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Flatness-based control of a quadrotor helicopter via feedforward linearization

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Abstract— **The problem of control law design for a small scale quadrotor helicopter is considered. The control problem is decomposed in an outer (position) loop and an inner (attitude) one. An approach based on the flatness property of the quadrotor position dynamics is proposed, while attitude control is dealt with by means of an (almost) globally stabilising control law.**

I. Introduction and motivation

Control of rotorcraft UAVs is a rapidly expanding research area, see, *e.g.*, the recent books [2], [15]. The control problems associated with such systems are particularly challenging for a number of reasons. First of all, rigid body dynamics is characterised by strong nonlinearities, which lead among other things to tight inter-axis coupling effects. In addition, most rotorcraft configurations (including both the conventional main/tail rotor one and the quadrotor one studied in this paper) are underactuated with respect to the six rigid body degrees of freedom. Finally, parametric uncertainty has to be taken into account because of, *e.g.*, varying payload characteristics. The state-of-the-art in *linear* control for small scale helicopters is given by approaches such as, *e.g.*, [12], in which modern robust control design is coupled to identification of linear rotorcraft models. More general approaches available in the literature, on the other hand, consider *nonlinear* trajectory planning and tracking techniques, which can be adapted to all the main operation modes of a rotorcraft UAV. Many methods have been proposed, covering, *e.g.*, control on nonlinear manifolds [13], adaptive control [2], dynamic inversion [4]. Of particular interest are methods for planning and tracking based on the flatness property of helicopter dynamics (see, *e.g.*, [18]) as well as procedures based on smoothing of a given trajectory (e.g., expressed as a sequence of way-points) by using a transcribed version of the dynamics expressed in terms of motion primitives (see for example [1] and references therein). In both cases the resulting trajectories are compatible by design with the vehicle model, embedded either in the algebraic flat model or in the motion primitives, and are trackable accurately by reflexive controllers.

Flatness of rigid body motion has been studied extensively in the literature and has been exploited in a number of contributions. More precisely, with specific reference to the rotorcraft literature, the attention has focused mainly on the trajectory planning and optimisation problem, see for example [3] and the references therein. Some contributions exist, however, in which control design has been considered, such as [11], where the problem of controlling a conventional main/tail rotor configuration using flatness-based techniques has been analysed. With respect to the cited work, in this paper feedforward-linearization control of position dynamics is studied, a novel yaw compensation via a rotation matrix is implemented and discussed (leading to a simpler design procedure and a simpler control law) and finally an almost global attitude control law is employed, based on a reparameterisation of the quadrotor attitude in terms of the Modified Rodrigues Parameters (MRPs).

The paper is organised as follows. Section II presents the equations of motion for the quadrotor helicopter studied in this paper, while Section III provides the relevant background on flatness and flatness-based control. The proposed control architecture is described in detail in Section IV, while some simulation results are presented and discussed in Section V.

II. Quadrotor description and modeling

The equations of motion for the quadrotor will be derived by relying on two reference frames: the Earth inertial reference frame (E-frame) and the Body-fixed reference frame (B-frame). The angular position (or attitude) of the quadrotor is defined by the orientation of the B-frame with respect to the E-frame, whereas position is defined on the E-frame. Furthermore, we will rely on the following assumptions:

1) the origin of the B-frame is located at the center of mass of the vehicle;
2) the body is rigid;
3) the axes of the B-frame coincide with the body principal axes of inertia (the inertia matrix is diagonal);
4) rotor thrust is proportional to the square of the rotor's angular rate (see, *e.g.*, [9]).

Even though every assumption is somehow restrictive, in this case the above hypotheses can be verified on a large number of present and past quadrotor models. In particular, the rigid body assumption allows to neglect all aeroelastic phenomena, that are not significant on such a small vehicle. Define the six degrees of freedom of the rigid body as $\mathbf{q} = [x, y, z, \phi, \theta, \psi]^T$, where the triple (x, y, z) represents the position of the center of mass (in the E-frame) and the "roll-pitch-yaw" (ϕ, θ, ψ) set of Euler angles is the representation of the orientation of the quadrotor in the same reference frame. The dynamical model can be derived by

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following the Lagrangian approach:

$$\begin{aligned}\ddot{x} &= U_1 \frac{(\cos\psi\cos\phi\sin\theta + \sin\psi\sin\phi)}{m} \\ \ddot{y} &= U_1 \frac{(\sin\psi\cos\phi\sin\theta - \sin\phi\cos\psi)}{m} \\ \ddot{z} &= U_1 \frac{(\cos\theta\cos\phi)}{m} - g\end{aligned} \quad (1)$$

$$\begin{aligned}\dot{p} &= \frac{(I_y - I_z)}{I_x} qr + \frac{1}{I_x} U_2 - \frac{J_m}{I_x} q\,\Omega_R \\ \dot{q} &= \frac{(I_z - I_x)}{I_y} pr + \frac{1}{I_y} U_3 + \frac{J_m}{I_y} p\,\Omega_R \\ \dot{r} &= \frac{(I_x - I_y)}{I_z} pq + \frac{d}{I_z} U_4\end{aligned} \quad (2)$$

$$\begin{aligned}\dot{\phi} &= p + \sin(\phi)\tan(\theta) q + \cos(\phi)\tan(\theta) r \\ \dot{\theta} &= \cos(\phi) q - \sin(\phi) r \\ \dot{\psi} &= \frac{\sin(\phi)}{\cos(\theta)} q + \frac{\cos(\phi)}{\cos(\theta)} r\end{aligned} \quad (3)$$

where p, q and r are the attitude velocities in the B-frame, m is the vehicle mass, I_x, I_y and I_z are the body principal moments of inertia, J_m is the motor inertia and Ω_R reads $\Omega_R = -\Omega_1 - \Omega_3 + \Omega_2 + \Omega_4$. The input vector $(U_1, U_2, U_3, U4)$ has been defined so that the state rates are linear in the control variables. The U_is are defined in terms of the motor angular rates Ω_i, $i = 1, \ldots, 4$ as

$$\begin{aligned}U_1 &= b \sum_{i=1}^{4} \Omega_i^2, \\ U_2 &= bl(\Omega_4^2 - \Omega_2^2), \\ U_3 &= bl(\Omega_3^2 - \Omega_1^2), \\ U_4 &= d(-\Omega_1^2 + \Omega_2^2 - \Omega_3^2 + \Omega_4^2),\end{aligned}$$

where b and d are, respectively, the thrust and drag factors of the vehicle. Note that the motor angular rates are not directly controllable in practice, however it can be assumed that each of the Ω_is is equal to its reference value in case the motor control bandwidth is sufficiently large with respect to the dynamics of the quadrotor.

III. Background

The theory of flat systems is a large and complex research area, for which overviews are presented in, *e.g.*, [6], [14] in both differential algebraic and differential geometric contexts. Formally, the following definition can be given.

Definition 1: A nonlinear system $\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t))$, with time $t \in \mathbb{R}$, state $\mathbf{x}(t) \in \mathbb{R}^n$ and input $\mathbf{u}(t) \in \mathbb{R}^m$, is said to be *(differentially) flat* if there exists a set of m differentially independent variables $\mathbf{w} = [w_1, \ldots, w_m]^T$, called *flat outputs*, such that:

$$\mathbf{w} = G(\mathbf{x}, \mathbf{u}, \dot{\mathbf{u}}, \ldots, \mathbf{u}^{(\delta)})$$

$$\mathbf{x} = f_x(\mathbf{w}, \dot{\mathbf{w}}, \ldots, \mathbf{w}^{(\rho)})$$

$$\mathbf{u} = f_u(\mathbf{w}, \dot{\mathbf{w}}, \ldots, \mathbf{w}^{(\rho+1)})$$

where G, f_x and f_u are smooth functions of their arguments, at least in an open subset of their domain, and δ, ρ are the maximum orders of derivatives of u and w needed to describe the system.

This property can be very important, *e.g.*, if a system is flat, a nominal input signal $\mathbf{u}^{\circ}$ can be algebraically computed from the flat output reference value or from the actual measurements of the outputs. The potential of flatness in motion control problems is then evident.

A possible way to exploit differential flatness for control of nonlinear systems is the exact feedforward linearization approach introduced in [7]. Such approach can be considered to design control laws which are not forced to exactly feedback linearize the nonlinear system. This strategy provides some good properties to the control system design. As a matter of fact, it is proved in [7] that a differentially flat system, to which a nominal feedforward $\mathbf{u}^{\circ}$ deduced from its flatness is applied, is equivalent, by change of coordinates, to a linear multivariable Brunovsky form without closing the loop, if the initial condition is consistent with the one considered in the design of the nominal trajectory. Moreover, if the initial condition which is taken into consideration for designing the nominal feedforward is sufficiently close to the true one, then a unique solution exists for the non-linear flat system in the vicinity of the desired solution of the aforementioned Brunovsky form. This also relates flatness to the existence of solutions of non-autonomous systems of differential equations.

Stability of this control scheme is demonstrated when using extended PID controls for the feedback part. By considering the stability result in [10], it can be shown that the absolute values of the control coefficients have to be traded off with the velocity of the desired trajectory. Thus, given reasonable bounded initial errors, non-linear flat systems can be stabilized around given desired trajectories by applying exact feedforward linearization and extended PID control. More recently, [8] have shown that such a control scheme is also robust with respect to time-varying parameters and exogenous perturbations.

IV. Flatness-based quadrotor control

It is apparent from (2) that inputs U_2, U_3 and U_4 act individually on ϕ, θ and ψ, respectively. Moreover, such inputs do not have a direct effect on the quadrotor position, since they act on (x, y, z) through the attitude coordinates only. It follows that (as already noticed in, *e.g.*, [11]) the control task can be decoupled in two subproblems concerning attitude control, handled via U_2, U_3 and U_4, and position control, that can be handled with U_1 and the reference values ϕ°, θ° and ψ°. The architecture of the quadrotor control system proposed in this paper is depicted in Figure 1. In detail, the overall scheme is composed by the following blocks.

- A *trajectory generation* block, that provides the quadrotor desired path in terms of the controlled degrees of freedom. Indeed, since the input vector has four

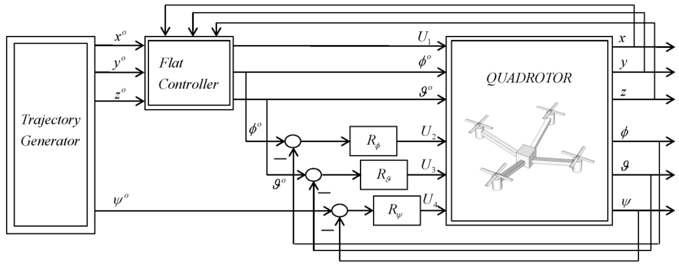

Fig. 1. Block diagram of the quadrotor guidance and control scheme based on differential flatness theory.

dimensions, only four degrees of freedom can be independently selected; in the present case, the quadruple (x, y, z, ψ) is given by this block, and the reason of this choice will become clear in the following Section. Note however that flatness-based design of the trajectory generation block has been dealt with in the classical paper [18] and is therefore not treated herein.

- A *position control block*, based on flatness of the position dynamics and exact feedforward linearization theory (see [7]), that computes the nominal input trajectories and compensates the position tracking errors. Note that, unlike the scheme proposed in [11] where flatness is applied to the quadruple (x, y, z, ψ), in this work the yaw angle is dealt separately in a simpler way, as illustrated in Section IV-B.
- Three passivity-based *attitude controllers*, which will be described in Section IV-C.

The position controller, the yaw compensator and the attitude controllers will be presented in the following subsections.

A. Flatness and feedforward linearization of the position dynamics

In view of the control architecture described in the previous Section, the position controller can be designed by considering the reduced-order model given by (1), where the state-vector is $\mathbf{x} = [x, \dot{x}, y, \dot{y}, z, \dot{z}]$, the input vector is $\mathbf{u} = [U_1, \phi, \theta]$ and m, g and ψ are system parameters. The following result holds.

Proposition 1: The system (1) is differentially flat.

Proof: Consider $\mathbf{w} = [w_1, w_2, w_3]^T = [x, y, z]^T$ as the flat output vector. The properties of the flat output can be verified as follows. First of all, $\mathbf{w}$ is a function of the state, specifically the linear selection

$$\mathbf{w} = G(\mathbf{x}) = G\mathbf{x}\ ,\ \ G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Obviously, first derivatives of $\mathbf{x}$ are linearly connected to the first derivative of $\mathbf{w}$ and $\mathbf{x} = f_x(\mathbf{w}, \dot{\mathbf{w}})$, with $f_x = I_{6\times 6}$. Moreover, concerning the input vector, it can be shown, after some computations, that

$$\sin(\phi) = \frac{m\ddot{x}}{U_1}\sin\psi - \frac{m\ddot{y}}{U_1}\cos\psi,$$

$$\sin(\theta)\cos(\phi) = \frac{m\ddot{x}}{U_1}\cos\psi + \frac{m\ddot{y}}{U_1}\sin\psi.$$

Since it holds that

$$U_1 = m\sqrt{\ddot{x}^2 + \ddot{y}^2 + (\ddot{z} + g)^2} = m\sqrt{\ddot{w_1}^2 + \ddot{w_2}^2 + (\ddot{w_3} + g)^2},$$

it follows that ϕ and θ are expressed as in (4). Therefore, $\mathbf{w} = G(\mathbf{x}) = G\mathbf{x}$ is such that $\mathbf{x}$ and $\mathbf{u}$ can be written as functions of $\mathbf{w}$ and its derivatives, hence $\mathbf{w}$ is a flat output with $\rho = 1$. ■

Assume now that the yaw angle ψ is zero, and recall that, according to [5], every flat system can be represented using the Brunovsky state ξ. Let ξ be defined as

$$\begin{aligned} \xi &= [\xi_{1,1}, \xi_{1,2}, \xi_{2,1}, \xi_{2,2}, \xi_{3,1}, \xi_{3,2}]^T = \\ &= [x, \dot{x}, y, \dot{y}, z, \dot{z}]^T. \end{aligned} \tag{5}$$

The simplified model of the system (1) with $\psi = 0$ can now be rewritten in Brunovsky form as

$$\begin{aligned} \dot{\xi}_{1,1} &= \xi_{1,2} \\ \dot{\xi}_{1,2} &= \frac{U_1}{m}\sin(\theta^\circ)\cos(\phi^\circ) \\ \dot{\xi}_{2,1} &= \xi_{2,2} \\ \dot{\xi}_{2,2} &= -\frac{U_1}{m}\sin(\phi^\circ) \\ \dot{\xi}_{3,1} &= \xi_{3,2} \\ \dot{\xi}_{3,2} &= \frac{U_1}{m}\cos(\theta^\circ)\cos(\phi^\circ) - g. \end{aligned} \tag{6}$$

Denoting with $\xi^\circ_{i,j}$ the desired trajectories of the new state variables, the corresponding tracking error system is given by

$$\begin{aligned} \dot{e}_{1,1} &= e_{1,2} \\ \dot{e}_{1,2} &= \frac{U_1}{m}\sin(\theta^\circ)\cos(\phi^\circ) - \dot{\xi}^\circ_{1,2} \\ \dot{e}_{2,1} &= e_{2,2} \\ \dot{e}_{2,2} &= -\frac{U_1}{m}\sin(\phi^\circ) - \dot{\xi}^\circ_{2,2} \\ \dot{e}_{3,1} &= e_{3,2} \\ \dot{e}_{3,2} &= \frac{U_1}{m}\cos(\theta^\circ)\cos(\phi^\circ) - g - \dot{\xi}^\circ_{3,2}, \end{aligned} \tag{7}$$

where $\mathbf{e} = [e_{1,1}, e_{1,2}, e_{2,1}, e_{2,2}, e_{3,1}, e_{3,2}]^T$ is the error vector. Now to design a control law that exactly linearizes the system by feedforward, take the system (7) and let

$$\begin{aligned} v_1 &= \frac{U_1}{m}\sin(\theta^\circ)\cos(\phi^\circ) \\ v_2 &= -\frac{U_1}{m}\sin(\phi^\circ) \\ v_3 &= \frac{U_1}{m}\cos(\theta^\circ)\cos(\phi^\circ) - g \end{aligned} \tag{8}$$

$$\phi = \arcsin\left(\frac{m\ddot{x}\sin\psi - m\ddot{y}\cos\psi}{U_1}\right) = \arcsin\left(\frac{\ddot{w}_1\sin\psi - \ddot{w}_2\cos\psi}{\sqrt{\ddot{w}_1{}^2 + \ddot{w}_2{}^2 + (\ddot{w}_3+g)^2}}\right),$$
$$\theta = \arcsin\left(\frac{m\ddot{x}\cos\psi + m\ddot{y}\sin\psi}{U_1\cos(\phi)}\right) = \arcsin\left(\frac{\ddot{w}_1\cos\psi + \ddot{w}_2\sin\psi}{\sqrt{\ddot{w}_1{}^2 + \ddot{w}_2{}^2 + (\ddot{w}_3+g)^2}\cos\left(\arcsin\left(\frac{\ddot{w}_1\sin\psi - \ddot{w}_2\cos\psi}{\sqrt{\ddot{w}_1{}^2 + \ddot{w}_2{}^2 + (\ddot{w}_3+g)^2}}\right)\right)}\right). \quad (4)$$

where v_i, $i = 1,2,3$ are the new control inputs. The original control variables U_1, ϕ° and θ° are easily obtained as functions of these new inputs as

$$\begin{aligned} U_1 &= m\sqrt{v_1^2 + v_2^2 + (v_3+g)^2} \\ \phi^\circ &= \arcsin\left(-\frac{v_2}{\sqrt{v_1^2+v_2^2+(v_3+g)^2}}\right) \\ \theta^\circ &= \arctan\left(\frac{v_1}{v_3+g}\right). \end{aligned} \quad (9)$$

The new inputs are then chosen as the combination of a feedforward part taking into account the desired rates of the Brunovsky states, returned by the trajectory generator, and a feedback part taking the tracking error into account $v_i = \dot{\xi}^o_{i,2} + \Lambda_i(e)$, $i = 1,2,3$. The "exact feedforward action" $\mathbf{u}^o$ is the one that satisfies (8), when **v** corresponds to the reference trajectory $\xi^o(\cdot)$; a correction term Λ is further added for correcting feedforward compensation errors. The latter part can be any type of control; extended PID control (see [7]) is considered in this work. Specifically, compensations of linearization errors are defined as

$$\Lambda_i(e) = \sum_{j=0}^{n_j} \lambda_{iij} e_{ij} + \sum_{i\neq j}\sum_{j=0}^{n_j} \mu_{ilj} e_{lj},\ i = 1,\ldots,n_j,$$

$$e_{i0} = \int_0^t e_{i1}(\tau)d\tau,\ i = 1,\ldots,n_j,$$

where λ_{iij} are the extended PID coefficients and n_j is the length of the chain i of integrators in the Brunovsky form $\xi_{i1},\ldots,\xi_{in_j}$ (with $i = 1,\ldots,m$ and m is the number of chains of integrators). In this case, $n_j = 2$, then the control law only includes the integral, proportional and derivative terms of the classical PID control law (the extended and the standard PID formulation coincide). Obviously, for the dynamics to be decoupled on the three axes, all μ_{ilj} must be zero.

B. Yaw compensator

Consider now the more general case in which $\psi \neq 0$. The error model (7) can be rewritten as

$$\begin{aligned} \dot{e}_{1,1} &= e_{1,2} \\ \dot{e}_{1,2} &= \frac{U_1}{m}(\sin(\psi^\circ)\sin(\phi^\circ) + \cos(\psi)\sin(\theta^\circ)\cos(\phi^\circ)) - \dot{\xi}^\circ_{1,2} \\ \dot{e}_{2,1} &= e_{2,2} \\ \dot{e}_{2,2} &= \frac{U_1}{m}(-\cos(\psi^\circ)\sin(\phi^\circ) + \sin(\psi)\sin(\theta^\circ)\cos(\phi^\circ)) - \dot{\xi}^\circ_{2,2} \\ \dot{e}_{3,1} &= e_{3,2} \\ \dot{e}_{3,2} &= \frac{U_1}{m}(\cos(\theta^\circ)\cos(\phi^\circ)) - g - \dot{\xi}^\circ_{3,2} \end{aligned} \quad (10)$$

Note that the effect of yaw on position dynamics can be interpreted geometrically as a rotation of the control action on the plane (x, y), so that the modification to the control law to account for nonzero yaw is simply the following. By keeping **v** as in (8), the error equation becomes

$$\begin{aligned} \dot{e}_{1,1} &= e_{1,2} \\ \dot{e}_{1,2} &= v_1\cos(\psi) - v_2\sin(\psi) - \dot{\xi}^\circ_{1,2} \\ \dot{e}_{2,1} &= e_{2,2} \\ \dot{e}_{2,2} &= v_2\cos(\psi) + v_1\sin(\psi) - \dot{\xi}^\circ_{2,2} \\ \dot{e}_{3,1} &= e_{3,2} \\ \dot{e}_{3,2} &= v_3 - \dot{\xi}^\circ_{3,2} \end{aligned} \quad (11)$$

that is, the new control vector is no longer given by $\mathbf{v} = [v_1\ v_2]^T$, but rather by $\mathbf{v}^* = [v_1^*\ v_2^*]^T$, where $\mathbf{v}^*$ is defined as

$$\begin{aligned} v_1^* &= v_1\cos(\psi) - v_2\sin(\psi) \\ v_2^* &= v_2\cos(\psi) + v_1\sin(\psi) \\ v_3^* &= v_3. \end{aligned} \quad (12)$$

It follows that the real control variables can still be recovered from **v**, but this one now has to be computed as

$$\begin{aligned} v_1 &= w_1\cos(\psi) + w_2\sin(\psi) \\ v_2 &= -w_1\sin(\psi) + w_2\cos(\psi) \\ v_3 &= w_3. \end{aligned} \quad (13)$$

The $\mathbf{v}^*$ vector can be used to place the eigenvalues of the linearized system as before via extended PID. Notice that in both the cases where $\psi = 0$ and $\psi \neq 0$, this approach allows complete freedom in the specification of the desired closed loop performance as the nominal eigenvalues are completely user-defined.

C. Kinematic transformation and passivity-based attitude control

In the previous subsections it has been assumed that the attitude control system is ideal (*i.e.*, reference and output attitude coincide). In all realistic cases, attitude control is one of the most tricky tasks, see [16], and the above "ideal" situation is far from being obtained. First of all, Euler angles are used since they are physically easy to understand and they allow one to design the flatness-based scheme described in Subsection IV-A. It is well-known though that the Euler representation of the attitude is not unique. As a matter of fact, to maintain a unique set of angle, one usually demands

$$0 \le \phi < 2\pi,\ -\pi/2 \le \theta \le \pi/2,\ 0 \le \psi < 2\pi.$$

Moreover, even if simple PD controllers can be easily tuned to assign the closed-loop dynamics to the linearized system, stability of the output trajectories cannot be guaranteed for the nonlinear system. In this subsection a solution to these two problems will be proposed, by means of a kinematic transformation. Consider the attitude representation given by the Modified Rodrigues Parameters (MRPs) σ. This new set of coordinates is such that $\sigma = e_E \tan(\alpha_E/4)$, where e_E and α_E are the Euler axis and angle that describe the three consecutive rotations roll, pitch and yaw (see again [16]). The set of kinematic and dynamic equations become respectively

$$\dot{\sigma} = G(\sigma)\omega, \quad \sigma(0) = \sigma_0, \tag{14}$$

$$J\dot{\omega} = S(\omega)J\omega + u, \quad \omega(0) = \omega_0, \tag{15}$$

where $\omega = [p, q, r]^T$, $u = [U_2, U_3, U_4]^T$ and J is the symmetric inertia matrix composed by I_x, I_y and I_z. The matrix $S(\omega)$ denotes a skew-symmetric matrix representing the cross product between two vectors, *i.e.*, $S(\omega)J\omega = -\omega \times J\omega$, whereas the kinematic matrix $G(\sigma)$ is defined as

$$G(\sigma) = \frac{1}{2}\left(\frac{1-\sigma^T\sigma}{2}I - S(\sigma) + \sigma\sigma^T\right).$$

Although MRPs still include the occurrence of discontinuous jumps in the parameter space when incrementing the rotation, this fact is now less critical, as only 2π-rotations are forbidden. Moreover, the following important result holds (see [17]).

Theorem 1: The linear Proportional Derivative (PD) control law $u = -K_p\sigma - K_d\omega$ almost globally asymptotically stabilizes (14) and (15) at the origin, for any positive value of K_p and K_d.

Proof: See [17]. ■

Note that *almost* global asymptotic stability can only be guaranteed, as obviously singular points are not considered. This is a general point, since the topological structure of the attitude motion is not a contractible space and it does not allow for globally continuously stabilizing control laws. However, the above theorem is very useful as it allows one to guarantee the stability of output trajectories also for nonlinear attitude dynamics with a simple PD controller for each degree of freedom. The linear controller can be designed based on the linearized model (*i.e.*, the representation for small angles) described by $\dot{\sigma} = 1/4\omega$ and $J\dot{\omega} = u$, by simply making the characteristic polynomial equal to a desired one via PD gains K_p and K_d. Notice that a feedforward action can also be added to improve dynamic performance without affecting the stability property.

V. SIMULATION RESULTS

In this Section, the proposed control strategy is tested in a simulation environment, where also actuators and sensors dynamics are modeled and values of the physical parameters are chosen as in Table I. The following second-order pole-assignment requirements have been considered: for attitude control loop, natural frequency of 4.24 Hz and unitary damping, whereas for position loop, natural frequency of 0.27 Hz and damping equal to 0.86.

The response of the closed-loop system to multi-step excitation of the reference flat outputs is illustrated in Figure 2 in the nominal case where the values of all parameters in Table I are known. Notice that transient and static requirements are satisfied for both ascending and descending excitations. In the same Figure the response of the system controlled without the yaw compensator can be seen and evaluated. It can be easily verified that the overall control scheme is linearized by the flatness-based control by testing different scaled step-excitations. As an example, in Figure 3, responses of the same control system are reported for reference signals of increasing amplitude. Robustness analysis in the simulation environment can be completed by showing the behaviour of the control strategy in the case of uncertanties for m and for the body inertias on the three axes. As an example, the case of strong inertia overestimation is shown in Figure 4, where robustness of the overall system is evident. Finally, sensitivity to mass variations can be also highlighted. Specifically, since θ° and ϕ° do not depend on vehicle mass, the effect of errors in mass estimation only affects the gravity compensation along the z-axis via U_1. Formally, by substituting expressions of ϕ° and θ° in the altitude error equation, the closed-loop dynamics

$$\dot{e}_{3,2} = \frac{U_1}{m}\cos(\theta^\circ)\cos(\phi^\circ) - g - \dot{\xi}^\circ_{3,2} = \frac{\hat{m}}{m}\Lambda_3(e) - g - \dot{\xi}^\circ_{3,2},$$

is obtained, where $\hat{m}$ is the estimate of m. On the error dynamics it can be verified that the constant exogenous term $(\hat{m} - m)g/m$ appears. This means that the PD action is not

name	symbol	value	m.u.
mass	m	0.5	kg
inertia on x	I_x	$5 \cdot 10^{-3}$	$N\ m\ s^2$
inertia on y	I_y	$5 \cdot 10^{-3}$	$N\ m\ s^2$
inertia on z	I_z	$9 \cdot 10^{-3}$	$N\ m\ s^2$
motor inertia	J_m	$3.4 \cdot 10^{-5}$	$N\ m\ s^2$
drag factor	d	$1.1 \cdot 10^{-5}$	$N\ m\ s^2$
thrust factor	b	$7.2 \cdot 10^{-5}$	$N\ s^2$
body center-propeller distance	l	0.25	m

TABLE I
NUMERICAL VALUES FOR THE MODEL PARAMETERS IN THE SIMULATOR.

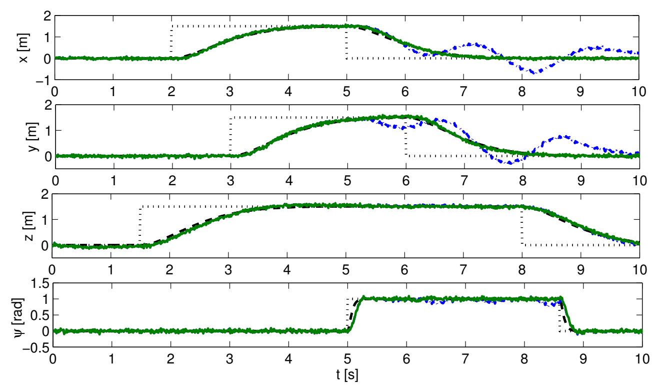

Fig. 2. Step response with and without yaw-compensator: reference signal (dotted line), ideal response (dashed line), actual output with yaw compensator (solid line) and actual output without yaw compensator (dash-dotted line).

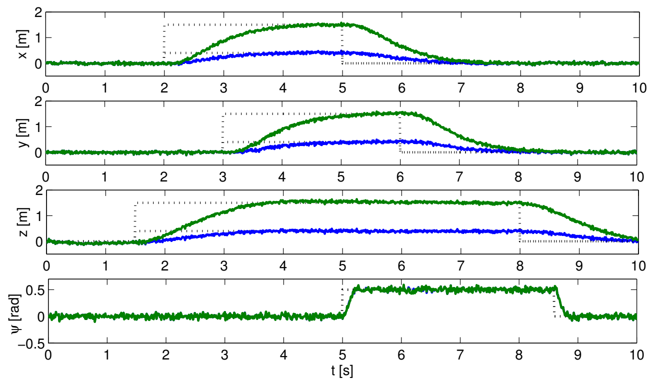

Fig. 3. Step response of positioning control loop for small and large reference variations: reference signal (dotted line), ideal response (dashed line, overlapped) and actual output (solid line).

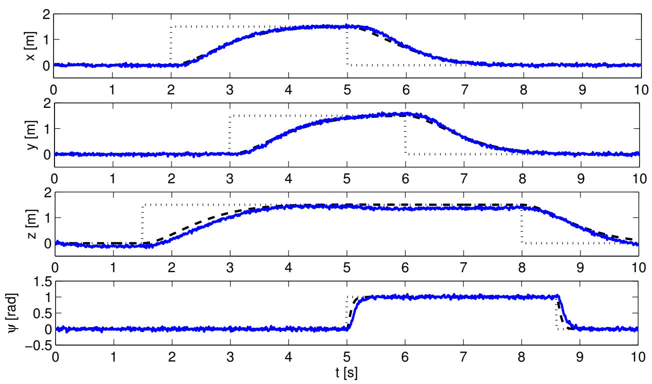

Fig. 4. The case of overestimated inertias (3 times the real value): reference signal (dotted line), ideal response (dashed line) and actual output (solid line).

sufficient to guarantee static performance, *i.e.*, null tracking error for $t \to \infty$ and constant reference signals. However, it is sufficient to add integral action into the altitude loop in order to counteract this effect. Obviously, this additional term does not affect anyway the case of correct mass estimation. In Figure 5, the case of 20% estimation-error is reported for both the cases with and without integral action.

VI. Conclusions

In this paper, the problem of designing a control system for a quadrotor helicopter has been addressed. Specifically, differential flatness of position dynamics has been exploited to linearize the system via feedforward and a passivity-based scheme has been employed to (almost) globally stabilize the attitude dynamics. Moreover, a suitable rotation of control variables has been implemented to compensate yaw variations, leading to a simpler design procedure. The proposed control solutions have been finally tested on a simulator, where also robustness to uncertainty on dynamic parameters has been shown.

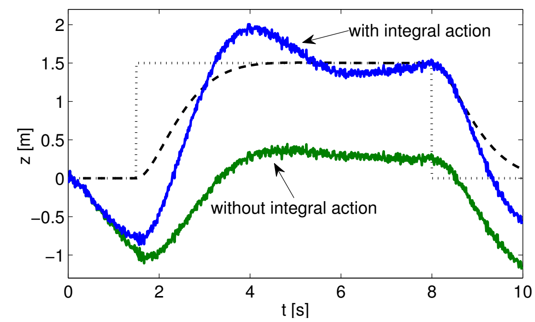

Fig. 5. Comparison between flat control system with and without integral action in case of underestimated mass (20% less than the real value): reference signal (dotted line), ideal response (dashed line) and actual output signals (solid lines).

References

[1] C. Bottasso, D. Leonello, and B. Savini. Path planning for autonomous vehicles by trajectory smoothing using motion primitives. *IEEE Transactions on Control Systems Technology*, 16:1152–1168, 2008.

[2] P. Castillo Garcia, R. Lozano, and A. Dzul. *Modelling and Control of Mini-Flying Machines*. Springer, 2010.

[3] I.D. Cowling, O.A. Yakimenko, J.F. Whidborne, and A.K. Cooke. Direct method based control system for an autonomous quadrotor. *Journal of Intelligent and Robotic Systems, 60, 285-316*, 2010.

[4] A. Das, K. Subbarao, and F. Lewis. Dynamic inversion with zero-dynamics stabilisation for quadrotor control. *IET Control Theory & Applications*, 3(3):303–314, 2009.

[5] E. Delaleau. Control of flat systems by quasi-static feedback of generalized states. *International Journal of Control*, 71(5):745–765, 1998.

[6] M. Fliess, J. Lévine, P. Martin, and P. Rouchon. Flatness and defect of non-linear systems: introductory theory and examples. *International Journal of Control*, 61(6):1327–1361, 1995.

[7] V. Hagenmeyer and E. Delaleau. Exact feedforward linearization based on differential flatness. *International Journal of Control*, 76(6):537–556, 2003.

[8] V. Hagenmeyer and E. Delaleau. Robustness analysis with respect to exogenous perturbations for flatness-based exact feedforward linearization. *IEEE Transactions on Automatic Control*, 55(3):727–731, 2010.

[9] G.M. Hoffmann, H. Huang, S.L. Waslander, and C.J. Tomlin. Quadrotor helicopter flight dynamics and control: Theory and experiment. In *Proceedings of the AIAA Guidance, Navigation, and Control Conference*, pages 2007–6461, 2007.

[10] M. Kelemen. A stability property. *IEEE Transactions on Automatic Control*, 31(8):766–768, 2002.

[11] T. J. Koo and S. Sastry. Differential flatness based full authority helicopter control design. In *IEEE Conference on Decision and Control, Phoenix, USA*, 1999.

[12] M. La Civita. *Integrated Modeling and Robust Control for Full-Envelope Flight of Robotic Helicopters*. PhD thesis, Carnegie Mellon University, 2003.

[13] T. Lee, M. Leok, and H. McClamroch. Geometric tracking control of a quadrotor UAV on SE(3). In *IEEE Conference on Decision and Control, Atlanta USA*, 2010.

[14] J. Levine. *Analysis and Control of Nonlinear Systems: A Flatness-based Approach*. Springer, 2009.

[15] K. Nonami, F. Kendoul, S. Suzuki, W. Wang, and D. Nakazawa. *Autonomous Flying Robots: Unmanned Aerial Vehicles and Micro Aerial Vehicles*. Springer, 2010.

[16] M.D. Shuster. A survey of attitude representations. *Journal of Austronautical Sciences*, 41(4):439–517, 1993.

[17] P. Tsiotras. Further passivity results for the attitude control problem. *IEEE Transactions on Automatic Control*, 43(11):1597–1600, 2002.

[18] M.J. Van Nieuwstadt and R.M. Murray. Real-time trajectory generation for differentially flat systems. *International Journal of Robust and Nonlinear Control*, 8(11):995–1020, 1998.