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Swing-up Control for a Two-Link Underactuated Robot with a Flexible Elbow Joint: New Results beyond the Passive Elbow Joint

Xin Xin

***Abstract*— This paper concerns the swing-up control problem for a two-link underactuated robot moving in the vertical plane with a single actuator at the base joint and a spring between the two links (flexible elbow joint). First, we present two new properties of such a flexible robot about the linear controllability at the UEP (upright equilibrium point, where two links are in the upright position) and the limitation of the PD control on the angle of the base joint. Second, for the robot which can be not locally stabilized about the UEP by the PD control, we study how to extend the energy-based control approach, which aims to control the total mechanical energy and the angle and angular velocity of the base joint of the robot, to design a swing-up controller. We provide a necessary and sufficient condition for avoiding the singularity in the controller. Third, we analyze the motion of the robot under the presented controller by studying the convergence of the total mechanical energy and clarifying the structure and stability of the closed-loop equilibrium points. We validate the presented theoretical results via numerical investigation. This paper not only unifies some previous results for the Pendubot (a two-link robot with a passive elbow joint), but also provides insight into the control and analysis of the underactuated robots with flexible joints.**

I. INTRODUCTION

Studies on underactuated mechanical systems (UMSs), which possess fewer actuators than degrees of freedom, have received considerable interest in recent years [1]–[3]. One of the important control problems for UMSs is the set-point control (regulation or stabilization) of a desired equilibrium point [2], [4]. Many researchers studied a particular problem of the set-point control called the swing-up and stabilizing control for two-link planar robots with a passive joint, see e.g., the Pendubot (with a passive elbow joint) in [5]–[7] and the Acrobot (with a passive base joint) in [8], [9]. Indeed, the swing-up control is to swing up the robot to a small neighborhood of the upright equilibrium point (denoted as UEP below), where two links are in the upright position, so that a locally stabilizing controller can be switched to stabilize the robot about the UEP. The swing-up control has become a benchmark problem for verifying the effectiveness of nonlinear design methods or techniques.

This paper concerns the swing-up control problem for a two-link underactuated planar robot moving in the vertical plane with a single actuator at the base joint and a spring between the two links. This indicates that the robot has an *active* base joint and a *flexible* elbow joint; and below we call it the AF robot, where "A" and "F" denote active and flexible, respectively. The flexibility of the link is a result of lightening a robot arm for example in space applications [10]. Indeed, a flexible link is modelled by some virtual rigid links connected by the joints consisting of springs and dampers [11]; and the AF robot is its most simplified model. Regarding the AF robot in the horizontal plane, [10] showed that the PD control on the angle of the base joint can globally stabilize the state of the robot at any desired value.

This paper presents two new properties of the AF robot in the presence of gravity. One property is about its linear controllability (the controllability of the linearized model of a nonlinear system around an equilibrium point [12]). We obtain a necessary and sufficient condition on the mechanical parameters of the robot and the spring constant such that the robot is linearly controllable at the UEP. We reveal that the linear controllability at the UEP is destroyed when the spring constant takes a value determined by the mechanical parameters of the two-links. This is a surprising result since the two extreme cases of the AF robot, the Pendubot and a one-link rigid robot (with the spring constant being 0 and ∞, respectively), are always linearly controllable at the UEP without any condition on their mechanical parameters. Another property is about the limitation of the PD control. We present a necessary and sufficient condition on the mechanical parameters such that the robot can be locally stabilized about the UEP by the PD control.

Next, for the AF robot which can not be locally stabilized about the UEP by the PD control, we study how to extend the energy-based control approach in [6], [7] for the Pendubot to design a swing-up controller for the AF robot. We provide a necessary and sufficient condition for avoiding the singularity in the presented controller. Third, we analyze the motion of the robot under the presented controller by studying the convergence of the total mechanical energy and clarifying the structure and stability of the closed-loop equilibrium points. As shown in this paper, it is much more difficult to carry the motion analysis of the AF robot than that of the Pendubot in [6], [7]. Indeed, different from the Pendubot, it is difficult to obtain analytically the equilibrium configuration of the AF robot due to the spring at the elbow joint. We validate the theoretical results via numerical investigation.

II. PRELIMINARY KNOWLEDGE

Consider the AF robot (a two-link planar robot with a spring between two links) moving in the vertical plane shown in Fig. 1. For the ith ($i = 1, 2$) link, m_i is its mass, l_i is its

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X. Xin is with Faculty of Computer Science and Systems Engineering, Okayama Prefectural University, 111 Kuboki, Soja, Okayama 719-1197, Japan, xxin@cse.oka-pu.ac.jp

length, l_{ci} is the distance from joint i to its center of mass (COM), and J_i is the moment of inertia around its COM.

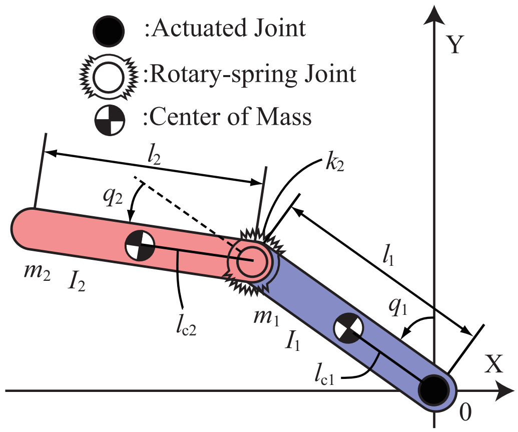

Fig. 1. AF robot: two-link flexible robot with a spring between two-links.

Let $q = \begin{bmatrix} q_1, & q_2 \end{bmatrix}^{\mathrm{T}}$ be the vector of the angles of two joints and τ_1 be the control input on the first joint. The motion equation of the robot is

$$M(q)\ddot{q} + H(q,\dot{q}) + G(q) + K(q) = B\tau_1, \tag{1}$$

where

$$M(q) = \begin{bmatrix} \alpha_1 + \alpha_2 + 2\alpha_3 \cos q_2 & \alpha_2 + \alpha_3 \cos q_2 \\ \alpha_2 + \alpha_3 \cos q_2 & \alpha_2 \end{bmatrix}, \tag{2}$$

$$H(q,\dot{q}) = \alpha_3 \begin{bmatrix} -2\dot{q}_1\dot{q}_2 - \dot{q}_2^2 \\ \dot{q}_1^2 \end{bmatrix} \sin q_2, \tag{3}$$

$$G(q) = \begin{bmatrix} -\beta_1 \sin q_1 - \beta_2 \sin(q_1+q_2) \\ -\beta_2 \sin(q_1+q_2) \end{bmatrix}, \tag{4}$$

where

$$\begin{cases} \alpha_1 = m_1 l_{c1}^2 + m_2 l_1^2 + J_1, \\ \alpha_2 = m_2 l_{c2}^2 + J_2, \quad \alpha_3 = m_2 l_1 l_{c2}, \\ \beta_1 = (m_1 l_{c1} + m_2 l_1) g, \quad \beta_2 = m_2 l_{c2} g, \end{cases} \tag{5}$$

and g is the acceleration of gravity; and

$$K(q) = \begin{bmatrix} 0 \\ k_2 q_2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{6}$$

where k_2 is the spring constant.

The energy of the robot is expressed as

$$E(q,\dot{q}) = \frac{1}{2}\dot{q}^{\mathrm{T}} M(q)\dot{q} + P(q), \tag{7}$$

where $P(q)$ is the potential energy and is defined as

$$P(q) = \beta_1 \cos q_1 + \beta_2 \cos(q_1+q_2) + \frac{1}{2}k_2 q_2^2. \tag{8}$$

Consider the open-loop equilibrium points of (1). Let $q^e = \begin{bmatrix} q_1^e, & q_2^e \end{bmatrix}^{\mathrm{T}}$ be an equilibrium configuration and τ_1^e be the equilibrium input. Putting $\ddot{q} = 0$, $\dot{q} = 0$, $q = q^e$, and $\tau_1 = \tau_1^e$ into (1) yields

$$-\beta_1 \sin q_1^e - \beta_2 \sin(q_1^e + q_2^e) = \tau_1^e, \tag{9}$$

$$-\beta_2 \sin(q_1^e + q_2^e) + k_2 q_2^e = 0. \tag{10}$$

Note that from (10) the equilibrium configuration of the Pendubot ($k_2 = 0$) is $\sin(q_1^e + q_2^e) = 0$, that is, its link 2 parallels to the vertical line (Y-axis). However, for the AF robot, it is unclear whether one can obtain an analytic solution of q_2^e in terms of q_1^e from (10). This makes the motion analysis in Section V difficult.

III. NEW PROPERTIES OF THE AF ROBOT

A. Linear Controllability of the AF Robot at the UEP

We study the linear controllability of the AF robot at the UEP. Let $x = \begin{bmatrix} q_1, & q_2, & \dot{q}_1, & \dot{q}_2 \end{bmatrix}^{\mathrm{T}}$ be the state-variable vector. The UEP is $x = 0$. The linearized model of the AF robot (1) around the UEP is

$$\dot{x} = Ax + N\tau_1, \tag{11}$$

where the formulae of A and N are omitted for brevity. Computing the determinant of the controllability matrix $U_{uu} = \begin{bmatrix} N, & AN, & A^2N, & A^3N \end{bmatrix}$ yields

$$|U_{uu}| = \frac{-(k_2(\alpha_2+\alpha_3) - \alpha_3\beta_2)^2}{(\alpha_3^2 - \alpha_1\alpha_2)^4}. \tag{12}$$

This gives the following result.

LEMMA 1: The linearized model of the AF robot (1) around the UEP is controllable if and only if

$$k_2 \neq \frac{\alpha_3\beta_2}{\alpha_2+\alpha_3}. \tag{13}$$

The following remark concerns Lemma 1.

Remark 1: From Lemma 1, the Pendubot [5], [6], which is an extreme case of the AF robot (1) with $k_2 = 0$, is always linearly controllable at $x = 0$; and the AF robot (1) in the absence of gravity, that is $\beta_2 = 0$, is also always linearly controllable at $x = 0$. Moreover, Lemma 1 indicates that there exists a spring with its spring constant satisfying

$$k_2 = k_{uc} := \frac{\alpha_3\beta_2}{\alpha_2+\alpha_3} \tag{14}$$

destroys the linear controllability of the robot around the UEP. Clearly, $k_{uc} < \beta_2$.

Below we assume that (13) holds. This guarantees that the robot can be stabilized about the UEP by an appropriate state-feedback controller.

B. Limitation of the PD Control

By using the following Lyapunov function

$$V(q,\dot{q}) = E(q,\dot{q}) + \frac{k_P}{2}q_1^2, \tag{15}$$

[10] showed that

$$\tau_1 = -k_P q_1 - k_D \dot{q}_1 \tag{16}$$

with $k_P > 0$ and $k_D > 0$ can globally stabilize the AF robot moving in the horizontal plane (without gravity) about $x = 0$. We will study whether the PD control is also effective for balancing the AF robot about the UEP in the presence of gravity. To this end, using the Routh-Hurwitz criterion, we check the stability of the UEP of the closed-loop system constituting of (1) and (16). We present the following result.

LEMMA 2: The UEP of the AF robot in (1) can be locally stabilized by the PD control (16) if and only if its mechanical parameters satisfy

$$k_2 > \beta_2; \tag{17}$$

and the control gains satisfy $k_D > 0$ and

$$k_P > \frac{\alpha_2\beta_1 + \alpha_1\beta_2 - k_2(\alpha_1 + \alpha_2 + 2\alpha_3)}{\alpha^2}, \tag{18}$$

$$k_P > \frac{-\beta_1\beta_2 + k_2(\beta_1 + \beta_2)}{k_2 - \beta_2}. \tag{19}$$

The following remark concerns Lemma 2.

Remark 2: Condition (17) in Lemma 2 means that the joint stiffness k_2 overcomes the gradient of the gravitational term β_2. If

$$k_2 \le \beta_2, \tag{20}$$

then the PD control (16) can not stabilize even locally the AF robot about the UEP. In what follows, we consider the swing-up control problem mainly for the robot satisfying (20).

IV. SWING-UP CONTROLLER FOR THE AF ROBOT

We study how to apply the energy-based control approach in [6], [7] to design a swing-up controller for the AF robot (1). The new result of this section is the presence of a necessary and sufficient condition for nonexistence of singularity in the presented controller.

For $E(q,\dot q)$, $\dot q_1$, and q_1, we aim to design τ_1 such that

$$\lim_{t\to\infty} E(q,\dot q) = E_r, \quad \lim_{t\to\infty} \dot q_1 = 0, \quad \lim_{t\to\infty} q_1 = 0, \tag{21}$$

where

$$E_r = \beta_1 + \beta_2 \tag{22}$$

is the potential energy of the robot at the UEP.

We use the following Lyapunov function candidate:

$$V = \frac{1}{2}(E - E_r)^2 + \frac{1}{2}k_D\dot q_1^2 + \frac{1}{2}k_P q_1^2, \tag{23}$$

where scalars $k_D > 0$ and $k_P > 0$ are control parameters.

Taking the time-derivative of V along the trajectories of (1), and using $\dot E = \dot q^{\mathrm{T}} B\tau_1 = \dot q_1\tau_1$, we obtain

$$\dot V = \dot q_1\left((E - E_r)\tau_1 + k_D\ddot q_1 + k_P q_1\right).$$

If we can choose τ_1 such that

$$(E - E_r)\tau_1 + k_D\ddot q_1 + k_P q_1 = -k_V\dot q_1 \tag{24}$$

for some constant $k_V > 0$, then we have

$$\dot V = -k_V\dot q_1^2 \le 0. \tag{25}$$

We discuss under what condition (24) is solvable with respect to τ_1 for any (q, $\dot q$). From (1), we obtain

$$\ddot q_1 = B^{\mathrm{T}}\ddot q = B^{\mathrm{T}}M^{-1}(B\tau_1 - H - G - K). \tag{26}$$

Substituting (26) into (24) yields

$$\Lambda(q,\dot q)\tau_1 = k_D B^{\mathrm{T}}M^{-1}(H + G + K) - k_V\dot q_1 - k_P q_1, \tag{27}$$

where

$$\Lambda(q,\dot q) = E(q,\dot q) - E_r + k_D B^{\mathrm{T}}M^{-1}B. \tag{28}$$

Therefore, when

$$\Lambda(q,\dot q) \ne 0, \quad \text{for } \forall q,\ \forall\dot q, \tag{29}$$

we obtain

$$\tau_1 = \Lambda^{-1}\left(k_D B^{\mathrm{T}}M^{-1}(H + G + K) - k_V\dot q_1 - k_P q_1\right). \tag{30}$$

We present the following lemma.

LEMMA 3: Consider the closed-loop system consisting of (1) and (30) with positive parameters k_D, k_P, and k_V. Then the controller (30) has no singularity for any (q, $\dot q$) if and only if

$$k_D > \max_{|q_2| \le 2\sqrt{\frac{\beta_1+\beta_2}{k_2}}} \left\{(E_r + \mu(q_2))\left(\alpha_1 - \frac{\alpha_3^2}{\alpha_2}\cos^2 q_2\right)\right\}, \tag{31}$$

where E_r is defined in (22), and

$$\mu(q_2) = -\frac{1}{2}k_2 q_2^2 + \sqrt{\beta_1^2 + \beta_2^2 + 2\beta_1\beta_2\cos q_2}. \tag{32}$$

In this case,

$$\lim_{t\to\infty} V = V^*, \quad \lim_{t\to\infty} E = E^*, \quad \lim_{t\to\infty} q_1 = q_1^*, \tag{33}$$

where V^*, E^*, and q_1^* are constants. Moreover, as $t \to \infty$, every closed-loop solution, $(q(t), \dot q(t))$, approaches the invariant set

$$W = \left\{(q,\dot q)\ \middle|\ q_1 \equiv q_1^*,\ \dot q_2^2 = \frac{2(E^* - P(q))}{\alpha_2}\Big|_{q_1 = q_1^*}\right\}, \tag{34}$$

where "$\equiv$" denotes the equality holds for all time.

V. MOTION ANALYSIS OF THE AF ROBOT

A. On Convergent Value of Lyapunov Function V

We now characterize the invariant set W in (34) by analyzing the convergent value, V^*, of the Lyapunov function V in (23). Since $\lim_{t\to\infty} V = 0$ is equivalent to (21), we separately analyze two cases: $V^* = 0$ and $V^* \ne 0$.

Define the set

$$W_r = \left\{(q,\dot q)\ \middle|\ q_1 \equiv 0,\ \dot q_2^2 = \frac{2\beta_2(1 - \cos q_2) - k_2 q_2^2}{\alpha_2}\right\}. \tag{35}$$

Consider the equilibrium points of the closed-loop system consisting of (1) and (30). In addition to (9) and (10), from (24) (for which we derive the controller (30)), we can see that the equilibrium configuration q^e must also satisfy

$$(P(q^e) - E_r)\tau_1^e + k_P q_1^e = 0. \tag{36}$$

Define the equilibrium set

$$\Omega = \{(q^e, 0) \mid q^e \text{ satisfies (9), (10), (36), and } P(q^e) \ne E_r\}. \tag{37}$$

We are ready to present the following theorem whose proof is much more involved than that in the case of $k_2 = 0$ in [7], [13].

Theorem 1: Consider the closed-loop system consisting of (1) and (30) with positive parameters k_D, k_P, and k_V. Suppose that k_D satisfies (31). Then, as $t \to \infty$, the closed-loop solution $(q(t), \dot{q}(t))$ approaches

$$W = W_r \cup \Omega, \quad \textit{with } W_r \cap \Omega = \emptyset, \tag{38}$$

where W_r, the set in (35), is for the case $V^* = 0$; and Ω, the equilibrium set in (37), is for the case $V^* \neq 0$.

Proof: Consider the case $V^* = 0$. From (23), we have $E^* = E_r$ and $q_1^* = 0$. Thus, putting these into the set W in (34), we obtain the set W_r in (35).

Consider the case $V^* \neq 0$. We show that q_2 is a constant on the invariant set W. As shown below, the proof is much more involved than that in the case of $k_2 = 0$ in [7], [13].

To start with, putting $E \equiv E^*$ and $q_1 \equiv q_1^*$ into (24) shows by contradiction that $E^* \neq E_r$. This shows that τ_1 is a constant, τ_1^*, satisfying

$$(E^* - E_r)\tau_1^* + k_P q_1^* = 0, \quad E^* \neq E_r. \tag{39}$$

Below we will eliminate the terms related to $\ddot{q}_2$, $\dot{q}_2^2$, and the linear terms of q_2 from (1) with $q_1 \equiv q_1^*$ and $\tau_1 \equiv \tau_1^*$ through the differentiation with respect to (w.r.t.) time t if necessary. To simplify the derivation and for the brevity of expression, we denote $S_2 = \sin q_2$, $C_2 = \cos q_2$, $S_{12} = \sin(q_1^* + q_2)$, $C_{12} = \cos(q_1^* + q_2)$, and

$$S_{122} = \sin(q_1^* + 2q_2), \quad C_{122} = \cos(q_1^* + 2q_2).$$

Putting $q_1 \equiv q_1^*$ and $\tau_1 \equiv \tau_1^*$ into (1) yields

$$(\alpha_2 + \alpha_3 C_2)\ddot{q}_2 - \alpha_3 S_2 \dot{q}_2^2 - \beta_2 S_{12} = \tau_1^* + \beta_1 \sin q_1^*, \tag{40}$$

$$\alpha_2 \ddot{q}_2 + k_2 q_2 = \beta_2 S_{12}. \tag{41}$$

From (41), we obtain

$$\ddot{q}_2 = a S_{12} - b q_2, \tag{42}$$

where $a = \beta_2/\alpha_2$ and $b = k_2/\alpha_2$. Putting (42) into (40) and using

$$C_2 S_{12} = \frac{1}{2}(S_{122} - \sin q_1^*), \tag{43}$$

we obtain

$$-S_2 \dot{q}_2^2 - b C_2 q_2 + \frac{a S_{122}}{2} - d q_2 = \lambda_1, \tag{44}$$

where

$$d = \frac{k_2}{\alpha_3}, \quad \lambda_1 = \frac{\tau_1^* + \beta_1 \sin q_1^*}{\alpha_3} + \frac{a \sin q_1^*}{2}.$$

Now, on the contrary, suppose that q_2 is not a constant, that is, $\dot{q}_2 \not\equiv 0$. By differentiating (44) w.r.t. time t and using repeatedly (42) to delete the terms related to $S_2 \dot{q}_2^2$, $C_2 q_2$, and $d q_2$, we can obtain

$$S_{122} = \sin(q_1^* + 2q_2) = 0 \tag{45}$$

holds for all time, which contradicts that q_2 is not a constant. To this end, differentiating (44) w.r.t. time t, we have

$$\dot{q}_2(-C_2 \dot{q}_2^2 - 2 S_2 \ddot{q}_2 + b S_2 q_2 - b C_2 + a C_{122} - d) = 0. \tag{46}$$

Since $\dot{q}_2 \not\equiv 0$, we have

$$-C_2 \dot{q}_2^2 - 2 S_2 \ddot{q}_2 + b S_2 q_2 - b C_2 + a C_{122} = d. \tag{47}$$

Putting (42) into (47) and using $S_2 S_{12} = (\cos q_1^* - C_{122})/2$ give

$$-C_2 \dot{q}_2^2 + 3b S_2 q_2 - b C_2 + 2a C_{122} = d + a \cos q_1^*. \tag{48}$$

Differentiating (48) w.r.t. time t yields

$$\dot{q}_2(S_2 \dot{q}_2^2 - 2 C_2 \ddot{q}_2 + 3b C_2 q_2 + 4b S_2 - 4a S_{122}) = 0. \tag{49}$$

This with (42) and (43) shows

$$S_2 \dot{q}_2^2 + 5b C_2 q_2 + 4b S_2 - 5a S_{122} = -a \sin q_1^*. \tag{50}$$

Summing (44) and (50) to delete $S_2 \dot{q}_2^2$, we have

$$4b C_2 q_2 + 4b S_2 - \frac{9a S_{122}}{2} - d q_2 = \lambda_1 - a \sin q_1^*. \tag{51}$$

To delete $4b C_2 q_2$ in (51), we use $\dot{q}_2 \not\equiv 0$ and differentiate (51) w.r.t. time t once and twice to obtain

$$-4b S_2 q_2 + 8b C_2 - 9a C_{122} = d, \tag{52}$$

and

$$-4b C_2 q_2 - 12b S_2 + 18a S_{122} = 0, \tag{53}$$

respectively. Summing (51) and (53) gives

$$-8b S_2 + \frac{27a S_{122}}{2} - d q_2 = \lambda_1 - a \sin q_1^*. \tag{54}$$

To obtain S_{122} in (45), using $\dot{q}_2 \not\equiv 0$ and differentiating (54) twice w.r.t. time t and yield

$$8b S_2 - 54a S_{122} = 0. \tag{55}$$

Summing (54) and (55) gives

$$-\frac{81a S_{122}}{2} - d q_2 = \lambda_1 - a \sin q_1^*. \tag{56}$$

Differentiating (56) twice w.r.t. time t and using $\dot{q}_2 \not\equiv 0$ yield

$$162a S_{122} = 0, \tag{57}$$

which shows (45). This contradicts the assumption that q_2 is not a constant. Thus, q_2 is constant on W in (34).

In conclusion, for the case $V^* \neq 0$, as $t \to \infty$, the closed-loop solution $(q(t), \dot{q}(t))$ approaches an equilibrium point of the equilibrium set Ω defined in (37). ■

B. On the Close-Loop Equilibrium Points

Let us consider the equilibrium point belonging to W_r in (35). Using $q_1^e = 0$, $P(q^e) = E_r$, (10), and (36), we obtain

$$-\beta_2 \sin q_2^e + k_2 q_2^e = 0, \tag{58}$$

$$2\beta_2(1 - \cos q_2^e) - k_2 (q_2^e)^2 = 0. \tag{59}$$

Denote $\gamma = k_2/\beta_2$ to rewrite (58) and (59) respectively as

$$\sin q_2^e = \gamma q_2^e, \tag{60}$$

$$\cos q_2^e = -\frac{1}{2}\gamma (q_2^e)^2 + 1. \tag{61}$$

By summing the square of each of the above two equations to delete $\sin q_2^e$ and $\cos q_2^e$, we obtain

$$\gamma(q_2^e)^2(\gamma(q_2^e)^2 + 4(\gamma - 1)) = 0.$$

Clearly, if $\gamma \geq 1$ ($k_2 \geq \beta_2$), then $q_2^e = 0$ is the unique solution to (58) and (59). If $0 < \gamma < 1$ ($0 < k_2 < \beta_2$), then in addition to $q_2^e = 0$, we can see that

$$q_2^e = \pm 2\sqrt{\frac{1}{\gamma} - 1} \tag{62}$$

are also solutions of (58) and (59) if and only if γ satisfies

$$f_1 := \sin 2\sqrt{\frac{1}{\gamma} - 1} - 2\sqrt{\gamma - \gamma^2} = 0, \tag{63}$$

$$f_2 := \cos 2\sqrt{\frac{1}{\gamma} - 1} - 2\gamma + 1 = 0, \tag{64}$$

which are obtained by putting q_2^e in (62) into (60) and (61).

In summary, we have the following lemma.

LEMMA 4: If $0 < k_2 < \beta_2$, (63) and (64) hold, then the set W_r in (35) has three equilibrium points including the UEP and two other equilibrium points of $q_1^e = 0$ and q_2^e in (62). Otherwise, the set W_r has a unique equilibrium point of the UEP.

We give the following remark about Lemma 4.

Remark 3: For the Pendubot ($k_2 = 0$), the W_r in (35) describes a homoclinic orbit with the UEP being its equilibrium point. For the AF robot, the W_r in (35) is not a homoclinic orbit, and it is different for the cases of $k_2 < \beta_2$ and $k_2 \geq \beta_2$. Suppose that the UEP is the unique equilibrium point in W_r. Then, $(q(t), \dot{q}(t))$ will have the UEP as an ω-limit point; that is, there exists a sequence of times t_m ($m = 1, \ldots, \infty$) such that $t_m \to \infty$ as $m \to \infty$ for which $\lim_{m\to\infty}(q(t_m), \dot{q}(t_m)) = (0, 0)$.

Now we consider the equilibrium set Ω in (37). If the set Ω contains a stable equilibrium point in the sense of Lyapunov, then the robot can not be swung up arbitrarily close to the UEP from some neighborhoods close to the stable equilibrium point. Since the set Ω does not contain the UEP due to the constraint $P(q^e) \neq E_r$, it is interesting to study how to find conditions about k_P such that Ω is an empty set or contains some isolated unstable equilibrium points. This should be studied further as a future subject.

VI. SIMULATION RESULTS

We simulated the AF robot (1) with the parameters α_1, α_2, α_3, β_1, and β_2 the same as those of the Pendubot in [14] (p. 15). Taking $g = 9.81$ yields $\alpha_1 = 0.0799$, $\alpha_2 = 0.0244$, $\alpha_3 = 0.0205$, $\beta_1 = 4.1326$, and $\beta_2 = 1.0428$.

From (14), we obtain $k_{uc} = 0.4761$. This shows that the AF robot with the above mechanical parameters and the spring constant $k_2 = 0.4761$ is not linearly controllable at the UEP. If $k_2 \leq \beta_2 = 1.0428$, the AF robot can not be stabilized around the UEP by the PD control (16).

For an initial condition

$$q_1(0) = -\frac{5\pi}{6},\ q_2(0) = 0,\ \dot{q}_1(0) = 0,\ \dot{q}_2(0) = 0,$$

we validated the presented swing-up controller by investigating cases $k_2 \leq \beta_2$ and $k_2 > \beta_2$, respectively.

A. Case $k_2 \leq \beta_2$

We took $k_2 = 1.0$ which does not satisfy (14). We verified that $\gamma = 0.9590$ satisfies neither (63) nor (64). This shows that the UEP is the unique equilibrium point belong to the set W_r in (35).

According to (31), the swing-up controller (30) does not contain any singularity if and only if $k_D > 0.7081$. The simulation results under the swing-up controller (30) with $k_D = 0.76$, $k_P = 8.63$, and $k_V = 3.36$ are depicted in Figs. 2–4. Fig. 2 shows that V and $E - E_r$ converged to 0. From Fig. 3, we know that q_1 converged to 0, while several swings brought q_2 quickly close to 0, and then there existed many periods of time such that q_2 remained close to 0. From Fig. 4, $(q_2(t), \dot{q}_2(t))$ approached the orbit described in the set W_r in (35), and τ_1 was large at the beginning and then became small. Since the AF robot was swung up and remained very close to the UEP, it is very easy to balance it about that point via a state-feedback stabilizing controller. Due to page limitations, we omit the details of the stabilizing controller, the switch condition from the swing-up controller to the stabilizing controller, and the simulation results.

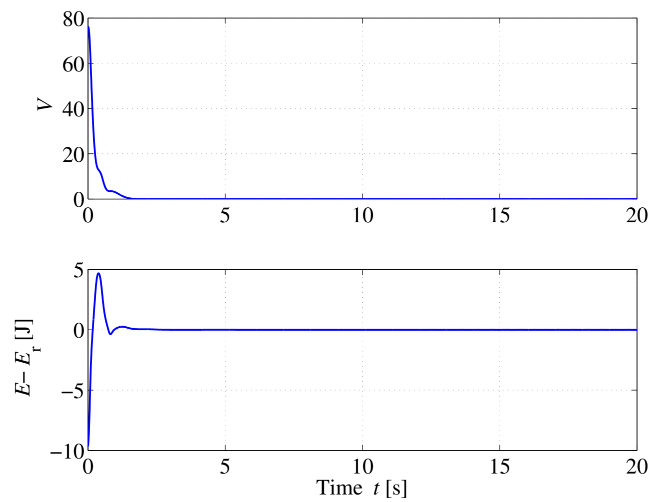

Fig. 2. Time responses of V and $E-E_r$ of the swing-up control: $k_2 = 1.0$.

B. Case $k_2 > \beta_2$

We took $k_2 = 1.5$ which does not satisfy (14). Since $k_2 > \beta_2$, the UEP is the unique equilibrium point belong to the set W_r in (35). According to (31), the swing-up controller (30) does not contain any singularity if and only if $k_D > 0.6899$. We took $k_D = 0.76$, $k_P = 22.40$, and $k_V = 4.60$.

Due to page limitations, we only present the simulation results of the time responses of q_1 and q_2 in Fig. 5 which shows that q_1 converged to 0, while several swings brought q_2 close to 0, and then q_2 remained very close to 0 with a very small oscillation (q_2 was not stabilized to 0). Since the AF robot was swung up and remained very close to the

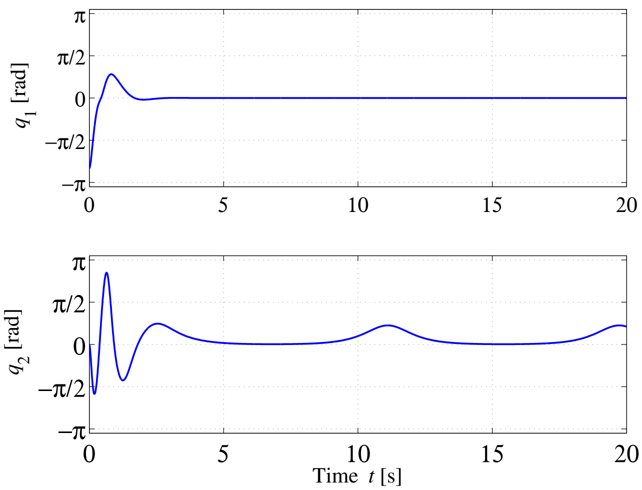

Fig. 3. Time responses of q_1 and q_2 of the swing-up control: $k_2 = 1.0$.

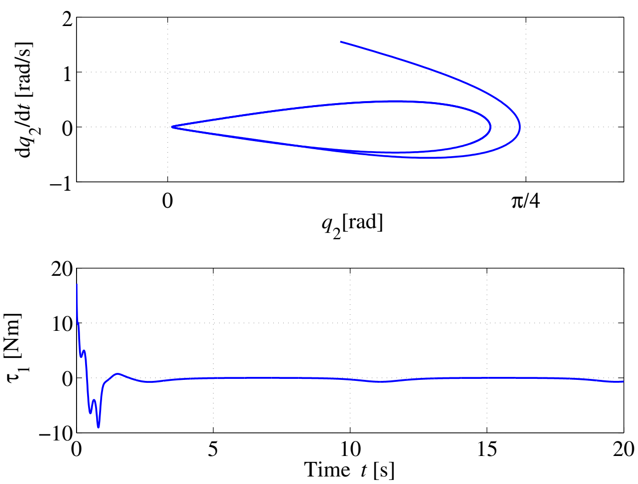

Fig. 4. Phase portrait of $(q_2, \dot{q}_2)$ in $2 \leq t \leq 20$ and time response of τ_1 of the swing-up control: $k_2 = 1.0$.

UEP, it is very easy to balance it about that point via a state-feedback stabilizing controller.

VII. CONCLUSION

This paper concerned the swing-up control for the AF robot moving in the vertical plane. First, we presented two new properties of the AF robot about the linear controllability at the UEP and the limitation of the PD control on the angle of the base joint. Second, for the AF robot which can be not locally stabilized about the UEP via the PD control, we showed how to extend the energy-based control approach to design a swing-up controller. We provided the necessary and sufficient condition for avoiding the singularity in the controller. Third, we analyzed the convergence of the total mechanical energy, and explored the structure of the closed-loop equilibrium points and clarified the stability of the UEP. We validated the theoretical results via numerical investigation. This paper not only unified some previous results for the Pendubot, but also provided insight the control and analysis of the underactuated robots with flexible joints.

A future research subject is to investigate whether the results obtained here can be generalized to multiple-degree-of-freedom underactuated robots with flexible joints.

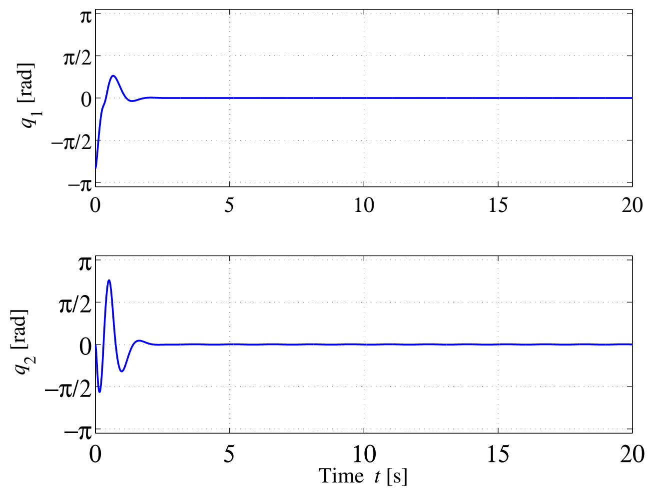

Fig. 5. Time responses of q_1 and q_2 under the swing-up control: $k_2 = 1.5$.

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