\right\} \leq \dfrac{\vartheta}{1-\lambda} := \varkappa(\lambda,\alpha) \end{array} \tag{20}$$

where λ is defined in (18) and $\vartheta := \limsup\limits_{n\to\infty}\bar{\beta}_n$.

D. Averaged attractive ellipsoid

From the relations (20) it follows that the *"averaged" attractive ellipsoid* $\mathcal{E}(0,P_\delta)$ is defined by the matrix $P_\delta = \frac{P_{11}}{\varkappa(\lambda,\alpha)}$. In other words, the following inequality holds:

$$\limsup_{n\to\infty} n^{-1}\sum_{k=1}^{n}x_k^T\left[\frac{P_{11}}{\varkappa(\lambda,\alpha)}\right]x_k \leq 1$$

IV. LMI representation of the constraint optimization problem

If one wishes to "minimize" the inventory production level x_n it make since to "maximize" $\dfrac{P_{11}}{\varkappa(\lambda,\alpha)}$ by K and L and other scalar parameters in some "matrix" sense. Usually (see, for example, [5], [12], the matrix gains K and L are suggested to be found as the solution of the following optimization problem

$$\frac{\operatorname{tr}P_{11}}{\varkappa(\lambda,\alpha)} \to \sup_{P_1>0,K,L;\lambda\in[0,1)} \tag{21}$$

Our aim here is to find the control gain matrix K and the observer gain matrix L, providing a "*good enough*" stabilization as well as state estimation of the system for a wide class of nonlinear systems (3). Obviously, the matrix inequality $\tilde{W} < 0$ is nonlinear with respect to the matrix arguments P_1, K, L even for fixed scalar parameters λ, α. Notice that all diagonal blocks contain the quadratic matrix terms which significantly complicates the numerical procedure resolving this inequality. Estimation from above each this block permits to simplify it up to a linear one. To do that the following consideration is applied. Now for the matrix (9), define the new matrix $\breve{W} := -\tilde{W}$ where now $\breve{W} > 0$, and using the Schur's complement for the symmetric block-matrix $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^{\intercal} & S_{22} \end{bmatrix}$ to conclude that $S > 0$ providing $S_{11} > 0$, $S_{22} > S_{12}^{\intercal} S_{11}^{-1} S_{12}$ which (among others matrix inequalities) leads to:

$$\begin{array}{c} \bar{w}_{22} = -\left[(BK)^{\intercal} P_{11} (BK) - (A - LC)^{\intercal} P_{22} (A - LC) \right] \\ + \lambda_{22} P_{22} > 0 \end{array} \tag{22}$$

Suppose that for some nonnegative matrix $Q_1 \in \mathbb{R}^{n \times n}$ we have $(BK)^{\intercal} P_{11} (BK) < Q_1$, or, equivalently (again by the Schur's complement implementation):

$$\begin{gathered} \begin{bmatrix} Q_1 & BK \\ (BK)^{\intercal} & P_{11}^{-1} \end{bmatrix} \geq 0 \Longleftrightarrow \begin{bmatrix} I_{n \times n} & 0 \\ 0 & P_{11} \end{bmatrix} \\ \begin{bmatrix} Q_1 & BK \\ (BK)^{\intercal} & P_{11}^{-1} \end{bmatrix} \begin{bmatrix} I_{n \times n} & 0 \\ 0 & P_{11} \end{bmatrix} \geq 0 \\ \overset{Y_1 = KP_1}{\Leftrightarrow} \begin{bmatrix} Q_1 & BY_1 \\ Y_1 B^{\intercal} & P_{11} \end{bmatrix} \geq 0 \end{gathered}$$

Also, by the same reasoning, fulfilling

$$(A - LC)^{\intercal} P_{22} (A - LC) < Q_2 \in \mathbb{R}^{2n \times 2n}$$

is equivalent to

$$\begin{gathered} \begin{bmatrix} Q_2 & A - LC \\ (A - LC)^{\intercal} & P_{22}^{-1} \end{bmatrix} \geq 0 \overset{Y_2 = P_2 L}{\Leftrightarrow} \\ \begin{bmatrix} Q_2 & AP_{22} - Y_2 C \\ (AP_{22} - Y_2 C)^{\intercal} & P_{22} \end{bmatrix} \geq 0 \end{gathered}$$

So, (22) becomes

$$\bar{w}_{22} > -2Q_1 - 2Q_2 + \lambda_{22} P_{22} > 0$$

By the same reasoning for $\bar{w}_{11}$ if

$$(A + BK)^{\intercal} P_{11} (A + BK) < Q_3$$

for some $Q_3 > 0$, then

$$\begin{gathered} \begin{bmatrix} Q_1 & A + BK \\ (A + BK)^{\intercal} & P_{11}^{-1} \end{bmatrix} \geq 0 \Leftrightarrow \\ \begin{bmatrix} Q_1 & AP_{11} + BY_1 \\ (AP_{11} + BY_1)^{\intercal} & P_{11} \end{bmatrix} \geq 0 \end{gathered}$$

which leads to

$$\bar{w}_{11} > -2Q_3 + \lambda_{11} P_{11} > 0$$

and for $2G^{\intercal} K^{\intercal} KG$ for some $Q_4 > 0$ we have

$$\begin{bmatrix} Q_4 & KG \\ G^{\intercal} K^{\intercal} & I_{2 \times 2} \end{bmatrix} \geq 0$$

Notice that these matrix inequalities for the fixed scalar parameters $\alpha, \lambda, \beta, \varepsilon_1, \varepsilon_2$ and ε_3 become to be LMIs. They can be solved using the MATLAB toolboxes LMItoolbox, SeDuMi and Yalmip. Our main optimization problem can be also solved using the following two-steps procedure:

first- we fix the scalar parameters $\alpha, \lambda, \beta, \varepsilon_1, \varepsilon_2$and ε_3 and solve the our problem with respect to the matrix variables which satisfy LMI- constraints.

second - for the found matrix variables Y_1, P_{11} and Y_2, P_{22} we solve our optimization problem only with respect to scalar parameters $\alpha, \lambda, \beta, \varepsilon_1, \varepsilon_2$and ε_3 (usually using a simple multidimensional grid search method).Iterating this process we finally find the solution $\alpha^*, \lambda^*, \varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^*$and $Y_1^* = KP_{11}^*, P_{11}^*$ and$Y_2^* = P_{22}^* L, P_{22}^*$ from which the optimal game matrices K^* and L^* can be found as

$$K^* = Y_1^* \left(P_{11}^* \right)^{-1}, \ L^* = \left(P_{22}^* \right)^{-1} Y_2^*$$

V. Numerical Examples

Take in (3) the sampling rate $\tau = 0.05$ and

$$\begin{gathered} A = \begin{bmatrix} 1 & -0.05 \\ 0 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 0.05 \\ 0 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 0 \end{bmatrix} \\ R_1 = \begin{bmatrix} 0 \\ 0.05 \end{bmatrix}, \ R = \begin{bmatrix} 0 \\ 0.025 \end{bmatrix}, \ \sigma = 0.5\sqrt{10} = 1.5811 \end{gathered}$$

The optimization procedure described above leads to

$$\begin{gathered} \alpha^* = 0.1599, \beta^* = 1.17, \lambda_1^* = 0.01, \lambda_2^* = 0.23 \\ \alpha_R^* = 0.021, \varepsilon_2^* = 0.41, \varepsilon_2^* = 2.76, \varepsilon_3^* = 3.22 \end{gathered}$$

so that the positive function $\varkappa$ is equal to

$$\begin{gathered} \varkappa(\lambda, \alpha) := \frac{1}{1 - \lambda^*} \left(Q_c + \lim_{n \to \infty} \sup \varepsilon_1^* \left\| F_n \right\|^2 + \tilde{U} \right) \\ = 2.2066, \ Q_c = \left(\alpha^* + \alpha_R^* \right) = 0.181 \end{gathered}$$

The matrix parameters P, K and L (obtained by the application of the suggested approach) realizing the robust output linear controller are as follows:

$$P^* = \begin{bmatrix} 0.1004 & 0.0083 & 0 & 0 \\ 0.0083 & 0.0522 & 0 & 0 \\ 0 & 0 & 0.0246 & 0.0062 \\ 0 & 0 & 0.0062 & 0.0454 \end{bmatrix}$$

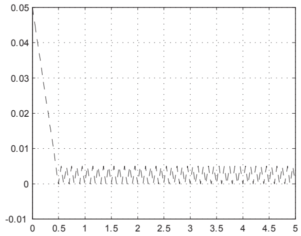
$$K^* = \begin{bmatrix} -0.3230 & -4.2294 \end{bmatrix}, \ L^* = \begin{bmatrix} 1.7980 \\ 0.0109 \end{bmatrix}$$

The fig 3 show the inventory production level x_1. and the similar behavior takes place for the state x_2 . It can be seen from these figures that the closed loop system has a *good performance* maintaining the control dynamics close to the state. The fig 4 shows that the trajectory of errors asymptotically converges to the corresponding ellipsoid. and shows that the trajectory of the inventory product level asymptotically converges to the corresponding ellipsoid.

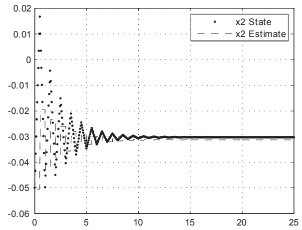
Introduce the cost function $h^0(y)$ defined by

$$h^0(y) = \frac{\ell_1}{2} \left[y - y^+ \right]_+^2 + \frac{\ell_2}{2} \left[-y \right]_+^2 \tag{23}$$

where the term $\left[y - y^+ \right]_+^2$ corresponds to the losses, related to an extra production storage, the term $[-y]_+^2$ reflects the losses due to a deficit and ℓ_1 , ℓ_2 are two nonnegative

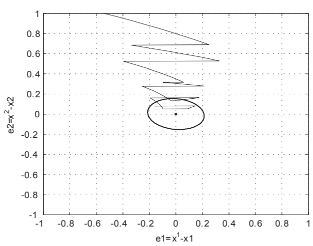


The inventory production level y_n.

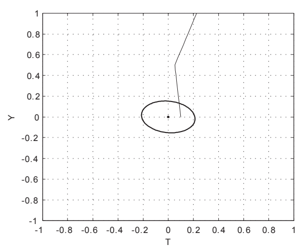


The states and the estimates x_2.

Fig. 2. Inventary production level and state estimates



The trajectory of errors e_1, e_2.



The trajectory of the inventory product.

Fig. 3. State estimate errors and inventory product trajectories

weighting parameters. To compare the quality of the obtained controller with K^* and L^*, consider 2 additional cases with K and L closed to K^*, L^* (referred below to as *Case 3*):

$$Case\ 1 : K_1 = \begin{bmatrix} -0.6574 & -8.6098 \end{bmatrix},\ L_1 = \begin{bmatrix} 4.6583 \\ 0.2821 \end{bmatrix}$$

$$Case\ 2 : K_2 = \begin{bmatrix} -0.9458 & -12.3861 \end{bmatrix},\ L_2 = \begin{bmatrix} 6.7014 \\ 0.0405 \end{bmatrix}$$

Fig.4 shows the cost function $h^0(y)$ (23)with $\ell_1 = \ell_2 = 4$.

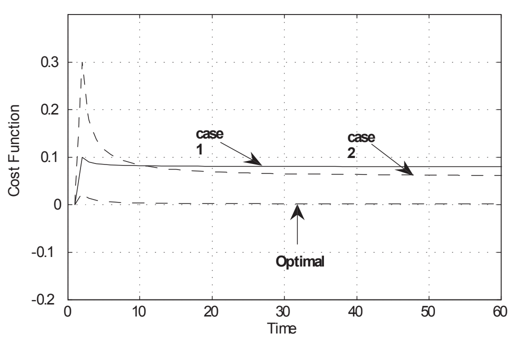


Fig. 4. The cost function $h^0(y)$.

One can see that selection $K = K^*$, $L = L^*$ leads to a better quality of the inventory production level.

REFERENCES

[1] H. Alazki and A. Poznyak, "*Probabilistic Analysis of Robust Attractive Ellipsoids for Quasi-Linear Discrete-Time Models*". In Proceedings of the 49th IEEE Conference on Decision and Control (CDC), Atlanta, GA, USA. 2010.

[2] M. Bensoubaya, A. Ferfera and A. Iggidr, "*Stabilization of Nonlinear Stochastic Discrete Systems*". Proceedings of the 38th Conference on Decision and Control, Phoenix, Arizona, pp. 3180-3181, 1999.

[3] D. Bertsekas, and I. Rhodes, "*On the minmax reachability of target set and target tubes"*. Automatic J.IFAC, 7: 233-247, 1971.

[4] F. Blanchini, "*Set invariance in control - a survey"*. Automatic J. IFAC, 35(11): 1747-1767, MA, 1999.

[5] S. Boyd, E. Feron and V. Balakrishnan, "*Linear Matrix Inequalities IN Systems and Control Theroy"*. SIAM, 2004.

[6] F.L. Chernousko, "*Ellipsoidal state estimation for dynamical systems Nonlinear Analysis*". Volume 63, Issues 5-7, Pages 872-879.querque, New Mexico, 2005.

[7] H.G. James, "*Production and Inventory Control Handbook*", McGraw-Hill, 1997.

[8] T.T Kalyan and V.R. Garrett, "*The theory and practice of revenue management*". Springer, 2005.

[9] A. Kurzhanski and I. Valyi, "*Ellipsoidal calculus for estimation and control"*. Birkhauser Boston, MA, 1997.

[10] K.S. Martin and M. David, "*Inventory control: theory and practice*", Prentice-Hall, 1962.

[11] K. Nagapal, J. Albedor and K. Poola, "*An LMI approch to peak-to-peak gain minmization: filtering and control"*. In Proceedings of the American Control Conference. Baltimor, Maryland, 1994.

[12] S.A. Nazin, B.T. Polyak and Topunov, M.V. "*Rejection of bounded exogenous disturbances by the method of invariant ellipsoids"*. Autom. Remote Control, no.3:467-486, 2007.

[13] B. Polyak, A.V. Nazin, M.V. Topunov and Nazin, S.A. "*Rejection of bounded disturbances via invariant ellipsoids technique*. in Proc. 45th IEEE Conf. Decision Contr., San Diego". USA, 1429-1434, 2006.

[14] E. L. Porteus, "*Stochastic inventory theory. In D. P. Heyman and M. J. Sobel, editors, Handbooks in OR and MS*", volume 2. Elsevier Science Publishers B.V., Amsterdam, The Netherlands, 1990.

[15] A.S. Poznyak, T.E. Duncan, B. Pasik-Duncan and V.G. Boltyansky, "*Robust stochastic maximum principle for multi-model worst case optimization*", INT. J. CONTROL, 75(13):1032-1048, 2002.

[16] A. Poznyak, A. Nazin and D. Murano, "*Observer Matrix Gain Optimazation for Stochastic Continuous Time Nonlinear Systems"*. Systems& Control Letters, 52(5):377-385, 2004.

[17] A. Poznyak, "*Advanced Mathematical Tools for Automatic Control Engineers Volume 2: Stochastic Techniques"*. Elsevier, Oxford-Amsterdam, 2009.

[18] M.g. Safonov, K. Goh and J.H. Ly, "*Control system synthesis via bilinear inequalities"*. Proceedings of the American Control Conference, Baltimore, MD, 1994.

[19] D.M. Stipanovic and Siljak, D.D. "*Robust Stability and Stabilization of Discrete-Time Nonlinear systems:The LMI Approach*". International Journal of Control, 74, 873–879, 2001.

[20] L.W.G. Strijbosch and J.J.A. Moors, "*Inventory control: the impact of unknown demand distribution*", Research Memorandum 770, Tilburg University, Faculty of Economics and Business Administration, 1998.

[21] M. Taksar and X. Zhou, "*Optimal risk and dividend control for a company with a debt liability*". Insurance: Math. Econom. 22, 105–122, 1998.

[22] O. Toshio, "*Dynamic management decision and stochastic control processes*", World Scientific, 1990.

[23] V.A. Ugrinovskii, "*Robust H_∞ Control in the Presence of Stochastic Uncertainty*". International Journal of Control, 71,2219-237, 1998.

[24] K. Zhou, K. Doyle and Glover, K. "*Robust and Optimal Control"*. Prentice Hall, Upper SaddelRiver, New York, 1996.

[25] X. Y. A Zhou, uni®ed treatment of maximum principle and dynamic programming in stochastic controls. Statistics and Stochastic Reports, 36, 137-161, 1991.

[26] P. H. Zipkin, "*Foundations of Inventory Management*". Irwin McGraw-Hill, 2000.