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Output-Feedback Sliding Mode Control for Global Tracking of Uncertain Nonlinear Time-Delay Systems

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***Abstract*— An output-feedback sliding mode controller is proposed for multivariable nonlinear systems with time-varying state delays and unmatched uncertainties. The control strategy is based on unit vector control and a novel norm observer for the unmeasured state of time-delay systems. This approach guarantees global stability of the closed-loop system, exponential convergence of the output error and exact tracking of the reference signal. In addition, less restrictive conditions on the high frequency gain matrix are obtained.**

***Index Terms*— Sliding mode control, Output-feedback, Time-delay systems, Uncertain systems, Nonlinear systems, Global stability.**

I. INTRODUCTION

The presence of time delay in dynamical systems is often the cause of instability and poor performance [1]. In addition, plant parameters may be unknown or imprecisely defined. Hence, time delay and uncertain plant parameters are serious obstacles to the stability and good performance of control systems. Thus, there is a crescent interest in the study of uncertain systems with time delay [2], [3], [4].

Sliding mode control (SMC) is an attractive methodology to uncertain systems [5], [6], [7]. The main advantages of SMC are robustness to parameter uncertainties and disturbances, fast response and good transient performance. However, time delay deteriorates the control performance since it is a cause of chattering and may destabilize the system.

The exponential stability of uncertain systems with state delay has been studied through the Lyapunov method and linear matrix inequalities (LMIs) in [8], assuming that the time delay is known and constant. In [9], exponential stability conditions are based on linear operator inequalities (LOIs) and applied to systems with unknown time-varying delay. In these works [8], [9] the system has no input or control signal.

Several approaches to control time-delay systems found in the literature are based on full-state measurement, e.g., [6], [7], [10], [1]. An alternative for output-feedback is the use of state observers as in [11], [12], which may be difficult to design for the class of uncertain time-varying nonlinear systems considered in the present paper. In [11], only nonlinear state-delayed systems with matched nonlinear terms are considered. On the other hand, the SMC in [12] is designed for nonlinear systems with known time-varying state delay and unmatched uncertainties, via the Lyapunov-Razumikhin approach. However, only local stability can be guaranteed. Reference [12, Remark 7] recognizes that the observer-based control scheme is dependent on the knowledge of the delay, which may limit its application.

Here we propose an output-feedback SMC for multivariable uncertain time-varying state delayed systems with unmatched nonlinear disturbances, which depend not only on the plant output but also on its unmeasurable state variables. Moreover, matched state dependent nonlinear disturbances are allowed to be, for instance, of polynomial type. This control scheme utilizes a *norm observer* [13], [14] for the unmeasured state vector. This is an advantage over previous output-feedback schemes, since *norm observers* are usually more robust to strong uncertainties than state observers. The resulting output-feedback controller guarantees global stability, exponential convergence of the output error and exact tracking. In addition, the use of unit vector control and part of the analysis method developed in [15] result in less restrictions imposed on the high frequency gain matrix of the system. To the best of our knowledge, such results are new in the context of SMC of nonlinear time-delay systems.

A. Notation and Terminology

The following notation and basic concepts are employed: **(1)** ISS means Input-to-State-Stable and classes $\mathcal{K}$, $\mathcal{K}_\infty$ functions are defined as in [16]. **(2)** The Euclidean norm of a vector x and the corresponding induced norm of a matrix A are denoted by $\|x\|$ and $\|A\|$, respectively. **(3)** As usual in SMC, Filippov's definition for solution of discontinuous differential equations is adopted [17].

II. PROBLEM FORMULATION

This paper considers the model-reference control of multi-input-multi-output (MIMO) nonlinear uncertain systems with time-varying delay, described by the state equations:

$$\dot{\eta} = \phi_0(\eta, \eta_d, y, y_d, t)\,, \qquad (1)$$

$$\dot{y} = \phi_1(\eta, \eta_d, y, y_d, t) + K_p u\,, \qquad (2)$$

where ϕ_0 and ϕ_1 are uncertain nonlinear functions, $u \in \mathbb{R}^l$ is the control input, $y \in \mathbb{R}^l$ is the measured output signal, and the state $\eta \in \mathbb{R}^{n-l}$ of the subsystem (1) is unmeasurable. To denote the time-delayed state vectors, the subscript d is introduced [12]: $\eta_d(t) := \eta(t - d(t))$ and $y_d(t) := y(t - d(t))$. As is usual in the time-delay systems framework [18, Sec. 1.2], the initial conditions are given by

$$\eta(t) = \eta_0(t)\,, \quad y(t) = y_0(t)\,, \quad t \in [-\bar{d}, 0]\,, \qquad (3)$$

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where $\eta_0(t)$ and $y_0(t)$ are vector functions continuous in $t \in [-\bar{d}, 0]$, and $\bar{d}$ is defined below in (A2).

The following assumptions regarding the system hold:

- **(A1)** For the high frequency gain (HFG) matrix K_p it is assumed that a matrix S_p is known such that $-K_p S_p$ is Hurwitz and $\|K_p^{-1}\| \leq c$, with $c > 0$ being a known constant.
- **(A2)** The time delay $d(t)$ is an uncertain piecewise-continuous function, and satisfies $0 < \underline{d} \leq d(t) \leq \bar{d} < +\infty$, where $\underline{d}$ and $\bar{d}$ are known bounds.
- **(A3)** The uncertain nonlinear functions ϕ_0 and ϕ_1 are piecewise continuous in t and locally Lipschitz in the other arguments.
- **(A4)** The norm of the state η of the subsystem (1) can be bounded by an exponentially stable *norm observer*.
- **(A5)** There exist *known* locally Lipschitz class $\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, $\alpha_3(\cdot)$ and a *known* non-negative function $\varphi_1(y, t)$ continuous in y, and piecewise continuous in t such that

$$\|\phi_1\| \leq \alpha_1(\|\eta\|) + \alpha_2(\eta_{\sup}) + \alpha_3(y_{\sup}) + \varphi_1(y, t)\,,$$

where $\eta_{\sup}(t) := \sup_{\tau \in [\underline{d}, \bar{d}]} \|\eta(t - \tau)\|$ and $y_{\sup}(t) := \sup_{\tau \in [\underline{d}, \bar{d}]} \|y(t - \tau)\|$.

The Hurwitz condition in (A1) is necessary and sufficient for the attractiveness of the sliding surface for unit vector SMC [15]. This assumption is less restrictive than usual conditions found in the literature of multivariable SMC, e.g., knowledge of the HFG as in [11], [12].

Here and in [9], the time delay $d(t)$ is allowed to be uncertain. From this point of view, assumption (A2) is less restrictive than the knowledge of the time delay assumed in [11], [12], [8].

Assumption (A3) guarantees the local existence and uniqueness of the solution of (1)–(2) for $u \equiv 0$. For each solution of (1)–(2) there exists a maximal time interval of definition given by $[0, t_M)$, where t_M may be finite or infinite. Thus, finite-time escape is not precluded *a priori*.

According to (A4), our output-feedback control scheme utilizes a norm observer [13], [14], instead of state observers, for the unmeasured state vector η, as will be explained in Section III. Even if state observers can be applied in output-feedback controllers, e.g., [11], [12], their design seems to be difficult for the class of uncertain time-varying nonlinear systems considered here. Norm observers are more advantageous than state observers since: **(1)** their structure is simpler than state observers; **(2)** norm observers give *"worst case"* upper bounds for the state norm and thus are more robust to uncertainties; **(3)** the design procedure of norm observers is independent of the order of the system, which is allowed to be uncertain [19] and; **(4)** norm observers have already been applied in sliding mode control schemes with global stability properties, e.g., [20], [14].

A first order exponentially stable norm observer for the subsystem (1) is a scalar dynamic system of the form (y is the plant output):

$$\dot{\bar{\eta}} = -\lambda_0 \bar{\eta} + c_0 \varphi_0(\|y\|, y_{\sup}, t)\,, \quad (4)$$

with input $\varphi_0(y, t)$ and output $\bar{\eta}$, such that: (i) $\lambda_0, c_0 > 0$ are constants; (ii) $\varphi_0(\|y\|, y_{\sup}, t)$ is a non-negative function continuous in $\|y\|$ and $y_{\sup}$, piecewise continuous and upper bounded in t; and (iii) for each initial states η_0 and $\bar{\eta}(0)$

$$\|\eta(t)\| \leq \bar{\eta}(t) + k_0\left(\eta_0^* + |\bar{\eta}(0)|\right) e^{-\lambda_0 t}\,, \quad (5)$$

$\forall t \in [0, t_M)$, with some constant $k_0 > 0$ and $\eta_0^* := \sup_{t \in [-\bar{d}, 0]} \|\eta_0(t)\|$.

In order to obtain a norm bound for ϕ_1 in (2), we additionally assume (A5). Note that (A5) is not too restrictive since ϕ_1 is assumed to be locally Lipschitz continuous in η, η_d, y and y_d. Furthermore, the bounding functions $\alpha_1, \alpha_2, \alpha_3$ and φ_1 do not impose particular growth conditions with respect to the nonlinear vector field ϕ_1. Thus, polynomial nonlinearities in η, η_d, y and y_d are not precluded.

A. Global Tracking Problem

The problem consists in designing an output-feedback control law u to drive the *output tracking error*

$$e(t) = y(t) - y_m(t) \quad (6)$$

asymptotically to zero (exact tracking), starting from any plant/controller initial conditions and maintaining uniform closed-loop signal boundedness. The *desired trajectory* $y_m(t)$ is assumed to be generated by the *reference model*:

$$\dot{y}_m = A_m y_m + r_m\,, \quad A_m = -\text{diag}\{a_1, \ldots, a_l\}\,, \quad (7)$$

where $a_i > 0, \forall i \in \{1, \ldots, l\}$, and $r_m, y_m \in \mathbb{R}^l$. The reference signal $r_m(t)$ is assumed piecewise continuous and uniformly bounded.

B. Error Equation

Subtracting (7) from (2), the error dynamics can be written as

$$\dot{e} = A_m e + K_p(u - u^*)\,, \quad (8)$$

where the *ideal control* $u^* := K_p^{-1}(-\phi_1 + A_m y + r_m)$ can be considered as a matched input disturbance in (8). From assumptions (A1)–(A5), the ideal control signal can be norm bounded by available signals, e.g.,

$$\begin{aligned} \|u^*\| \quad &\leq \quad c\,[\alpha_1(2|\bar{\eta}|) + \alpha_2(2\bar{\eta}_{\sup}) + \alpha_3(y_{\sup}) + \\ &+ \quad \varphi_1(y, t) + \|A_m y + r_m\|] + \pi_1\,, \end{aligned} \quad (9)$$

where c is given in (A1) and

$$\bar{\eta}_{\sup}(t) := \sup_{\tau \in [\underline{d}, \bar{d}]} |\bar{\eta}(t - \tau)|\,. \quad (10)$$

The term $\pi_1 := k_1\left(\eta_0^* + |\bar{\eta}(0)|\right) e^{-\lambda_0 t}$ ($k_1 > 0$ is an appropriate constant) bounds exponentially decaying signals due to initial conditions as in (5). To develop (9), we have considered the fact that α_1 and α_2 are locally Lipschitz and $\psi(a + b) \leq \psi(2a) + \psi(2b)$, $\forall a, b \geq 0$ and $\forall \psi \in \mathcal{K}_\infty$ [21, p. 94].

Then, the global tracking problem can be reformulated as the regulation problem described as follows. Find an output-feedback sliding mode control law u in such a way that, for all initial conditions $(y_0, \eta_0, e(0), \bar{\eta}(0))$: (i) the solutions of (1)–(2), (4) and (8) are bounded and (ii) $e(t)$ tends at least asymptotically to zero as $t \to +\infty$.

III. Norm Observers for Time-delay Systems

In this paper, we have assumed that one can obtain a norm observer of the form (4) for the state η of the subsystem (1). In this section, we characterize a class of MIMO nonlinear plants where a linear growth condition is required only *w.r.t.* the unmeasured state η and for which such exponentially stable norm observers can be implemented.

A. An Illustrative Class of Nonlinear Time-Delay Systems

We consider the class of nonlinear MIMO systems (1)–(2) with the function ϕ_0 given by

$$\phi_0(\eta,\eta_d,y,y_d,t) = A_0\eta + f_0(\eta,t) + f_1(\eta_d,t) + \bar{\phi}_0(y,y_d,t)\,, \tag{11}$$

where the matrix A_0 and nonlinear functions $f_0, f_1, \bar{\phi}_0$ can be uncertain.

In particular, the approach to design an exponentially stable norm observer developed here considers that A_0 is Hurwitz with *stability margin* [19] given by

$$\gamma_0 := -\max_i\{\mathrm{Re}(\gamma_i)\}\,, \tag{12}$$

where $\{\gamma_i\}$ are the eigenvalues of A_0. Moreover, the nonlinear functions $f_0(\eta,t)$ and $f_1(\eta_d,t)$ are bounded by linear growth functions, such as

$$\|f_0(\eta,t)\| \leq \mu_0\|\eta\|\,, \qquad \|f_1(\eta_d,t)\| \leq \mu_1\|\eta_d\|\,, \tag{13}$$

where μ_0 and μ_1 are positive known constants. We also assume that the output dependent nonlinear function $\bar{\phi}_0(y,y_d,t)$ is norm bounded by $\varphi_0(\|y\|,y_{\mathrm{sup}},t)$ as in (4), *i.e.*,

$$\|\bar{\phi}_0(y,y_d,t)\| \leq \varphi_0(\|y\|,y_{\mathrm{sup}},t)\,. \tag{14}$$

B. Norm Observer Design

In what follows, we show that the nonlinear subsystem (1) with ϕ_0 of the form (11) may admit an exponentially stable norm observer of the form (4).

Applying Lemma 1 (see Appendix A) and the upper bounds (13)–(14) to the system (1) and (11) one has:

$$\begin{aligned}\|\eta(t)\| &\leq c_1 e^{-\gamma_0 t} * \left\|f_0(\eta,t) + f_1(\eta_d,t) + \bar{\phi}_0(y,y_d,t)\right\| \\ &+ c_2\eta_0^* e^{-\gamma_0 t} \\ &\leq c_1 e^{-\gamma_0 t} * [\mu_0\|\eta\| + \mu_1\|\eta_d\| + \varphi_0(\|y\|,y_{\mathrm{sup}},t)] \\ &+ c_2\eta_0^* e^{-\gamma_0 t} \leq r(t)\,, \quad \forall t\in[0,t_M)\,,\end{aligned} \tag{15}$$

where

$$\begin{aligned} r(t) &= c_1 e^{-\gamma_0 t} * [\mu_0\|\eta\| + \mu_1\eta_{\mathrm{sup}} + \varphi_0(\|y\|,y_{\mathrm{sup}},t)] \\ &+ c_2\eta_0^* e^{-\gamma_0 t}\,,\end{aligned} \tag{16}$$

for some constants $c_1, c_2 > 0$, γ_0 defined in (12) and reminding that $\eta_0^* := \sup_{t\in[-\bar{d},0]}\|\eta_0(t)\|$ and $\eta_{\mathrm{sup}}(t) := \sup_{\tau\in[\underline{d},\bar{d}]}\|\eta(t-\tau)\|$. For design purposes, the constants c_1 and γ_0 can be computed for the uncertain matrix A_0 using the method proposed in [19]. The scalar signal $r(t)$ is the solution of the differential equation

$$\begin{aligned}\dot{r} &= -\gamma_0 r + c_1\left[\mu_0\|\eta\| + \mu_1\eta_{\mathrm{sup}} + \varphi_0(\|y\|,y_{\mathrm{sup}},t)\right]\,, \\ r(0) &= c_2\eta_0^*\,.\end{aligned} \tag{17}$$

Since $r(t)\geq\|\eta(t)\|$, $\forall t\in[0,t_M)$, then

$$\eta_{\mathrm{sup}}(t) \leq \sup_{\tau\in[\underline{d},\bar{d}]} r(t-\tau)\,. \tag{18}$$

In addition, $\dot{r} \geq -\gamma_0 r$ and by using the Comparison Theorem [17, Theorem 7], one has from (18) that

$$\begin{aligned}\eta_{\mathrm{sup}}(t) &\leq \sup_{\tau\in[\underline{d},\bar{d}]} e^{-\gamma_0[(t-\tau)-t]} r(t) \\ &\leq e^{\gamma_0\bar{d}} r(t)\,, \quad \forall t\in[0,t_M)\,.\end{aligned} \tag{19}$$

Upon substituting $\|\eta\|$ by r and η_{sup} by $e^{\gamma_0\bar{d}}r$ in (17) we get the differential equation

$$\begin{aligned}\dot{\bar{r}} &= (c_1\mu_0 + c_1\mu_1 e^{\gamma_0\bar{d}} - \gamma_0)\bar{r} + c_1\varphi_0(\|y\|,y_{\mathrm{sup}},t)\,, \\ \bar{r}(0) &= r(0) = c_2\eta_0^*\,,\end{aligned} \tag{20}$$

which satisfies $\bar{r}(t) \geq r(t) \geq \|\eta(t)\|$, $\forall t \in [0,t_M)$. We conclude that if

$$\lambda_0 = \gamma_0 - c_1\mu_0 - c_1\mu_1 e^{\gamma_0\bar{d}} > 0\,, \tag{21}$$

then (20) and, consequently, (4) is ISS *w.r.t.* the known nonlinear function $\varphi_0(\|y\|,y_{\mathrm{sup}},t)$. Hence, from (20), we equivalently obtain (4) by setting

$$c_0 \geq c_1\,, \tag{22}$$

and, finally, we can write (5).

Notice that the positive condition of λ_0, implies in a transcendental inequality

$$\gamma_0 > c_1\mu_0 + c_1\mu_1 e^{\gamma_0\bar{d}}\,. \tag{23}$$

Such inequality is useful to determine the maximum delay allowed by the stability margin γ_0 of A_0. Rewriting (23) as $e^{\gamma_0\bar{d}} \leq (\gamma_0 - c_1\mu_0)/c_1\mu_1$ and then applying $\ln(\cdot)$ in both sides yields

$$\bar{d} \leq \frac{1}{\gamma_0}\ln\frac{\gamma_0 - c_1\mu_0}{c_1\mu_1}\,. \tag{24}$$

As in [8], we can realize that the maximum time-delay can be increased by tuning the stability margin γ_0 of A_0, but there is a natural physical limitation imposed by the argument of $\ln(\cdot)$ function which must be positive, *i.e.*, $\gamma_0 > c_1\mu_0$.

The design of the norm observer for time-delay systems proposed here may be conservative for some systems. Therefore, less conservative procedures could explore favorable characteristics of the system of interest and optimization techniques, for instance, following the ideas in [19], [8].

In the next section, we discuss the design of the proposed output-feedback controller based on exponentially stable norm observers.

IV. Output-feedback Sliding Mode Controller

From assumption (A1), there exists a known pre-compensator matrix S_p which assures that $-K_pS_p$ is Hurwitz. Thus, the Lyapunov equation $(K_pS_p)^T P + P(K_pS_p) = I$ has a solution $P = P^T > 0$.

$$\dot{\eta} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \eta + \begin{bmatrix} -0.5\eta_1 \sin\eta_2 + 0.2\eta_{1d}^{\frac{1}{3}}\eta_{2d}^{\frac{2}{3}} - 1y_1^2 \\ 0.5\eta_2\cos\eta_1 - 0.2\eta_{1d} + 0.2\eta_{2d} + 1.5y_2^2 \end{bmatrix} \tag{25}$$

$$\dot{y} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \eta + \begin{bmatrix} -0.2\eta_2^5 + 0.7y_1^3 \\ -1.1\eta_{1d}^4 + 0.8y_2^2 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ -2 & 1.5 \end{bmatrix} u \tag{26}$$

In this case, one can apply the *unit vector control*[1] (UVC) law [15]

$$u = -S_p \varrho \frac{e}{\|e\|}, \quad e \neq 0, \tag{27}$$

to (8) and verify that, if the modulation function ϱ satisfies

$$\varrho \geq c_d\|u^*\| + c_e\|e\| + \delta, \quad \delta \geq 0, \tag{28}$$

modulo the exponentially decaying term $c_d\pi_1$ (π_1 from (9)), then the time Dini derivative of $V = \sqrt{e^T P e}$ along the solutions of (8) satisfies:

$$\dot{V} \leq -\lambda_m V + \frac{c_d \pi_1}{2\sqrt{\lambda_{min}(P)}}, \quad \forall t \in [0, t_M), \tag{29}$$

where $0 < \lambda_m < \min_i\{a_i\}$, $i = 1, \ldots, l$ in (7),

$$c_d \geq 2\|PK_p\|, \quad c_e \geq \|A_m^T P + PA_m\| + \lambda_m. \tag{30}$$

Moreover, if $a_i = \lambda_m$ ($\forall i$), then one can choose $c_e = 0$ [15, Corollary 1].

Hence, by using the Comparison Theorem [17, Theorem 7], one has:

$$\|e(t)\| \leq \|e(0)\|e^{-\lambda_m t} + \pi_2, \quad \forall t \in [0, t_M), \tag{31}$$

where $\pi_2 := \Psi_2(|\bar{\eta}(0)| + \eta_0^*)e^{-\lambda_c t}$, $0 < \lambda_c < \min\{\lambda_0, \lambda_m\}$ and $\Psi_2 \in \mathcal{K}$ (for details, see [15, Lemma 1]). Thus, the tracking control objective can be attained.

Using inequality (9), one alternative modulation function which satisfies inequality (28) is

$$\begin{aligned} \varrho &= c_d c\, [\alpha_1(2|\bar{\eta}|) + \alpha_2(2\bar{\eta}_{\sup}) + \alpha_3(y_{\sup}) + \\ &+ \varphi_1(y,t) + \|A_m y + r_m\|] + c_e\|e\| + \delta, \end{aligned} \tag{32}$$

where the scalar signals $\bar{\eta}$ and $\bar{\eta}_{\sup}$ are generated by a norm observer in (4) and (10).

The stability results are summarized in the next section.

V. Stability Results

In the following stability analysis, the inverse dynamics state η and the norm observer state $\bar{\eta}$ are treated as exogenous signals in the error system (8). The main result is now stated.

Theorem 1: **(*Main Result*)** Consider nonlinear time-delay systems transformable into the form (1)–(2) with UVC law (27) and modulation function (32) constructed by using the norm observer (4). Assume that (A1)–(A5) hold. Then, the error system (8) is globally asymptotically stable and ultimately exponentially convergent to zero. Moreover, all signals in the closed-loop system remain uniformly bounded.

Proof: See Appendix B. ■

Finite frequency chattering is avoided and an ideal sliding mode is produced according to the following corollary.

Corollary 1: **(*Ideal Sliding Mode*)** Additionally to the results of Theorem 1, if $\delta > 0$ in (28), then the sliding mode on the manifold $e = 0$ starts in finite-time.

Proof: See Appendix C. ■

VI. Numerical Example

Consider the uncertain nonlinear time-varying delayed system described in (25)–(26), where $\eta = [\eta_1, \eta_2]^T \in \mathbb{R}^2$, $\eta_d = [\eta_{1d}, \eta_{2d}]^T \in \mathbb{R}^2$, $y = [y_1, y_2]^T \in \mathbb{R}^2$ and $u \in \mathbb{R}^2$.

The selected reference model and the reference signal used in the simulations are given by

$$\dot{y}_m = \begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix} y_m + \begin{bmatrix} \sin(10t) \\ \cos(10t) \end{bmatrix}. \tag{33}$$

The knowledge of the time-varying delay

$$d(t) = 0.3\sin(t) + 0.5 \tag{34}$$

(in seconds) is not needed in the control system design. However, the upper bound $\bar{d} = 0.8\,\mathrm{s}$ is required to design the norm observer (see eq. (21)).

The matrix $S_p = I$ was chosen such that $-K_p S_p$ is Hurwitz and (A1) is verified. From this assumption, we also have the constant $c = 1.4 \geq \|K_p^{-1}\|$ used in the modulation function (32).

The nonlinear terms in η and in η_d present in (25) satisfy (13) with $\mu_0 = 0.5$ and $\mu_1 = 0.2$.

To satisfy (23) and (24), $\gamma_0 = 1$, $c_1 = 1$, and $\lambda_0 = 0.0549\,\mathrm{rad/s}$. The coefficient $c_0 = 1.1$ is chosen to satisfy (22).

Due to (30), $c_d = 35$ and $c_e = 6$. The constant $\delta = 2$ was arbitrarily chosen to guarantee finite-time convergence of the error signal to zero.

Figures 1 and 2 display the simulation results for this control system. Note that y reaches y_m in finite-time and, therefore, the output tracking error becomes null as expected.

To illustrate the effect of time-varying delay, Fig 3 shows the actual state variable η_1 of the subsystem (25) and the corresponding delayed state variable (η_{1d}). The delayed signal is distorted due to the time-varying nature of the delay given by (34). It is remarkable that the controller stability and performance are robust to this unknown delay.

[1]Since $e = 0$ corresponds to a set of zero Lebesgue measure then, according to Filippov's theory, the control value at this point is irrelevant. However, for the sake of mathematical consistency, we assume that $u = 0$ if $e = 0$.

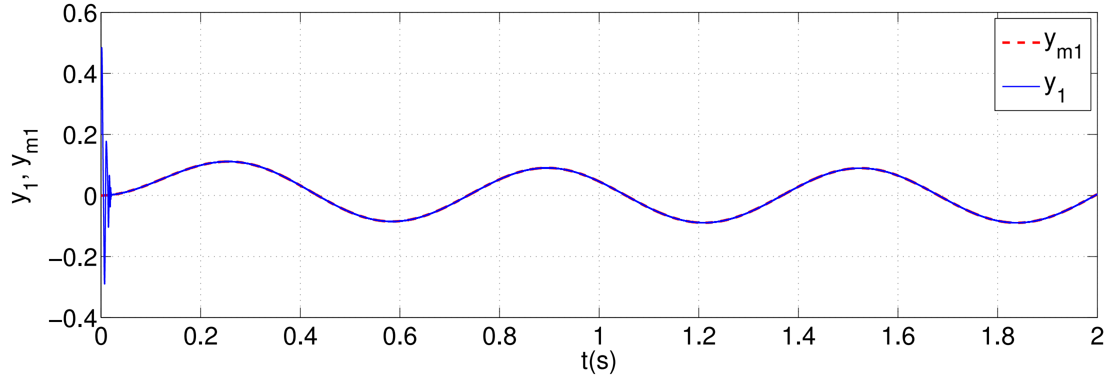

Fig. 1. System output $y_1(t)$ and reference model output $y_{m1}(t)$.

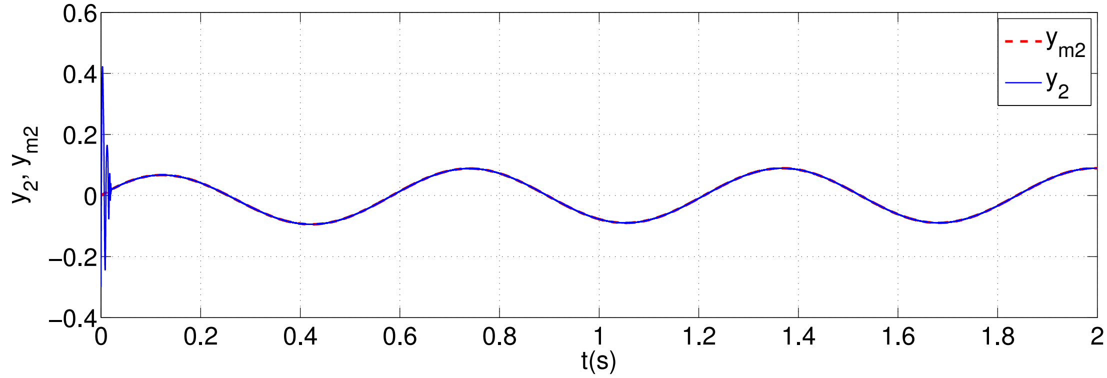

Fig. 2. System output $y_2(t)$ and reference model output $y_{m2}(t)$.

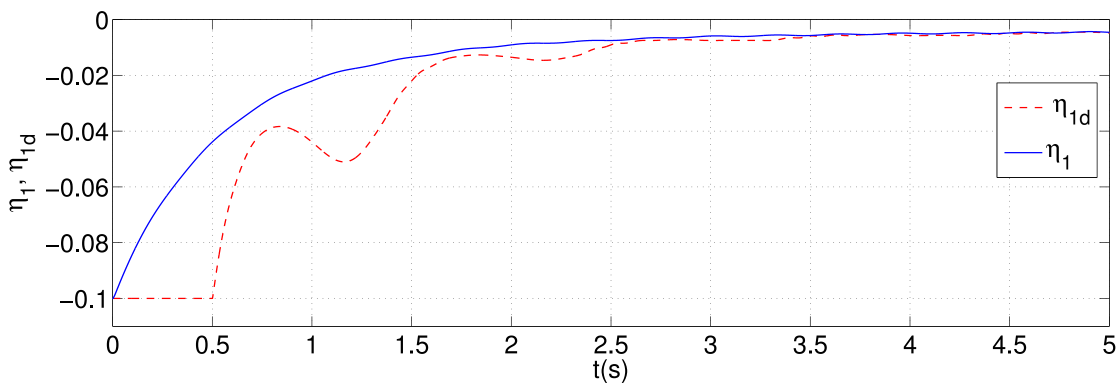

Fig. 3. Actual state variable $\eta_1(t)$ of the subsystem (25) and the corresponding delayed state variable $\eta_{1d}(t) = \eta_1(t - d(t))$.

VII. Conclusion

An output-feedback controller was developed for uncertain multivariable systems with time-varying delay and unmatched state dependent nonlinearities. The usual assumption of uniform norm boundedness with respect to unmeasured states is not required. Based on unit vector sliding mode control and estimation of the norm of the unmeasurable state, the controller leads to global stability and finite-time convergence of the output tracking error to zero. To the best of our knowledge, such results are new in the sliding mode control literature concerning time-delay systems.

Appendix

A. Auxiliary Lemmas

Lemma 1: Consider the system

$$\dot{\eta}(t) \quad = A_0\eta(t) + f(t)\,, \tag{35}$$

where $\eta, f \in \mathbb{R}^{n-l}$. Let $\gamma_0 := \min_i \{-Re(\gamma_i)\}$ be the stability margin of A_0, where $\{\gamma_i\}$ are the eigenvalues of A_0 and $\gamma := \gamma_0 - \delta$ with $\delta > 0$ being an arbitrary constant. Let $\bar{f}(t)$ be an instantaneous upper bound of $f(t)$, *i.e.*, $\|f(t)\| \leq \bar{f}(t)$, $\forall t \geq 0$. Then, $\exists c_1, c_2 > 0$ such that the matrix exponential satisfies $\|e^{A_0 t}\| \leq c_1 e^{-\gamma t}$ and the following inequalities hold $\forall t \geq 0$

$$\|e^{A_0 t} * f(t)\| \leq c_1 e^{-\gamma t} * \bar{f}(t)\,, \quad (36)$$
$$\|\eta(t)\| \leq c_1 e^{-\gamma t} * \bar{f}(t) + c_2 e^{-\gamma t}\|\eta(0)\|\,. \quad (37)$$

Proof: See proof of [20, Lemma 2]. ∎

Lemma 2: Let $r(t)$ be an absolutely continuous scalar function. Suppose $r(t)$ is non-negative and while $r > 0$ it satisfies $\dot{r} \leq -\delta - \gamma r + Re^{-\lambda t}$, where $\delta, \gamma, \lambda, R$ are non-negative constants. Then, one can conclude that: **(a)** $r(t)$ is bounded by $r(t) \leq [r(0) + cR]\, e^{-\lambda_1 t}, \forall t \geq 0$, where $c > 0$ is an appropriate constant and $\lambda_1 < \min(\lambda, \gamma)$; **(b)** if $\delta > 0$ then $\exists t_1 < +\infty$ such that $r(t) = 0$, $\forall t \geq t_1$.

Proof: The proof presented in [22, Lemma 3] is based on the Comparison Theorem [17, Theorem 7]. ∎

B. Proof of Theorem 1

In order to fully account for all initial conditions in the closed-loop system, let $z(t) := [z^{0T}(t), e^T(t)]^T$, where the vector signal $z^0(t)$ represents the transient terms of the states $\eta(t)$ and $\bar{\eta}(t)$, which can be bounded by $\|z^0(t)\| \leq c_z(\eta_0^* + |\bar{\eta}(0)|)e^{-\lambda_z t}$, with $\eta_0^* := \sup_{t \in [-\bar{d}, 0]} \|\eta_0(t)\|$ and $c_z, \lambda_z > 0$ being appropriate constants.

The proof is carried out in two parts as follows.

1) Global Stability: Once it is assumed that the modulation function ϱ in (27) is implemented in order to satisfy (28), the exponential convergence of the tracking error $e(t)$ to zero can be proved as shown in (31), using the Razumikhin Theorem [18, pp. 13–15]. In this case, it is clear that $\|z(t)\| \leq \Psi_1(\|z(0)\|)$, $\forall t \in [0, t_M)$, where $\Psi_1 \in \mathcal{K}$. Thus, given any $R > 0$, for $\|z(0)\| < R_0$ with $R_0 \leq \Psi_1^{-1}(R)$, one has that $\|z(t)\|$ is bounded away from R as $t \to t_M$. This implies that $z(t)$ is uniformly bounded and cannot escape in finite-time, *i.e.*, $t_M \to +\infty$. Hence, stability with respect to any ball of radius R is guaranteed for $z(0)$ in the R_0-ball. Since R and thus R_0 can be chosen arbitrarily large, global stability is concluded.

2) Closed-loop signal boundedness and exponential convergence: Since $e(t)$ converges to zero exponentially, then, one concludes that $z(t)$ will converge to zero at least exponentially. Reminding that $y = e + y_m$ and y_m is uniformly bounded, then, from (A4) and from the ISS property of the $\bar{\eta}$-dynamics in (4) *w.r.t.* the output dependent nonlinear function φ_0, we can conclude that $\bar{\eta}$, η, and, consequently, u are also uniformly bounded. Thus, one concludes that all closed-loop system signals are uniformly bounded. ∎

C. Proof of Corollary 1

In what follows, $k_i > 0$ are appropriate constants not depending on the initial conditions.

Analogously to what was shown in Section IV, if the modulation function ϱ in the control law (27) satisfies (28) with $\delta > 0$, then, the time Dini derivative of $V = \sqrt{e^T P e}$ along the solutions of (8) is such that:

$$\dot{V} \leq -k_1 \delta - k_2 V + k_3 \pi\,,$$

where π is an exponentially decaying term. Therefore, from Lemma 2 (see Appendix A), one can further conclude that $\exists t_1 < +\infty$ such that $V(t) = 0$, $\forall t \geq t_1$, hence, the sliding mode at $e = 0$ starts in some finite time t_s, $0 \leq t_s \leq t_1$. ∎

REFERENCES

[1] A. Si-Ammour, S. Djennoune, and M. Bettayeb, "A sliding mode control for linear fractional systems with input and state delays," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 5, pp. 2310–2318, May 2009.

[2] J. P. Richard, "Time-delay systems: an overview of some recent advances and open problems," *Automatica*, vol. 39, no. 10, pp. 1667–1694, 2003.

[3] M. Sbarciog, R. Keyser, S. Cristea, and C. Prada, "Nonlinear predictive control of processes with time delay. a temperature control case study," in *Proc. Int. Conf. on Control Applications*, San Antonio (TX), 2008, pp. 1001–1006.

[4] G. Liu, A. Zinober, and Y. B. Shtessel, "Second-order SM approach to SISO time-delay system output tracking," *IEEE Trans. Ind. Electronics*, vol. 56, no. 9, pp. 3638–3645, September 2009.

[5] V. I. Utkin, *Sliding Modes and Their Application in Variable Structure Systems*. Moscow: MIR Publishers, 1978.

[6] N. Luo and M. Sen, "State feedback sliding mode control of a class of uncertain time delay systems," in *Proc. Control Theory and Applications*, vol. 140, July 1993, pp. 261–274.

[7] M. S. Mahmoud and N. F. Al-Muthairi, "Quadratic stabilization of continuous time systems with state-delay and norm-bounded time-varying uncertainties," in *IEEE Trans. Aut. Contr.*, vol. 39, no. 10, October 1994, pp. 2135–2139.

[8] P. Nam, "Exponential stability criterion for time-delay systems with nonlinear uncertainties," *Applied Mathematics and Computation*, vol. 214, no. 2, pp. 374–380, August 2009.

[9] E. Fridman and Y. Orlov, "Exponential stability of linear distributed parameter systems with time-varying delays," *Automatica*, vol. 45, no. 1, pp. 194–201, January 2009.

[10] M. Basin, J. Gonzalez, and L. Fridman, "Optimal and robust control for linear state-delay systems," *Journal of the Franklin Institute*, vol. 344, no. 6, pp. 830–845, September 2007.

[11] Y. Niu, J. Lam, X. Wang, and D. Ho, "Observer-based sliding mode control for nonlinear state-delayed systems," *International Journal of Systems Science*, vol. 35, no. 2, pp. 139–150, February 2004.

[12] X.-G. Yan, S. K. Spurgeon, and C. Edwards, "Sliding mode control for time-varying delayed systems based on a reduced-order observer," *Automatica*, vol. 46, no. 8, pp. 1354–1362, August 2010.

[13] M. Krichman, E. D. Sontag, and Y. Wang, "Input-output-to-state stability," *SIAM J. Contr. Optim.*, vol. 39, no. 6, pp. 1874–1928, 2001.

[14] T. R. Oliveira, A. J. Peixoto, and L. Hsu, "Sliding mode control of uncertain multivariable nonlinear systems with unknown control direction via switching and monitoring function," *IEEE Trans. Aut. Contr.*, vol. 55, no. 4, pp. 1028–1034, 2010.

[15] J. P. V. S. Cunha, L. Hsu, R. R. Costa, and F. Lizarralde, "Output-feedback model-reference sliding mode control of uncertain multivariable systems," *IEEE Trans. Aut. Contr.*, vol. 48, no. 12, pp. 2245–2250, 2003.

[16] H. K. Khalil, *Nonlinear Systems*, 3^{rd} ed. Prentice Hall, 2002.

[17] A. F. Filippov, "Differential equations with discontinuous right-hand side," *American Math. Soc. Translations*, vol. 42, no. 2, pp. 199–231, 1964.

[18] K. Gu, V. L. Kharitonov, and J. Chen, *Stability of Time-Delay Systems*. Birkhäuser, 2003.

[19] J. P. V. S. Cunha, R. R. Costa, and L. Hsu, "Design of first-order approximation filters for sliding-mode control of uncertain systems," *IEEE Trans. Ind. Electronics*, vol. 55, no. 11, pp. 4037–4046, 2008.

[20] L. Hsu, R. R. Costa, and J. P. V. S. Cunha, "Model-reference output-feedback sliding mode controller for a class of multivariable nonlinear systems," *Asian Journal of Control*, vol. 5, no. 4, pp. 543–556, 2003.

[21] E. D. Sontag, "Comments on integral variants of ISS," *Systems & Contr. Letters*, vol. 34, no. 1-2, pp. 93–100, May 1998.

[22] L. Hsu, J. P. V. S. Cunha, R. R. Costa, and F. Lizarralde, "Multivariable output-feedback sliding mode control," in *Variable Structure Systems: Towards the 21st Century*, X. Yu and J.-X. Xu, Eds. Berlin: Springer-Verlag, 2002, pp. 283–313.