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Stability and Invariance Analysis of Uncertain PWA Systems Based on Linear Programming

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Abstract— **This paper analyzes stability of discrete-time uncertain piecewise-affine systems whose dynamics are defined on a bounded set $\mathcal{X}$ that is not necessarily invariant. The objective is to prove the uniform asymptotic stability of the origin and to find an invariant domain of attraction. This goal is attained by defining a suitable extended dynamics (which is partially fictitious), and by using a numerical procedure based on linear programming. The theoretical results are based on the definition of a piecewise-affine, possibly discontinuous, Lyapunov function.**

I. Introduction

In the last decade, the interest in piecewise affine (PWA) systems has increased, due to their ability to represent a useful modeling framework for hybrid systems and to approximate nonlinear systems [1], [2].

Analyzing the stability of PWA systems is fundamental to describe the properties of an autonomous hybrid system, or to check a-posteriori the stability of a given closed-loop system [3], [4]. In particular, stability analysis becomes fundamental when a PWA control law is synthesized without a-priori guarantees of stability. This can occur, for instance, when explicit model predictive control (MPC) laws [5], are approximated in order to reduce their complexity [6], [7].

The most widely used methods for stability analysis of discrete-time PWA systems are based on piecewise quadratic (PWQ) Lyapunov functions [8]. Such methods rely on the solution of a semi-definite program to get a stability certificate. As highlighted in [7], the search for a PWQ Lyapunov function can be overly conservative, even with the use of the so-called S-procedure [9]. A valid alternative are PWA Lyapunov functions, that are calculated by solving a linear program (LP) [10]. Of course, other types of Lyapunov functions can be used for the same purpose, such as piecewise polynomial Lyapunov functions [11].

In most of the literature the considered set where the PWA system is defined is assumed invariant, because, as remarked in [10], the notion of stability has no practical relevance if the state trajectory exits the defined set of states. More technically, the decay condition for the Lyapunov function associated to a discrete-time system cannot be defined in a set that is not invariant. However, in many cases the system to be analyzed is not defined in an invariant set. In this case, a possible approach is to perform a reachability analysis to find the maximum positively invariant set (see [12], [13, Chap. 4-5] and the references therein) to establish, using a recursive procedure, an invariant subset of the given set. However, this procedure can lead to very involved solutions due to the exponential complexity of reachability analysis, and in many cases searching the maximum invariant set is an undecidable problem. An alternative solution is proposed in [14], where an invariant set is determined a-posteriori, permitting the stability analysis of systems defined in non-invariant sets.

Another important issue is stability analysis in the presence of disturbances, which consists of determining if the state will converge to the origin despite parametric uncertainties affecting the process. This problem was almost ignored in the literature, mostly due to the complexity of uncertain switching systems, and the focus has been mainly only on nominal stability. Only some classical results appeared in case of linear parameter varying systems [13, Chap. 7], for linear switched systems [15], where quadratic stabilizability is analyzed in case of two discrete states and polytopic parametric uncertainties, and in [16], where the synthesis of switching control laws is tackled, assuring that the state is ultimately bounded within a given set, in case of both parametric uncertainties and external disturbances.

This paper proposes a stability analysis framework for discrete-time PWA systems subject to parametric uncertainties which are unknown but bounded and defined in polytopic sets, extending the solution proposed in [14] for systems without uncertainties. The proposed method is based on the use of PWA Lyapunov functions synthesized by linear programming (LP). The system dynamics are defined only in a closed polytopic region, which is not necessarily required to be invariant. The proposed method can determine an invariant subset of the region where the system dynamics are defined, exploiting an extended system with partially fictitious dynamics. The invariant subset (region of attraction) for the original system is calculated a-posteriori based on the definition of the PWA Lyapunov function. Finally, discontinuities on the boundaries of the polytopic sets are tackled for both the system dynamics and the PWA Lyapunov function, in order to broaden the range of applicability of the proposed approach and to reduce the conservativeness due to the imposition of continuity. The presence of discontinuities, however, requires additional attention on technical conditions [17].

This paper is organized as follows. After the preliminary concepts introduced in Section II, Section III introduces the

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class of considered uncertain PWA systems. Section IV is devoted to the one-step reachability analysis of the system, while the main results on the analysis of the extended system are formulated in Section V. The region of attraction is obtained in Section VI, where the analysis of the original system is performed. Section VII shows simulation examples, and a few conclusions are gathered in Section VIII.

II. Basic notations and definitions

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the sets of reals, non-negative reals, integers and non-negative integers, respectively. Given a countable set $\mathcal{S}$, $\mathrm{card}(\mathcal{S})$ denotes its cardinality. The symbol $\|\cdot\|$ represents any p-norm of a vector. Given a discrete-time signal $v : \mathbb{Z}_+ \to \mathbb{R}^{n_v}$, the sequence of the values of v from the zero instant to the k-th instant is denoted by $\mathbf{v}_{[k]}$. Given a set $\mathcal{A} \subseteq \mathcal{R}^n$, its interior is denoted by $int(\mathcal{A})$, its closure by $\bar{\mathcal{A}}$, and its convex hull by $conv(\mathcal{A})$. If $\mathcal{A}$ is a *polyhedron*, the set of the vertices of $\bar{\mathcal{A}}$ is denoted by *vert*$(\bar{\mathcal{A}})$. A bounded polyhedron is called *polytope*. A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is called *$\mathcal{K}$-function* if it is continuous, positive definite, and strictly increasing. A function $\phi : \mathbb{R}_+ \times \mathbb{Z}_+ \to \mathbb{R}_+$ is a *$\mathcal{KL}$-function* if, for each fixed $k \geq 0$, $\phi(\cdot, k)$ is a $\mathcal{K}$ function, for each fixed $c \geq 0$, $\phi(c, \cdot)$ is decreasing, and $\phi(c, k) \to 0$ as $k \to \infty$.

Consider a generic discrete-time nonlinear system

$$x(k+1) = f(x(k), w(k)) \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, while the input $w \in \mathcal{W} \subset \mathbb{R}^q$ represents all the model uncertainties, $\mathcal{W}$ being a compact set.

Definition 1 (One-step reachable set): Given a set $\mathcal{X} \subset \mathbb{R}^n$ and system dynamics (1), the one step reachable set from $\mathcal{X}$ is

$$\mathcal{R}(\mathcal{X}) \triangleq \{y \in \mathbb{R}^n : y = f(x, w),\ w \in \mathcal{W},\ x \in \mathcal{X}\}$$

□

Definition 2 (RPI set): A set $\mathcal{F} \subset \mathbb{R}^n$ is called *robustly positively invariant* (RPI) with respect to dynamics (1) if, for all $x \in \mathcal{F}$ and all $w \in \mathcal{W}$, $f(x, w) \in \mathcal{F}$. □

Definition 3 (Uniform asymptotic stability): Given a set $\mathcal{X} \subseteq \mathbb{R}^n$ with $0 \in \mathcal{X}$, system (1) is uniformly asymptotically stable in $\mathcal{X}$ (UAS($\mathcal{X}$)) if there exists a $\mathcal{KL}$-function ϕ such that, for all the initial conditions $x(0) \in \mathcal{X}$ and for all the sequences $\mathbf{w}_{[k]}$ with $w(i) \in \mathcal{W}$, $i = 0, ..., k$, $\|x(k)\| \leq \phi(\|x(0)\|, k)$, for all $k \in \mathbb{Z}_+$. □

III. Problem formulation

Consider the autonomous discrete-time uncertain PWA system

$$x(k+1) = A_i(w(k))x(k) + a_i(w(k)) \tag{2}$$

where $x(k) \in \mathcal{X}_i \subset \mathbb{R}^n$, $w(k) \in \mathcal{W} \subset \mathbb{R}^q$, with

$$A_i(w) \triangleq A_{i,0} + \sum_{r=1}^{q} A_{i,r} w_r \tag{3}$$

$$a_i(w) \triangleq a_{i,0} + \sum_{r=1}^{q} a_{i,r} w_r \tag{4}$$

$$\mathcal{W} \triangleq \left\{ w \in \mathbb{R}^q : \sum_{r=1}^{q} w_r = 1,\ w_r \geq 0 \right\} \tag{5}$$

$A_{i,r} \in \mathbb{R}^{n \times n}$, $a_{i,r} \in \mathbb{R}^n$, with $r = 0, ..., q$, and $k \in \mathbb{Z}_+$. The sets $\mathcal{X}_i$, $i \in \mathcal{I} \triangleq \{1, ..., s\}$, are polytopes such that $int(\mathcal{X}_i) \neq \emptyset$, $\mathcal{X}_i \cap \mathcal{X}_j = \emptyset$, $\forall\, i, j \in \mathcal{I}$ with $i \neq j$, and such that $\mathcal{X} \triangleq \bigcup_{i=1}^{s} \mathcal{X}_i$ is a closed polytope. The interior of each set $\mathcal{X}_i$ is defined as

$$int(\mathcal{X}_i) \triangleq \{x : H_i x < h_i\},\ i \in \mathcal{I} \tag{6}$$

where H_i and h_i are a constant matrix and a constant vector, respectively, of suitable dimensions. The sets $\mathcal{X}_i$ can be open, closed, or neither open nor closed.

Note that dynamics (2) may not be continuous with respect to x on the boundaries of the sets $\mathcal{X}_i$.

Assumption 1: Given the PWA system (2), there exists an index $i \in \mathcal{I}$ such that $0 \in$ *vert*$(\bar{\mathcal{X}}_i)$, $0 \in int(\mathcal{X})$. □

Assumption 1 can be always satisfied. In fact, if the origin is not on a vertex of any polyhedron $\mathcal{X}_i$, it is always possible to further partition $\mathcal{X}$ to obtain new sets $\mathcal{X}_i$ which fulfill the assumption. Note also that the state trajectories may not be persistent in time, since $\mathcal{X}$ is not necessarily an RPI set, and the dynamics is not defined outside $\mathcal{X}$.

This paper addresses the following problem: given the uncertain PWA system (2) only defined in the compact set $\mathcal{X}$ (which is not necessarily an RPI set), prove the asymptotical stability of the origin and find an RPI subset $\mathcal{P} \subseteq \mathcal{X}$ of its domain of attraction.

IV. Reachability analysis

A. One-step reachability analysis

Since the set $\mathcal{X}$ is not assumed to be RPI with respect to dynamics (2), we must take into account that the trajectories may possibly leave $\mathcal{X}$, and be therefore defined only on a finite time interval, $k \in [0, k_{max}]$. Define the one-step reachable set from $\mathcal{X}$

$$\mathcal{R}(\mathcal{X}) \triangleq \{A_i(w)x + a_i(w) :\ w \in \mathcal{W},\ x \in \mathcal{X}_i,\ i \in \mathcal{I}\}$$

and

$$\mathcal{R}_\cup(\mathcal{X}) \triangleq \mathcal{R}(\mathcal{X}) \cup \mathcal{X} \tag{7}$$

which represents an extension of $\mathcal{X}$, including all the state values that can be reached in one time step starting from $\mathcal{X}$. The set $\mathcal{R}(\mathcal{X})$ can be computed as the union of the one-step reachable sets from all the $\mathcal{X}_i$, defined as

$$\mathcal{R}(\mathcal{X}_i) \triangleq \{A_i(w)x + a_i(w),\ w \in \mathcal{W}, x \in \mathcal{X}_i\}$$

which is not a convex set in general. Note that the terms in (3) and (4) can be equivalently expressed as

$$A_i(w) = \sum_{r=1}^{q} (A_{i,0} + A_{i,r})\, w_r \triangleq \sum_{r=1}^{q} \tilde{A}_{i,r} w_r$$
$$a_i(w) = \sum_{r=1}^{q} (a_{i,0} + a_{i,r})\, w_r \triangleq \sum_{r=1}^{q} \tilde{a}_{i,r} w_r$$

Relying on the results in [13, Chap. 6], we can compute the convex hulls of the sets $\mathcal{R}(\mathcal{X}_i)$ as

$$\begin{aligned} &conv\,(\mathcal{R}(\mathcal{X}_i)\} \\ &\quad = conv\left(\tilde{A}_{i,r} v_{i,h} + \tilde{a}_{i,r},\ r = 1, ..., q_i,\ h = 1, ..., m_i\right) \end{aligned}$$

where $v_{i,h}$ represents each of the m_i vertices of $\bar{\mathcal{X}}_i$. Let

$$\tilde{\mathcal{R}}_{\cup}(\mathcal{X}) \triangleq \bigcup_{i=1}^{s} \left(conv\,(\mathcal{R}(\mathcal{X}_i))\right) \cup \mathcal{X} \supseteq \mathcal{R}_{\cup}(\mathcal{X}) \tag{8}$$

be an over-approximation of $\mathcal{R}_{\cup}(\mathcal{X})$.

B. Fake dynamics and extended system

As dynamics (2) is not defined outside $\mathcal{X}$, the proposed strategy consists in defining a "fake" dynamics on $\tilde{\mathcal{R}}_{\cup}(\mathcal{X}) \setminus \mathcal{X}$, analogously to [14], but for uncertain systems. Let $\mathcal{X}_H \supseteq \tilde{\mathcal{R}}_{\cup}(\mathcal{X})$ be the bounding box of $\tilde{\mathcal{R}}_{\cup}(\mathcal{X})$, i.e., the smallest closed hyper-rectangle containing $\tilde{\mathcal{R}}_{\cup}(\mathcal{X})$, and consider the dynamics

$$x(k+1) = \rho x(k),\ \text{if } x(k) \in \mathcal{X}_E \triangleq \mathcal{X}_H \setminus \mathcal{X} \tag{9}$$

where $\rho \in [0, 1)$. The region $\mathcal{X}_E$ can be divided into convex polyhedral regions as in [5, Th. 3]. As a result, new regions $\mathcal{X}_i$, $i = s+1, ..., \tilde{s}$, are created. Let $\tilde{\mathcal{I}} \triangleq \{1, ..., \tilde{s}\}$. The dynamics of the extended system on $\mathcal{X}_H$ is

$$x(k+1) = \begin{cases} A_i(w)x(k) + a_i(w) & \text{if } x(k) \in \mathcal{X}_i,\ i \in \mathcal{I} \\ \rho x(k) & \text{if } x(k) \in \mathcal{X}_E \end{cases} \tag{10}$$

For convenience, we define $A_i(w) = \rho I$, $a_i(w) = 0$ for $i \in \tilde{\mathcal{I}} \setminus \mathcal{I}$. Note that the dynamics for $i \in \tilde{\mathcal{I}} \setminus \mathcal{I}$ do not depend on w.

Lemma 1: The set $\mathcal{X}_H$ is an RPI set with respect to the extended dynamics (10).

Proof: If $x \in \mathcal{X}_H$, then either $x \in \mathcal{X}$ or $x \in \mathcal{X}_E$. If $x \in \mathcal{X}$ then the successor state $A_i(w)x + a_i(w) \in \tilde{\mathcal{R}}_{\cup}(\mathcal{X}) \subseteq \mathcal{X}_H$ by definition of $\mathcal{X}_H$. If $x \in \mathcal{X}_E$, the successor state $\rho x \in \mathcal{X}_H$, because $\mathcal{X}_H$ is a hyper-rectangle including the origin. ■

Other choices of $\mathcal{X}_H$ and of the dynamics (9) are possible, provided that Lemma 1 holds.

Let $x(k) \in \mathcal{X}_i$ and $x(k+1) \in \mathcal{X}_j$, $(i, j) \in \tilde{\mathcal{I}} \times \tilde{\mathcal{I}}$. To characterize transitions we define the *region transition map* $\mathcal{S}$

$$\mathcal{S}_{i,j} \triangleq \begin{cases} 1 & \text{if } conv\,(\mathcal{R}(\mathcal{X}_i)) \cap \bar{\mathcal{X}}_j \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

which states (in a conservative way) whether there exists a state $x \in \bar{\mathcal{X}}_i$ and an uncertain vector $w \in \mathcal{W}$ such that $A_i(w)x + a_i(w) \in \bar{\mathcal{X}}_j$. For any pair $(i, j) \in \tilde{\mathcal{I}} \times \tilde{\mathcal{I}}$, we define

$$\mathcal{X}_{i,j} \triangleq \begin{cases} \bar{\mathcal{X}}_i & \text{if } \mathcal{S}_{(i,j)} = 1 \\ \emptyset & \text{if } \mathcal{S}_{(i,j)} = 0 \end{cases} \tag{12}$$

that we refer to as *transition set*, representing an overestimate of all the points that can possibly end up in $\mathcal{X}_j$ in one step under dynamics i.

V. PWA Lyapunov analysis for the extended system

By recalling basic results of stability of nonlinear discrete-time systems (see [17]), assume the origin is an equilibrium point for (10). Lyapunov stability is guaranteed by the existence of a function $V : \mathcal{X}_H \to \mathbb{R}$ satisfying the conditions

$$V(x) \geq \alpha_1 \|x\| \tag{13a}$$

$$V(f(x, w)) - \lambda V(x) \leq 0 \tag{13b}$$

$\forall x \in \mathcal{X}_i$ and $\forall w \in \mathcal{W}_i$ $(i \in \tilde{\mathcal{I}})$, where $f : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^n$ is the PWA state update function defined in (10), $\alpha_1 > 0$, $\lambda \in (0, 1)$. Note that (13) implies the condition $V(0) = 0$.

Remark 1: Condition (13b) could be replaced by

$$V(f(x, w)) - V(x) \leq -\alpha_3 \|x\| \tag{14}$$

where $\alpha_3 = (1 - \lambda)\alpha_1 > 0$. In fact, by (13), it follows that $V(f(x, w)) - V(x) \leq -(1 - \lambda)V(x) \leq -(1 - \lambda)\alpha_1 \|x\|$. Also, note that the imposition of an upperbound on $V(x)$, usually found in the literature (i.e. imposing $V(x) \leq \alpha_2 \|x\|$, $\alpha_2 > 0$), is not necessary here, as V is defined over the bounded set $\mathcal{X}_H$. As a consequence, it is always possible to find a-posteriori $\alpha_2 > 0$ such that $V(x) \leq \alpha_2 \|x\|$, $\forall x \in \mathcal{X}_H$. □

Note that the fulfillment of (13a) and (13b) (or (14)), even if the resulting Lyapunov function is discontinuous, according to [17], is a sufficient condition for the asymptotic stability of (10).

The goal is to synthesize a PWA Lyapunov function for system (10) satisfying (13). Consider the candidate function $V : \mathcal{X}_H \to \mathbb{R}$

$$V(x) = \max_{i \in \mathcal{N}(x)} V_i(x) \tag{15a}$$

where

$$\mathcal{N}(x) \triangleq \{i \in \tilde{\mathcal{I}} :\ x \in \bar{\mathcal{X}}_i\} \tag{15b}$$

and let $V_i : \bar{\mathcal{X}}_i \to \mathbb{R}$ be defined as

$$V_i(x) \triangleq F_i x + g_i \tag{15c}$$

for $i \in \tilde{\mathcal{I}}$, where in (15c) $F_i \in \mathbb{R}^{1 \times n}$ and $g_i \in \mathbb{R}$ are coefficients to be determined. Note that simply $V(x) = F_i x + g_i$ for $x \in int(\mathcal{X}_i)$. The rationale for using the max in (15a) is that for numerical reasons we want to consider closed sets $\bar{\mathcal{X}}_i$ and $V_i(x)$, $V_j(x)$ may not coincide on common boundaries $\bar{\mathcal{X}}_i \cap \bar{\mathcal{X}}_j$ unless very conservative continuity conditions are imposed.

Since $\bar{\mathcal{X}}_i$ is a convex set and V_i is affine on the corresponding set $\bar{\mathcal{X}}_i$, for all $i \in \tilde{\mathcal{I}}$, it will be shown that it is enough to impose the Lyapunov conditions (13a) only at *vert*($\bar{\mathcal{X}}_i$), and (13b) only at *vert*($\mathcal{X}_{i,j}$):

$$F_i v_{i,h} + g_i \geq \alpha_1 \|v_{i,h}\| \tag{16a}$$

for all m_i vertices $v_{i,h} \in$ *vert*($\bar{\mathcal{X}}_i$), $i \in \tilde{\mathcal{I}}$, $h = 1, \dots, m_i$, and

$$F_j(\tilde{A}_{i,r} v_{ij,h} + \tilde{a}_{i,r}) + g_j - \lambda(F_i v_{ij,h} + g_i) \leq 0 \tag{16b}$$

for all $v_{ij,h} \in$ *vert*($\mathcal{X}_{i,j}$), with $h = 1, \dots, m_i$, for all $A_{i,r}$, $a_{i,r}$ with $r = 1, ..., q$. Note that the set generated by the convex combination of the points $A_{i,r} v_{ij,h} + a_{i,r}$ with respect to the vertices of $\bar{\mathcal{X}}_i$ coincides with $conv\left\{\mathcal{R}\left(\bar{\mathcal{X}}_{i,j}\right)\right\}$. As a consequence, we can consider only the vertices of $\mathcal{X}_{i,j}$ generating the vertices of $conv\left\{\mathcal{R}\left(\mathcal{X}_{i,j}\right)\right\}$ to impose the decreasing condition (16b). The resulting constraints (16) define a linear feasibility problem in the unknowns F_i, g_i, α_1, for a fixed decay rate λ, and a feasible solution can be determined by linear programming.

As for the computational burden, the LP (16) has a number of variables equal to $n_v = 1 + \tilde{s}(n+1)$. One inequality is imposed for each vertex of each region $\mathcal{X}_i$, $i = 1, ..., \tilde{s}$ to fulfill (16a). Moreover, to fulfill (16b), for each vertex of each region one has to impose a number of inequalities equal to the number of regions $\mathcal{X}_j$ such that $\mathcal{S}_{i,j} \neq 0$, multiplied by the value of q. More concisely, the overall number of scalar constraints is

$$n_c = \sum_{i=1}^{\tilde{s}} m_i \left(1 + q \cdot \text{card}\left(\left\{j \in \tilde{\mathcal{I}} : \mathcal{S}_{i,j} = 1\right\}\right)\right)$$

Lemma 2: Let Assumption 1 hold, and let the LP (16) associated with the autonomous uncertain PWA dynamics (10) and the candidate Lyapunov function (15) be feasible. Then system (10) is $\text{UAS}(\mathcal{X}_H)$.

Proof: (sketch) As for the positive definiteness of the Lyapunov function, since functions V_i are affine functions defined on convex sets $\mathcal{X}_i$, the satisfaction of (16a) for all $v_{i,h} \in$ *vert*($\bar{\mathcal{X}}_i$), with $i \in \tilde{\mathcal{I}}$, $h = 1, \dots, m_i$, for $x \in \bar{\mathcal{X}}_i$ leads to

$$\alpha_1 \|x\| = \alpha_1 \left\| \sum_{h=1}^{m_i} \beta_{i,h} v_{i,h} \right\| \leq F_i x + g_i \tag{17}$$

where $\beta_{i,h} \geq 0$, $\sum_{h=1}^{m_i} \beta_{i,h} = 1$, are a set of coefficients defining x as a convex combination of the vertices of $\bar{\mathcal{X}}_i$. For this reason, for $x \in int(\mathcal{X}_i)$, since $V_i(x) = F_i x + g_i$, (13a) holds. Moreover, on the boundaries of $\bar{\mathcal{X}}_i$, according to (15a), one has $\alpha_1 \|x\| \leq F_i x + g_i$ for all $i \in \mathcal{N}(x)$, and therefore $\alpha_1 \|x\| \leq \max_{i \in \mathcal{N}(x)} \{F_i x + g_i\} = V(x)$. This implies that (13a) holds for all $x \in \mathcal{X}_H$, since $\mathcal{X}_H = \bigcup_{i \in \tilde{\mathcal{I}}} \bar{\mathcal{X}}_i$.

As for the decay of the Lyapunov function, note that

$$\begin{aligned} V(f(x,w)) = F_j \Bigg[\left(\sum_{r=1}^{q} \tilde{A}_{i,r} w_r \right) \left(\sum_{h=1}^{m_i} \beta_{i,h} v_{ij,h} \right) \\ + \sum_{r=1}^{q} \tilde{a}_{i,r} w_r \Bigg] + g_j \end{aligned} \tag{18}$$

Recalling that (16b) holds for all the vertices of $\mathcal{X}_{ij}$, and that $\sum_{h=1}^{m_i} \beta_{i,h} = \sum_{r=1}^{q} w_r = 1$, from (18), it is possible to prove that $V(f(x,w)) \leq \lambda V(x)$, which proves that (13b) holds for all $x \in int\left(\mathcal{X}_{i,j}\right)$. Also, on the boundaries of $\bar{\mathcal{X}}_i$, the decreasing condition (13b) is imposed for all $(i,j) \in \mathcal{N}(x) \times \mathcal{N}(f(x,w))$, and therefore

$$\max_{j \in \mathcal{N}(f(x,w))} \Big(F_j(f(x,w)) + g_j\Big) \leq \lambda \max_{i \in \mathcal{N}(x)} (F_i x + g_i)$$

Since for the definition of the $\mathcal{X}_{i,j}$ we have that $\mathcal{X}_H = \bigcup_{i=1}^{\tilde{s}} \bigcup_{j=1}^{\tilde{s}} \mathcal{X}_{i,j}$, (13b) holds for all $x \in \mathcal{X}_H$. As a result, (13) hold for all $x \in \mathcal{X}_H$, which, following in spirit the proof of Theorem 2.2.1 in [18, Chap. 2] for deterministic systems, guarantees that system (10) is $\text{UAS}(\mathcal{X}_H)$ according to Definition 3. ■

Remark 2: In case the LP (16) is infeasible, besides increasing the value of λ, a possibility is to increase the number of sets $\mathcal{X}_i$ of $\mathcal{X}_H$, therefore providing more flexibility in synthesizing the PWA Lyapunov function. This can be done, for instance, computing the Delaunay triangulation [19] for each set $\mathcal{X}_i$, and performing the synthesis of the PWA Lyapunov function by replacing the sets $\mathcal{X}_i$ with the elements of the obtained simplicial partition. However, in case the LP remains unfeasible after a given number of iterations, no result is given on the stability of the system. □

VI. Invariance analysis

So far the properties of the extended system (10) were analyzed. We want to derive conditions on the original system (2). Consider again system (10) in $\mathcal{X}_H$, assume that a feasible solution to (16) exists, define

$$V_E^- \triangleq \inf_{x \in \mathcal{X}_E} V(x) \tag{19}$$

and consider the subset $\mathcal{P}$ of $\mathcal{X}$

$$\mathcal{P} \triangleq \{x \in \mathcal{X} : \; V(x) < V_E^-\} \tag{20}$$

The set $\mathcal{P}$ may not be convex, not even connected.

The results for the extended system (10) proved in Lemma 2 and the definition of $\mathcal{P}$ in (20) are exploited next to state the main result of the paper.

Theorem 1: Consider system (2), whose dynamics are defined on $\mathcal{X}$, and assume that the extended dynamics (9) are defined in $\mathcal{X}_E$. If a Lyapunov function for system (10) is found by solving the LP (16), $\mathcal{P} \subseteq \mathcal{X}$ defined in (20) is an RPI set for (2). Moreover, (2) is $\text{UAS}(\mathcal{P})$. □

Proof: The proof consists in showing that the PWA Lyapunov function

$$V_{\text{UAS}}(x) \triangleq V(x), \; \forall x \in \mathcal{P} \tag{21}$$

where $V(x)$ is found as in Lemma 2 for the extended system (10) in $\mathcal{X}_H$, is a Lyapunov function for (2) over the set $\mathcal{P}$. First of all, considering that $\mathcal{P}$ is an RPI set for (10) in $\mathcal{X}_H$, one can note that the state update $f(x,w) \in \mathcal{P}$ for $x \in \mathcal{P}$ is always given by (2). Then, $\mathcal{P}$ is an RPI set for (2), because the dynamics (9) is never executed. Considering that $\mathcal{P} \subseteq \mathcal{X}_H$, if (13) are satisfied for all $x \in \mathcal{X}_H$ (and then for all

$x \in \mathcal{P}$), we conclude that $V_{\mathrm{UAS}}(x)$ is a Lyapunov function for system (2) in $\mathcal{P}$, and that system (2) is UAS($\mathcal{P}$). ■

The overall procedure proposed in this paper for uniform asymptotic stability is summarized in Algorithm 1.

Algorithm 1:

Input: PWA system (2)
Output: Region of attraction $\mathcal{P}$ and certificate of UAS($\mathcal{P}$) for system (2)
repeat
  if *a given maximum number of iterations is exceeded* **then**
    $\mathcal{P}$ undefined;
    STOP
  Obtain new regions $\mathcal{X}_i$ splitting the existing regions;
  Compute $\tilde{\mathcal{R}}_{\cup}(\mathcal{X})$ in (8) and $\mathcal{X}_H$;
  Define the fake dynamics (9);
  Find $\mathcal{S}$ in (11), and the $\mathcal{X}_{i,j}$ in (12);
  Solve the LP feasibility problem (16);
until *the LP has a solution* ;
Find the region of attraction $\mathcal{P} \subseteq \mathcal{X}$;
System (2) is UAS($\mathcal{P}$)

As a practical procedure to represent the set $\mathcal{P}$, one can define the polyhedra

$$\mathcal{X}_i^{\mathcal{P}} \triangleq \{x \in \mathcal{X}_i : \; V(x) < V_E^-\}, \; i = 1, ..., \mathcal{I}, \qquad (22)$$

and define the invariant set $\mathcal{P}$ as

$$\mathcal{P} = \bigcup_{i=1}^{\mathcal{I}} \mathcal{X}_i^{\mathcal{P}} \qquad (23)$$

VII. Simulation Example

An interesting application of PWA stability analysis is to analyze a-posteriori closed-loop system stability. In particular, the proposed stability and invariance analysis procedure is tested on the closed-loop system composed by a discrete-time PWA system and a switched explicit linear MPC controller. The second-order open-loop PWA system is defined by

$$x(k+1) = A_j(w)x(k) + B_j u(k) \;\; \text{if } x(k) \in \Omega_j,$$

where $j = 1, 2$, $x(k) \in \Omega_i \subset \mathbb{R}^2$, $w(k) \in \mathcal{W} \subset \mathbb{R}^2$, with

$$A_j(w) \triangleq A_{j,0} + A_{j,1} w_{j,1} + A_{j,2} w_{j,2}$$
$$\mathcal{W} \triangleq \left\{ w \in \mathbb{R}^2 : w_1 + w_2 = 1, \; w_r \geq 0, \; r = 1, 2 \right\}$$

More specifically, we have

$$A_{1,0} = \begin{bmatrix} 0.7 & 0.7 \\ 0 & 0.7 \end{bmatrix}, A_{2,0} = \begin{bmatrix} 0.6 & 0.6 \\ 0 & 0.6 \end{bmatrix}$$
$$A_{1,1} = \begin{bmatrix} 0 & 0.1 \\ 0 & 0 \end{bmatrix}, A_{2,1} = \begin{bmatrix} 0 & -0.2 \\ 0 & 0 \end{bmatrix}$$
$$A_{1,2} = \begin{bmatrix} 0 & 0 \\ 0 & -0.1 \end{bmatrix}, A_{2,2} = \begin{bmatrix} 0 & 0 \\ 0 & 0.2 \end{bmatrix}$$
$$B_1 = \begin{bmatrix} 1.1 \\ 0 \end{bmatrix}, B_2 \begin{bmatrix} 0.9 \\ 0 \end{bmatrix}$$

The regions Ω_i are defined by $\Omega_1 = \{x \in \mathbb{R}^2 : H_1 x \leq h_1\}$, $\Omega_2 = \{x \in \mathbb{R}^2 : H_2 x \leq h_2\} \setminus \Omega_1$, with

$$H_1 = \begin{bmatrix} 0 & 0.1 \\ 0 & -0.1 \\ -0.1 & 0 \\ 1 & 0 \end{bmatrix}, H_2 = \begin{bmatrix} 0 & 0.1 \\ 0 & -0.1 \\ -1 & 0 \\ 0.1 & 0 \end{bmatrix}$$
$$h_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, h_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

The switched explicit linear MPC controller is defined by computing an explicit MPC control law [5] $u_j(x)$ for each nominal linear system

$$x(k+1) = A_{j,0}x(k) + B_j u(k)$$

and by setting

$$u(k) = u_j(k) \;\; \text{if } x(k) \in \Omega_j$$

with a prediction horizon $N = 5$, a control horizon $N_u = 2$, weight matrices

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}, \quad R = 0.1$$

on the state and the control, respectively, and terminal weight matrices $P_j \in \mathbb{R}^2$ found as the solution of the Lyapunov equation $A'_{j,0} P A_{j,0} - P = -Q$ on each set Ω_j. The control constraints are $u \in [-4, 4]$, while the soft state constraints are defined by the set $\mathcal{X}$. The overall closed-loop system, which does not have any a priori stability properties, can be written in form (2), with a suitable definition of the sets $\mathcal{X}_i$, automatically generated by the multiparametric program. The set $\mathcal{X} = \Omega_1 \cup \Omega_2$ is not invariant for the closed-loop system. For instance, starting at an initial condition $x(0) = \begin{bmatrix} -10 & -9.5 \end{bmatrix}' \in \mathcal{X}$, the explicit MPC control variable is $u(0) = 4$. With a disturbance vector $w(0) = \begin{bmatrix} 0.9 & 0.1 \end{bmatrix}'$, it yields $x(1) = \begin{bmatrix} -10.11 & -6.56 \end{bmatrix} \notin \mathcal{X}$. Therefore, we find $\tilde{\mathcal{R}}_{\cup}(\mathcal{X})$ in (8) and its bounding box $\mathcal{X}_H$ (Fig. 1), with the extended dynamics (10) defined with $\rho = 0.99$.

According to Algorithm 1, we found the transition map $\mathcal{S}$ in (11), the transition sets $\mathcal{X}_{i,j}$ in (12), and solved the LP in (16) with $\lambda = 0.99$. The LP is composed of 691 constraints and 76 variables, and solved using LINPROG in Matlab in 0.87 s on a 2.4 GHz processor. The corresponding Lyapunov function is shown in Figure 2. The regions obtained using the switched explicit MPC, together with the extension given by $\mathcal{X}_H \setminus \mathcal{X}$ and the invariant set $\mathcal{P}$, are shown in Figure 3.

VIII. Conclusions

This paper has addressed the problem of determining the uniform asymptotic stability of (possibly discontinuous) uncertain discrete-time PWA systems using (possibly discontinuous) PWA Lyapunov functions, together with the determination of an invariant subset of the region of attraction. The complexity of the resulting LP has been determined, and a simulation example on a simple second-order system showed the effectiveness of the proposed method.

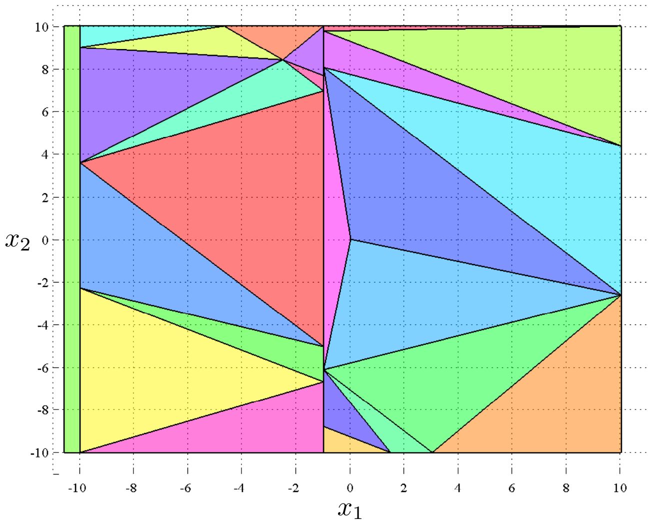

Fig. 1. The invariant set $\mathcal{X}_H$ is constituted by the union of the regions of the explicit MPC and the box $\mathcal{X}_H \setminus \mathcal{X}$ (the green rectangle on the left)

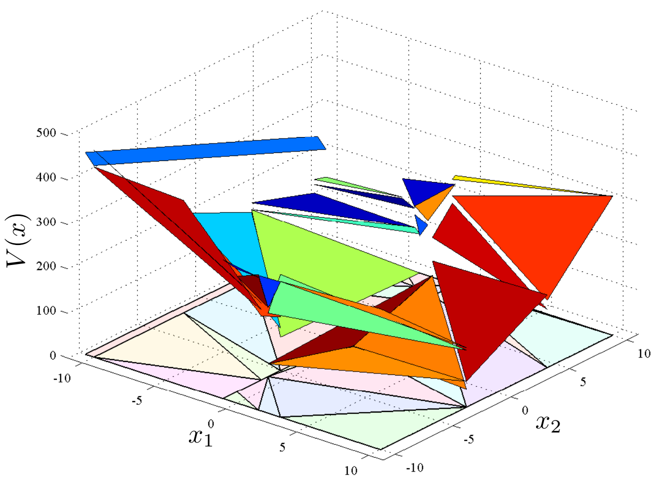

Fig. 2. The PWA Lyapunov function for the extended system

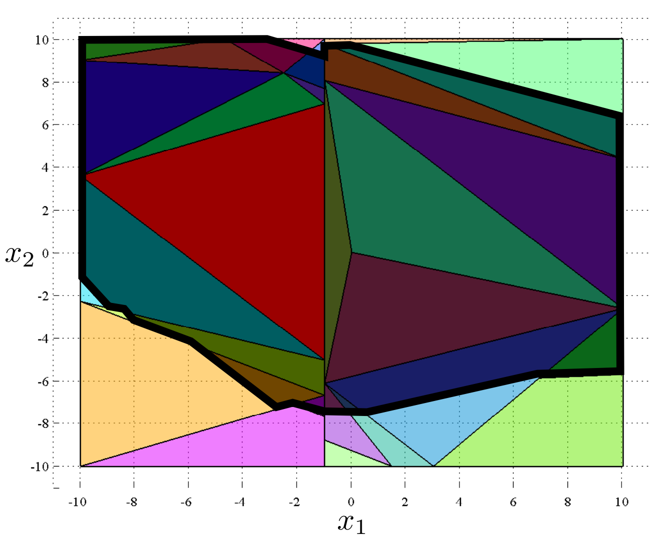

Fig. 3. The invariant set $\mathcal{P}$ obtained for $\lambda = 0.99$

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