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Semidiscrete approximation schemes for LQR control of equations in thermoelasitcity

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***Abstract*— We consider finite dimensional semidiscrete approximation of an LQR control problem for a model in linear thermoelasticity. A test problem is constructed for which the exact solution of the associated algebraic Riccati equation is known. We can then determine the exact feedback functional gain, and compare convergence behavior of different semidiscrete Galerkin approximation schemes. Numerical results are presented.**

I. INTRODUCTION

Let us consider the following equations, which arise in the modeling of thermoelastic damping in flexible structures:

$$\begin{aligned} y_{tt}(t,x) &= y_{xx}(t,x) - \gamma\theta_x(t,x) + b(x)u(t) \\ \theta_t(t,x) &= \theta_{xx}(t,x) - \gamma y_{tx}(t,x), \end{aligned} \tag{1}$$

with initial conditions

$$y(0,x) = y_0(x),\ y_t(0,x) = v_0(x), \quad \theta(0,x) = \theta_0(x),$$

and boundary conditions

$$y(t,0) = y_x(t,1) = 0, \quad \theta_x(t,0) = \theta(t,1) = 0. \tag{2}$$

Here $y(t,x)$ represents displacement (longitudinal or transverse, depending upon the application) at time t and position x along the interval $[0,1]$, and $\theta(t,x)$ represents temperature at time t and position x. The small positive constant γ is the thermomechanical coupling parameter, and the function $b(x)$ characterizes a one dimensional distributed controller. We will consider an LQR control problem for these dynamics and will construct a quadratic cost functional for which the exact solution to the associated algebraic Riccati equation can be determined. With this exact solution we may also determine the exact feedback gain operator, and the related feedback functional gains. This constitutes the main contribution of the paper. Finally we use this test problem to numerically investigate the performance of certain semidiscrete approximation schemes.

A natural setting for approximation and control is to reformulate the dynamics as a Cauchy problem on the energy space. It is convenient to introduce the Sobolev spaces

$$H_L^1(0,1) = \{f \in L^2(0,1) :\ f' \in L^2(0,1),\ f(0) = 0\},$$

and

$$H_R^1(0,1) = \{f \in L^2(0,1) :\ f' \in L^2(0,1),\ f(1) = 0\}.$$

For the model (1), we define the energy space

$$X = H_L^1(0,1) \times L^2(0,1) \times L^2(0,1),$$

equipped with the energy norm defined by

$$\|(y,v,\theta)\|_X^2 = \int_0^1 |y'(x)|^2 + |v(x)|^2 + |\theta(x)|^2\, dx.$$

Next define the operator $\mathcal{A}$ on the domain

$$\begin{aligned} \operatorname{dom}\mathcal{A} = \quad & \big\{(y,v,\theta) \in X :\ y \in H^2(0,1),\ y'(1) = 0, \\ & \theta \in H^2(0,1) \cap H_R^1,\ \theta'(0) = 0, \\ & v \in H_L^1(0,1)\big\}, \end{aligned}$$

by

$$\mathcal{A}(y,v,\theta) = (v, y'' - \gamma\theta', \theta'' - \gamma v').$$

Also define the control operator $B : U \to X$ by $Bu = (0, b(x)u, 0)$, where $U = \mathbb{C}$ is a one dimensional control space. If we set

$$z(t) = (y(t,x), y_t(t,x), \theta(t,x)),$$

then the system (1)-(2) can be reformulated as the Cauchy problem

$$\begin{aligned} \frac{d}{dt}z(t) &= \mathcal{A}z(t) + Bu(t), \\ z(0) &= (y_0, v_0, \theta_0) \end{aligned} \tag{3}$$

evolving on the energy space X. It is known that $\mathcal{A}$ is the infinitesimal generator of an exponentially stable C_0-semigroup $T(t)$ on X (see e.g. [1], [2], [3], [4]). We observe that for all $z = (y,v,\theta) \in \operatorname{dom}\mathcal{A}$,

$$\begin{aligned} \operatorname{Re}\langle \mathcal{A}z, z\rangle_X &= \operatorname{Re}\left\{\int_0^1 (v'\overline{y'} + y''\overline{v}) - \gamma\theta'\overline{v} \right. \\ & \qquad \left. -\gamma v'\overline{\theta} - |\theta'|^2\, dx\right\} \\ &= \int_0^1 -|\theta'|^2\, dx \\ &\le 0, \end{aligned} \tag{4}$$

so by (4) $\mathcal{A}$ is dissipative.

We consider the LQR problem of minimizing the cost functional

$$J(u) = \int_0^\infty \langle Wz(t), z(t)\rangle_X + \|u(t)\|_U^2\, dt \tag{5}$$

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subject to dynamics governed by (3). Here $W : X \to X$ is a self-adjoint, nonnegative definite, bounded linear operator which we shall specify below. Under appropriate assumptions, it is known that the solution of the LQR problem (3),(5) is given in feedback form by

$$u(t) = -Kz(t), \qquad K = B^*\Pi,$$

where Π is the unique nonnegative definite solution of the algebraic Riccati equation

$$\Pi\mathcal{A} + \mathcal{A}^*\Pi - \Pi BB^*\Pi + W = 0. \tag{6}$$

In section II we show how to define W so that the exact solution Π of (6) is known, and hence the exact gain K is known.

II. Exact solution of Riccati equation

In this section we construct an operator W for which the exact solution to the algebraic Riccati equation (6) can be determined. To begin, one can check that the adjoint operator $\mathcal{A}^*$ is defined on the domain

$$\operatorname{dom}\mathcal{A}^* = \operatorname{dom}\mathcal{A}$$

by

$$\mathcal{A}^*(f,g,h) = (-g, -f'' + \gamma h', h'' + \gamma g').$$

Because of the nature of the boundary conditions in the model, it will be convenient to introduce the following operators. Define the operator D_L on the domain $\operatorname{dom} D_L = H_L$ by

$$D_L f = f',$$

and the operator D_R on the domain $\operatorname{dom} D_R = H_R$ by

$$D_R f = -f'.$$

Also define the operator $S = D_L D_R$ on the domain

$$\operatorname{dom} S = \{f \in H^2(0,1) : \ f \in H_R, \ f' \in H_L\},$$

and the operator $T = D_R D_L$ on the domain

$$\operatorname{dom} T = \{f \in H^2(0,1) : \ f \in H_L, \ f' \in H_R\}.$$

Thus $Tf = -f''$ with boundary conditions associated with y in the thermoelastic model, and $Sf = -f''$ with boundary conditions associated with θ in the model. All of these are closed, densely defined, linear operators on $L^2(0,1)$, with bounded inverses. Note that $D_L^* = D_R$ and $D_R^* = D_L$, so T and S are self-adjoint, positive definite operators. Also observe that

$$\mathcal{A}(y,v,\theta) = (v, -Ty + \gamma D_R\theta, -S\theta - \gamma D_L v)$$

and

$$\mathcal{A}^*(f,g,h) = (-g, Tf - \gamma D_R h, -Sh + \gamma D_L g).$$

The operators D_L, D_R, S, T, and their bounded inverses are used in our construction and subsequent analysis. To proceed, define the linear operator $W : X \to X$ by

$$W(f,g,h) = (w_1, w_2, w_3),$$

where

$$\begin{aligned}
w_1 &= c\gamma^2 T^{-1}T^{-1}b(x) + 2\gamma T^{-1}D_L^{-1}h + 2\gamma^2 T^{-1}f \\
w_2 &= cT^{-1}b(x) \\
w_3 &= c\gamma S^{-1}D_R^{-1}b(x) + 2S^{-1}h + 2\gamma D_R^{-1}f.
\end{aligned}$$

Here $c = c(f,g,h)$ is a bounded linear functional on X defined by

$$c = \int_0^1 b(x)[T^{-1}g + \gamma T^{-1}D_L^{-1}h + \gamma^2 T^{-1}f]\,dx. \tag{7}$$

In the interest of space we leave out the straightforward but lengthy calculation which shows that W is self-adjoint on X (but see below the similar argument where we show that Π is self-adjoint). For this and other calculations it is useful to observe that the inner product on X can be written as

$$\begin{aligned}
&\langle (f,g,h),(y,v,\theta)\rangle_X \\
&\quad= \int_0^1 [f'(x)\overline{y'(x)} + g(x)\overline{v(x)} + h(x)\overline{\theta(x)}]\,dx \\
&\quad= \int_0^1 [D_L f\overline{D_L y} + g(x)\overline{v(x)} + h(x)\overline{\theta(x)}]\,dx.
\end{aligned}$$

We claim that W is nonnegative definite. To see this, for all $(f,g,h) \in X$ we have

$$\begin{aligned}
&\langle W(f,g,h),(f,g,h)\rangle_X = \\
&\quad\int_0^1 D_L[c\gamma^2 T^{-1}T^{-1}b(x) + 2\gamma T^{-1}D_L^{-1}h \\
&\qquad\qquad +2\gamma^2 T^{-1}f]D_L\overline{f}\,dx \\
&\quad+\int_0^1 cT^{-1}b(x)\overline{g}\,dx \\
&\quad+\int_0^1 [c\gamma S^{-1}D_R^{-1}b(x) + 2S^{-1}h + 2\gamma D_R^{-1}f]\overline{h}\,dx.
\end{aligned}$$

Since $D_L^* = D_R$, we have in general

$$\begin{aligned}
\int_0^1 D_L T^{-1} y D_L\overline{f}\,dx &= \int_0^1 D_L D_L^{-1}D_R^{-1} y D_L\overline{f}\,dx \\
&= \int_0^1 D_R^{-1} y D_L\overline{f}\,dx \\
&= \int_0^1 D_L^* D_R^{-1} y\overline{f}\,dx \\
&= \int_0^1 y\overline{f}\,dx.
\end{aligned} \tag{8}$$

So we may continue from above and get

$$
\begin{aligned}
&\langle W(f,g,h),(f,g,h)\rangle_X = \\
&\quad \int_0^1 [c\gamma^2 T^{-1} b(x) + 2\gamma D_L^{-1} h + 2\gamma^2 f] D_L \overline{f}\, dx \\
&\quad + \int_0^1 c T^{-1} b(x) \overline{g}\, dx \\
&\quad + \int_0^1 [c\gamma S^{-1} D_R^{-1} b(x) + 2 S^{-1} h + 2\gamma D_R^{-1} f] \overline{h}\, dx \\
&\quad = c \int_0^1 b(x) [\gamma^2 T^{-1} \overline{f} + T^{-1} \overline{g} + \gamma D_L^{-1} S^{-1} \overline{h}]\, dx \\
&\qquad\qquad + 2\gamma^2 \int_0^1 |f|^2\, dx + 4\gamma \mathrm{Re} \int_0^1 D_L^{-1} h \overline{f} \\
&\qquad\qquad + 2 \int_0^1 S^{-1} h \overline{h}\, dx.
\end{aligned}
$$

Now use the fact that $D_L^{-1} S^{-1} = T^{-1} D_L^{-1}$, so the first integral is just $\overline{c}$, and that $\int_0^1 S^{-1} h \overline{h}\, dx = D_R^{-1} D_L^{-1} h \overline{h}\, dx = \int_0^1 D_L^{-1} h D_L^{-1} \overline{h}\, dx$. We continue from above and get

$$
\begin{aligned}
&\langle W(f,g,h),(f,g,h)\rangle_X = \\
&\quad c\overline{c} + 2\gamma^2 \int_0^1 |f|^2\, dx + 4\gamma \mathrm{Re} \int_0^1 D_L^{-1} h \overline{f} \\
&\qquad + 2 \int_0^1 |D_L^{-1} h|^2\, dx \\
&\quad = |c|^2 + 2 \int_0^1 |\gamma f + D_L^{-1} h|^2\, dx \\
&\quad \geq 0.
\end{aligned}
$$

Thus W is nonnegative definite, and this choice of W defines the cost functional (5) and the algebraic Riccati equation (6). We claim that the exact solution of (6) is given by the operator $\Pi : X \to X$ defined by

$$\Pi(f,g,h) = (\pi_1, \pi_2, \pi_3),$$

where

$$
\begin{aligned}
\pi_1 &= [I + \gamma^2(1+\gamma^2) T^{-1}] T^{-1} f + \gamma^2 T^{-1} T^{-1} g \\
&\qquad + \gamma(1+\gamma^2) T^{-1} T^{-1} D_L^{-1} h \\
\pi_2 &= \gamma^2 T^{-1} f + T^{-1} g + \gamma T^{-1} D_L^{-1} h \\
\pi_3 &= \gamma(1+\gamma^2) S^{-1} D_R^{-1} f + \gamma S^{-1} D_R^{-1} g \\
&\qquad + (1+\gamma^2) S^{-1} S^{-1} h
\end{aligned}
$$

(While this operator seems unwieldy, we shall see later that the functional gains are quite accessible). Let us first check that Π is self-adjoint. For all $(f,g,h),(y,v,\theta) \in X$, we have

$$\langle \Pi(f,g,h),(y,v,\theta)\rangle_X = \int_0^1 [D_L \pi_1 D_L \overline{y} + \pi_2 \overline{v} + \pi_3 \overline{\theta}]\, dx.$$

We collect the f, g, and h terms together. For example, the first f term is

$$
\begin{aligned}
\int_0^1 D_L T^{-1} f D_L \overline{y}\, dx &= \int_0^1 D_L D_L^{-1} D_R^{-1} f D_L \overline{y}\, dx \\
&= \int_0^1 D_R^{-1} f D_L \overline{y}\, dx \\
&= \int_0^1 f (D_R^{-1})^* D_L \overline{y}\, dx \\
&= \int_0^1 f \overline{y}\, dx \\
&= \int_0^1 D_L^{-1} D_L f \overline{y}\, dx \\
&= \int_0^1 D_L f D_R^{-1} \overline{y}\, dx \\
&= \int_0^1 D_L f D_L D_L^{-1} D_R^{-1} \overline{y}\, dx \\
&= \int_0^1 D_L f D_L T^{-1} \overline{y}\, dx.
\end{aligned}
$$

Similar manipulations occur for the other terms, which we list without details of the derivations. The remaining f terms are

$$
\begin{aligned}
&\int_0^1 D_L \gamma^2 (1+\gamma^2) T^{-1} T^{-1} f D_L \overline{y}\, dx = \\
&\qquad \int_0^1 D_L f D_L \gamma^2 (1+\gamma^2) T^{-1} T^{-1} \overline{y}\, dx
\end{aligned}
$$

$$\int_0^1 \gamma^2 T^{-1} f \overline{v}\, dx = \int_0^1 D_L f D_L \gamma^2 T^{-1} T^{-1} \overline{v}\, dx$$

$$
\begin{aligned}
&\int_0^1 \gamma(1+\gamma^2) S^{-1} D_R^{-1} f \overline{\theta}\, dx = \\
&\qquad \int_0^1 D_L f D_L \gamma (1+\gamma^2) T^{-1} T^{-1} D_L^{-1} \overline{\theta}\, dx.
\end{aligned}
$$

The g terms are

$$\int_0^1 D_L \gamma^2 T^{-1} T^{-1} g D_L \overline{y}\, dx = \int_0^1 g \gamma^2 T^{-1} \overline{y}\, dx$$

$$\int_0^1 T^{-1} g \overline{v}\, dx = \int_0^1 g T^{-1} \overline{v}\, dx$$

$$
\begin{aligned}
\int_0^1 \gamma S^{-1} D_R^{-1} g \overline{\theta}\, dx &= \int_0^1 g \gamma D_L^{-1} S^{-1} \overline{\theta}\, dx \\
&= \int_0^1 g \gamma T^{-1} D_L^{-1} \overline{\theta}\, dx.
\end{aligned}
$$

The h terms are

$$\int_0^1 D_L\gamma(1+\gamma^2)T^{-1}T^{-1}D_L^{-1}hD_L\overline{y}\,dx$$
$$= \int_0^1 h\gamma(1+\gamma^2)D_R^{-1}T^{-1}\overline{y}\,dx$$
$$= \int_0^1 h\gamma(1+\gamma^2)S^{-1}D_R^{-1}\overline{y}\,dx$$

$$\begin{aligned}\int_0^1 \gamma T^{-1}D_L^{-1}h\overline{v}\,dx &= \int_0^1 h\gamma D_R^{-1}T^{-1}\overline{v}\,dx \\ &= \int_0^1 h\gamma S^{-1}D_R^{-1}\overline{v}\,dx\end{aligned}$$

$$\int_0^1 (1+\gamma^2)S^{-1}S^{-1}h\overline{\theta}\,dx = \int_0^1 h(1+\gamma^2)S^{-1}S^{-1}\overline{\theta}\,dx.$$

It follows that

$$\begin{aligned}&\langle \Pi(f,g,h),(y,v,\theta)\rangle_X = \\ &\quad\int_0^1 D_L f D_L[(I+\gamma^2(1+\gamma^2)T^{-1})T^{-1}\overline{y} \\ &\qquad +\gamma^2 T^{-1}T^{-1}\overline{v}+\gamma(1+\gamma^2)T^{-1}T^{-1}D_L^{-1}\overline{\theta}]\,dx \\ &\quad+\int_0^1 g[\gamma^2 T^{-1}\overline{y}+T^{-1}\overline{v}+\gamma T^{-1}D_L^{-1}\overline{\theta}]\,dx \\ &\quad+\int_0^1 h[\gamma(1+\gamma^2)S^{-1}D_R^{-1}\overline{y}+\gamma S^{-1}D_R^{-1}\overline{v} \\ &\qquad +(1+\gamma^2)S^{-1}S^{-1}\overline{\theta}]\,dx \\ &\quad= \langle (f,g,h),\Pi(y,v,\theta)\rangle_X.\end{aligned}$$

Thus Π is self-adjoint. A lengthy but straightforward calculation, which we omit, shows that Π is nonnegative definite. It remains to verify that Π solves the algebraic Riccati equation (6). To see this, first note that $B^*: X \to U$ is defined by

$$B^*(f,g,h) = (\int_0^1 b(x)\overline{g(x)}\,dx.$$

Consequently $BB^*: X \to X$ is given by

$$BB^*(f,g,h) = (0, b(x)\int_0^1 b(x)\overline{g(x)}\,dx, 0).$$

It follows that

$$\Pi BB^*\Pi(f,g,h) = c(\gamma^2 T^{-1}T^{-1}b, T^{-1}b, \gamma S^{-1}D_R^{-1}b), \tag{9}$$

where c is defined by (7). It is straightforward to check that for all $(f,g,h) \in \operatorname{dom}\mathcal{A}$,

$$\begin{aligned}&(-\Pi\mathcal{A}-\mathcal{A}^*\Pi)(f,g,h) = \\ &\quad(2\gamma T^{-1}D_L^{-1}h+2\gamma^2 T^{-1}f, 0, 2S^{-1}h+2\gamma D_R^{-1}f).\end{aligned} \tag{10}$$

From (9), (10), and the definition of W, it follows that

$$\Pi BB^*\Pi - \Pi\mathcal{A} - \mathcal{A}^*\Pi - W = 0 \tag{11}$$

on $\operatorname{dom}\mathcal{A}$. Since the operators W, $-\Pi\mathcal{A}-\mathcal{A}^*\Pi$, and $\Pi BB^*\Pi$ are all bounded and $\operatorname{dom}\mathcal{A}$ is dense in X, (11) extends to hold on all of X. Thus Π is the solution to (6).

With this solution Π we may define the exact gain operator

$$K = B^*\Pi : X \to U.$$

Since the control space $U = \mathbb{C}$ is one-dimensional, the gain operator is in fact a bounded linear functional. Thus by the Riesz representation theorem there exists $(k_1,k_2,k_3) \in X$ such that

$$\begin{aligned}K(f,g,h) &= \langle (k_1,k_2,k_3),(f,g,h)\rangle_X \\ &= \int_0^1 k_1'(x)\overline{f'(x)}\,dx + \int_0^1 k_2(x)\overline{g(x)}\,dx \\ &\qquad + \int_0^1 k_3(x)\overline{h(x)}\,dx.\end{aligned} \tag{12}$$

We refer to the functions $k_1 \in H_L^1(0,1)$, $k_2, k_3 \in L^2(0,1)$ as the feedback functional gains. To calculate these functional gains, observe that

$$\begin{aligned}K(f,g,h) &= B^*\Pi(f,g,h) \\ &= \int_0^1 b(x)[\gamma^2 T^{-1}\overline{f}+T^{-1}\overline{g} \\ &\qquad +\gamma T^{-1}D_L^{-1}\overline{h}]\,dx.\end{aligned} \tag{13}$$

It follows by comparing (12) and (13) that the exact feedback functional gains are

$$\begin{aligned}k_1(x) &= \gamma^2 T^{-1}T^{-1}b(x) \\ k_2(x) &= T^{-1}b(x) \\ k_3(x) &= \gamma D_R^{-1}T^{-1}b(x).\end{aligned}$$

This construction is possible for any choice of the control function $b(x)$. With these exact solutions, it is possible to test convergence behavior for semidiscrete approximation schemes applied to the original LQR problem. In the next section we present preliminary numerical results in this direction.

III. Semidiscrete approximation of LQR problem

In this section we give very preliminary numerical results from a semidiscrete approximation scheme for the LQR problem (3), (5). We use cubic B-splines to define basis functions on which to construct Galerkin approximations, and then solve the resulting finite dimensional LQR problem. From this we get finite dimensional approximations to the functional feedback gains, which we can compare with the exact gains given above. Detailed discussion of the general methodology for matrix representations of such LQR approximation schemes can be found in [5] and [6], for example. If we recall that $k_2(x) = T^{-1}b(x)$, then $k_2''(x) = -b(x)$. In the example we present here, we take $b(x) \equiv 1$ and give numerical results for approximations to $-k_2'' = b(x)$.

To proceed, for each N define a partition of $[0,1]$ by $x_j = j/N$ for $j = 0, 1, \ldots, N$. From the usual set of $N+3$ cubic B-spline basis functions for this partition, let $\{\beta_j^L(x)\}_{j=1}^{N+2}$ be a set of basis functions which satisfy the boundary condition (for example, see [7])

$$\beta_j^L(0) = 0,$$

and let $\{\beta_j^R(x)\}_{j=1}^{N+2}$ be a set of basis functions which satisfy the boundary condition

$$\beta_j^R(1) = 0.$$

Define

$$X_L^N = \text{span}\,\{\beta_j^L\}_{j=1}^{N+2} \quad X_R^N = \text{span}\,\{\beta_j^R\}_{j=1}^{N+2},$$

and observe that $X_L^N \subset H_L^1$ and $X_R^N \subset H_R^1$. Define

$$X^N = X_L^N \times X_L^N \times X_R^N,$$

and observe that $X^N \subset V = H_L^1 \times H_L^1 \times H_R^1$. For $j = 1, \ldots, N+2$, define $e_j = (\beta_j^L(x), 0, 0)$, $e_{N+2+j} = (0, \beta_j^L(x), 0)$, and $e_{2(N+2)+j} = (0, 0, \beta_j^R(x))$. Thus $\{e_j\}_{j=1}^{3(N+2)}$ is a basis for X^N. On the space V define the sesquilinear form $\sigma : V \times V \to \mathbb{C}$ by

$$\sigma((y,v,\theta),(f,g,h)) = \int_0^1 [v'\overline{f}' - y'\overline{g}' - \gamma\theta'\overline{g} - \theta'\overline{h} - \gamma v'\overline{h}]\,dx.$$

The form σ is related to $\mathcal{A}$ by

$$\sigma((y,v,\theta),(f,g,h)) = \langle \mathcal{A}(y,v,\theta),(f,g,h)\rangle_X$$

for all $(y,v,\theta) \in \text{dom}\,\mathcal{A}$, $(f,g,h) \in V$. The Galerkin method is used with the form σ to define the finite dimensional operators $\mathcal{A}^N : X^N \to X^N$ by

$$\langle \mathcal{A}^N z, \tilde{z}\rangle_X = \sigma(z, \tilde{z})$$

for all $z, \tilde{z} \in X^N$. We do not provide details, but it can be shown that one obtains Trotter-Kato type convergence for both the semigroup $T(t)$ and its adjoint. That is,

$$\begin{aligned} e^{\mathcal{A}^N t} &\to T(t) \\ e^{(\mathcal{A}^N)^* t} &\to T^*(t) \end{aligned}$$

in the Trotter-Kato sense. For approximation of an LQR problem as we are doing here, typical convergence theorems (for the feedback control and the gain) generally require this type of convergence as well as some sort of uniform exponential stability condition (for example, see [8], [9], [10]). It turns out there is numerical evidence that this scheme lacks any reasonable uniform preservation of stability, similar to the example in [11]. This lack of uniform stability manifests itself in Figure 1, which shows a kind of weak convergence in the approximation of the second derivative of the feedback functional gain $k_2(x)$.

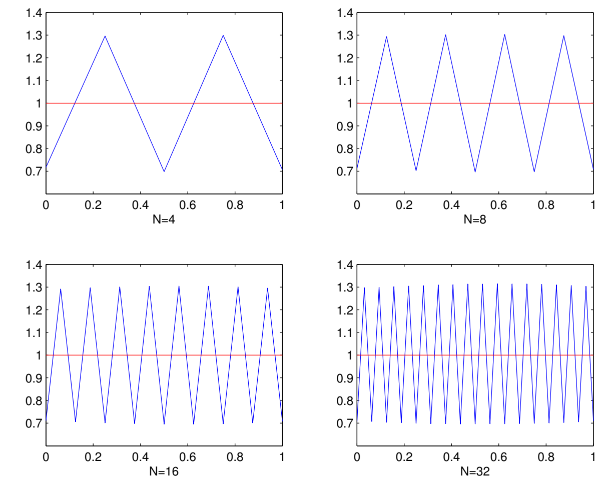

Fig. 1. Functional gain - approximation of $-k_2''(x) \equiv 1$

Similar behavior (weak convergence of functional gain approximations) was seen for examples in delay equations in [12], [13], [14]. In [14] it was seen that using a different norm for the Galerkin construction could mitigate the problem of weak convergence, and it remains an issue for future research to apply this strategy to thermoelastic models.

IV. Conclusion

We have considered renorming and approximation issues for an LQR control problem with dynamics governed by a model in linear thermoelasticity. A cost functional was constructed for which the exact solution of the corresponding algebraic Riccati equation can be found. Consequently the exact feedback gain and functional gain can be determined and used to test behavior of approximation schemes. Preliminary numerical results were given.

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