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Constrained stabilization of a continuous stirred tank reactor via smooth control Lyapunov R-functions

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Abstract— **The constrained stabilization of the multivariable nonlinear model of a continuous stirred tank reactor is addressed. In this control setting, the novel class of control Lyapunov R-functions is proposed to design a smooth control Lyapunov function having external level set in accordance to the state constraints and non-homothetical inner level sets arbitrarily close to the desired (optimal) ones. The flexibility in shaping the composite control Lyapunov function is obtained by introducing a novel composition rule. The simulation of an exothermic chemical process shows the benefits of using a control Lyapunov R-function together with a gradient-based nonlinear controller, in fact a much larger controlled invariant state space set is obtained, moreover better state convergence and smoother control inputs are recovered.**

I. INTRODUCTION

As modern-day chemical processes are continuously faced with the requirements of becoming safer, more reliable, and more economical in operation, the need for a rigorous, yet practical, approach for the design of effective chemical process control systems that can meet these demands, becomes increasingly apparent [1]. However, the control design problem is highly non trivial because most chemical processes are inherently Multi-Input Multi-Output (MIMO) and nonlinear [2], and the use of controllers only designed on the basis of the approximate linearized process can lead to conservative, besides reduced, control performances. Moreover, the unavoidable presence of physical constraints on the process variables and in the capacity of control actuators not only limit the nominal performance of the controlled system, but also can affect the stability of the overall system. Therefore, the stabilization of nonlinear processes is one of the most attractive research areas for the chemical and control engineering community [2].

The Model Predictive Control (MPC), also known as Receding Horizon Control (RHC) [3], is one of the few control methods for handling state and control input constraints within an optimal control setting [4]. Recently, in [4], an interesting Lyapunov-based MPC approach has been proposed for the control of an exothermic chemical reaction, taking place in a Continuous Stirred Tank Reactor (CSTR). In particular, a Quadratic Control Lyapunov Function (QCLF) is used together with an MPC strategy. However, an ellipsoidal set can not accurately shape the polyhedral state constraints describing the limits on the admissible concentration of the chemical reactant and on the temperature in the reactor. In other words, a QCLF is actually not conclusive for a large part of the controllable invariant set.

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Mainly for this reason, Truncated Ellipsoid (TE) Control Lyapunov Functions (CLFs) [5], [6], and smooth TE CLFs [7], [8] have been proposed to enlarge the estimate of the controlled invariant state space region, which is one of the focuses of this paper.

This paper extends the results of [7] and [8] to the class of constrained nonlinear control affine systems. The main contribution is the definition of a novel composition rule in the setting of Control Lyapunov R-Functions (CLRFs), that allows the design of a smooth CLF in accordance to both the state constraints and the optimal shape for the level sets. This is a substantial novelty, because, unlike switching control strategies [9], both constraints and optimality arguments can be handled by a unique smooth CLF, together with a continuous control.

The paper is organized as follows. Next section describes the motivating case of study for the control of a chemical reaction inside a CSTR. Sections III, IV present the main theoretical results. The simulation results are shown in Section V, where the benefits of using the proposed CLRF approach are pointed out. In last section we conclude the paper and outline some interesting future lines of research.

A. Notation

I_n denotes the $n \times n$ identity matrix and $diag(\cdot)$ denotes a diagonal matrix. The closed k-level set of a continuous function $V : \mathcal{X} \subseteq \mathbb{R}^n \to \mathbb{R}$, i.e. $\{x \in \mathcal{X} : \; V(x) \leq k\}$, is denoted by $\mathcal{L}[V, k]$. A set $\mathcal{S} \subseteq \mathbb{R}^n$ is called $\mathcal{C}$-set if it is a convex and compact set including the origin in its interior [10]. $\mathbb{I}_r$ denotes $\{n \in \mathbb{Z}^+ : \; n \leq r\}$.

II. A CONTINUOUS STIRRED TANK REACTOR AS A MOTIVATING CASE OF STUDY

Let us consider a continuous stirred tank reactor where an irreversible, exothermic first-order reaction of the form $A \xrightarrow{k} B$ takes place. The inlet stream consists of pure A at flow rate F, concentration C_{A0} and temperature T_{A0} [2]. The mathematical model of the process takes the form

$$
\begin{aligned}
\dot{C_A} &= \frac{F}{V}\left(C_{A0} - C_A\right) - k_0 \exp\left(-\frac{E}{RT_R}\right) C_A \\
\dot{T_R} &= \frac{F}{V}\left(T_{A0} - T_R\right) - \frac{\Delta H}{\rho c_p} k_0 \exp\left(-\frac{E}{RT_R}\right) C_A + \frac{Q}{\rho c_p V},
\end{aligned} \tag{1}
$$

where C_A denotes the concentration of the species A, T_R denotes the temperature of the reactor, Q is the heat input to the reactor, V is the volume of the reactor, k_0, E, ΔH are, respectively, the pre-exponential constant, the activation

TABLE I

PROCESS PARAMETERS AND STEADY STATE VALUES

V	0.1	m^3
R	8.314	$kJ/(kmol\ ^\circ K)$
C_{A0_s}	1	$kmol/m^3$
T_{A0_s}	310	$^\circ K$
Q	0	kJ/min
ΔH	$-4.78 \cdot 10^4$	$kJ/kmol$
k_0	$72 \cdot 10^9$	$1/min$
E	$8.314 \cdot 10^4$	$kJ/kmol$
c_p	0.239	$kJ/(kg\ ^\circ K)$
ρ	1000	kg/m^3
F	0.1	m^3/min
T_{R_s}	395.3	$^\circ K$
C_{A_s}	0.57	$kmol/m^3$

energy and the enthalpy of the reaction, c_p and ρ are, respectively, the heat capacity and the fluid density in the reactor. The numerical values of the process parameters, taken from [4], are shown in Table I.

The nonlinear model (1) of the reactor has to be stabilized at the unstable equilibrium point $\bar{x} = [C_{A_s}, T_{A_s}]^\top = [0.57\ kmol/m^3,\ \ 395.3\ ^\circ K]^\top$, $\bar{u} = 0$, according to the state constraints

$$|C_A - C_{A_s}| \leq 0.16\ kmol/m^3, \quad |T_R - T_{R_s}| \leq 3\ ^\circ K. \quad (2)$$

The control variables u are the variation of the inlet concentration of species A, $u_1 = \Delta C_{A0} = C_{A0} - C_{A0_s}$, and the heat input to the reactor $u_2 = Q$. These manipulated control inputs are constrained as follows:

$$|\Delta C_{A0}| \leq 1\ kmol/m^3, \quad |Q| \leq 1\ kJ/h = 0.0167\ kJ/min. \quad (3)$$

It is indeed desired to find a *time-continuous* constrained control law $u(t)$ that drives the state $x(t) = [C_A(t), T_R(t)]^\top$ to $\bar{x}$, in accordance to the state constraints. In fact, discontinuous and/or chattering control laws, such as the ones usually obtained by switching controllers, are actually not well implementable on real actuators.

A. Standard Lyapunov-based state feedback control

Since the state feedback stabilization of a nonlinear system of the kind (1) is equivalent to the design of a CLF [11], a typical control approach is based on the use of a CLF, together with a static state feedback controller and/or together with an MPC-like controller [3]. Moreover, a minimization objective is also considered in standard control approaches for multivariable chemical processes [12], [13], [4], where a (piecewise) QCLF is designed. This particular choice is motivated by the fact that the gradient-based control $u(x(t)) = -R^{-1}B^\top P^* x(t)$, being $P^* \succ 0$ the solution of the Algebraic Riccati Equation (ARE) $A^\top P + PA + Q - PBR^{-1}B^\top P = 0$, asymptotically stabilizes the unconstrained linearized system $\dot{\tilde{x}}(t) = A\tilde{x}(t) + B\tilde{u}(t)$, where $\tilde{x} = x - \bar{x}$ and $\tilde{u} = u - \bar{u}$, by minimizing the quadratic performance cost

$$J(\tilde{x}, \tilde{u}) = \int_0^{+\infty} \left(\tilde{x}(\tau)^\top Q \tilde{x}(\tau) + \tilde{u}(\tau)^\top R \tilde{u}(\tau)\right) d\tau. \quad (4)$$

In the case of constrained systems, the particular choice of the CLF is a critical point in the control design, since the largest (indeed non conservative) estimate for the controllable state space set should be provided. Considering the matrix parameters $Q = I_2, R = 4I_2$, a candidate QCLF is $\tilde{x}^\top P^* \tilde{x}$, with

$$P^* = \begin{bmatrix} 61.8634 & 3.9425 \\ 3.9425 & 0.3507 \end{bmatrix} \quad (5)$$

solution of the corresponding ARE for the linearized unconstrained system. As shown in Figure 1, since a quadratic function can not fit well the polyhedral state constraints, only a *shrunk* QCLF can be used for the control design, in order to guarantee the state constraints.

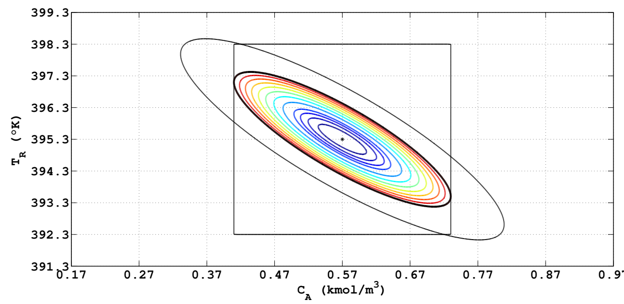

Fig. 1. In solid line the 1-level set of the optimal QCLF and the polyhedral state constraints. The controlled invariant set is geometrically obtained by considering the set where the QCLF is inside the state constraints.

B. Enlarging the controlled invariant state space region

A possible way to obtain a larger estimate of the controllable set is the use of a Truncated Ellipsoid (TE) [5] [6] CLF, shown in Figure 2, whose shape actually takes into account the presence of the state constraints. However, since the standard TE CLF is not differentiable and thus (optimal) nonlinear gradient-based controllers can not be used [14], a smoothing technique has been proposed in [8] via the framework of CLRFs.

Ideally, it should be designed a modified TE CLF with external level sets in accordance to the maximal controllable set and to the state constraints, and with internal level sets *close* to the (quadratic) optimal ones, for instance the one in Figure 3, together with a gradient-based (MPC) nonlinear controller.

In the next section a novel composition rule within the framework of R-function is introduced in order to obtain such kind of smooth CLF having non-homothetic level sets of the desired shape.

III. PRELIMINARIES ON LYAPUNOV R-FUNCTIONS

The framework of R-functions has been introduced in [15] for geometric applications of logic algebra. In the setting of state feedback stabilization of control systems, the use of R-functions has been firstly proposed in [7], [8]. Recalled the basic notions on R-functions, a useful novel composition rule is defined.

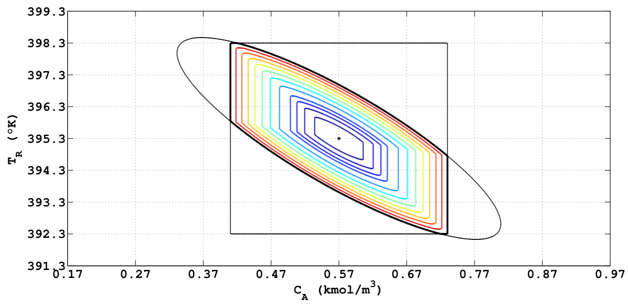

Fig. 2. The truncated ellipsoid control Lyapunov function is obtained by intersecting the optimal quadratic function and the polyhedral state constraints.

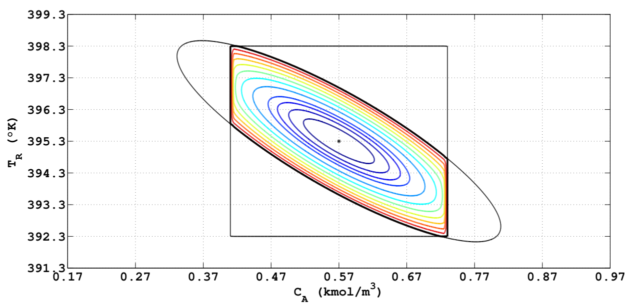

Fig. 3. The proposed control Lyapunov R-function has inner level sets close to the quadratic optimal ones.

TABLE II

NOVEL R-COMPOSITION RULES

BOOLEAN		R-COMPOSITION
NOT	$\neg$	$-r$
AND	$\overset{\alpha}{\wedge}_\phi$	$\dfrac{\phi r_1 + r_2 - \sqrt{(\phi r_1)^2 + r_2^2 - 2\alpha\phi r_1 r_2}}{\phi + 1 - \sqrt{\phi^2 + 1 - 2\alpha\phi}}$
OR	$\overset{\alpha}{\vee}_\phi$	$\dfrac{\phi r_1 + r_2 + \sqrt{(\phi r_1)^2 + r_2^2 - 2\alpha\phi r_1 r_2}}{\phi + 1 + \sqrt{\phi^2 + 1 - 2\alpha\phi}}$

Definition 1: A function $r : \mathbb{F}^n \subseteq \mathbb{R}^n \to \mathbb{R}$ is an R-function if there exists a Boolean function $\mathcal{R} : \mathbb{B}^n \to \mathbb{B}$, where $\mathbb{B} = \{0, 1\}$, such that the following equality is satisfied:

$$h\left(r\left(x_1, x_2, \ldots, x_n\right)\right) = \mathcal{R}\left(h\left(x_1\right), h\left(x_2\right), \ldots, h\left(x_n\right)\right), \tag{6}$$

where $h(\cdot)$ is the standard Heaviside step function.

Informally, a real function r is an R-function if it can change its sign only when some of its arguments change their sign [8]. The parallelism between logic functions and R-functions is shown in Table II, in view of classic Boolean operators.

For instance, the interpretation of the AND composition is that the composed function $r_{\overset{\alpha}{\wedge}_\phi}$ is positive when evaluated in x if and only if both functions $r_1(x)$ *and* $r_2(x)$ are positive in x. The result is obtained by exploiting the triangle inequality and the law of cosines; it holds for all values of the parameters $\alpha \in [0, 1] \subset \mathbb{R}$ [16], and also $\phi \in \mathbb{R}^+$. The terms at the denominator in Table II are normalizing factors such that

$r_1(x) \overset{\alpha}{\wedge}_\phi r_2(x) = r_{\overset{\alpha}{\wedge}_\phi}(x) = 1 \Leftrightarrow \{r_1(x) = 1 \wedge r_2(x) = 1\}$

and also

$r_1(x) \overset{\alpha}{\vee}_\phi r_2(x) = r_{\overset{\alpha}{\vee}_\phi}(x) = 1 \Leftrightarrow \{r_1(x) = 1 \vee r_2(x) = 1\}$.

Remark 1: In [7], [8] the standard R-composition $\overset{\alpha}{\wedge}$, actually with $\phi = 1$, has been used. The additional parameter ϕ preserves functions $r_{\overset{\alpha}{\wedge}_\phi}$ and $r_{\overset{\alpha}{\vee}_\phi}$ to be suitable R-functions, because $sign\,(r_1) = sign\,(\phi r_1)\ \forall \phi \in \mathbb{R}^+$.

Remark 2: Let $\alpha = 1$. Then $r_1 \overset{1}{\wedge}_\phi r_2 = \min\{\phi r_1, r_2\}$ and $r_1 \overset{1}{\vee}_\phi r_2 = \max\{\phi r_1, r_2\}$.

In the following, we consider only the AND composition rule, since only 0-symmetric controlled $\mathcal{C}$-sets are dealt with.

Proposition 1: Consider $\epsilon \in (0, 1) \subset \mathbb{R}^+$ and the set $\mathcal{S} = \{x \in \mathbb{R}^n :\ r_1(x) \geq \epsilon,\ r_2(x) \geq \epsilon\}$. The composed function $r_{\overset{\alpha}{\wedge}_\phi}$ converges pointwise to r_2 (r_1) as the shape parameter ϕ approaches to infinity (zero).

$$\lim_{\phi \to +\infty} r_{\overset{\alpha}{\wedge}_\phi}(x) = r_2(x) \quad \forall \alpha \in [0, 1],\ \forall x \in \mathcal{S} \tag{7}$$

$$\lim_{\phi \to 0^+} r_{\overset{\alpha}{\wedge}_\phi}(x) = r_1(x) \quad \forall \alpha \in [0, 1],\ \forall x \in \mathcal{S}. \tag{8}$$

Proof: For ease of notation, let r_1 and r_2 denote, respectively $r_1(x)$ and $r_2(x)$. The assumption that $r_1 > 0$ and $r_2 > 0$ allows the division by r_1 and/or r_2.

$$\begin{aligned} &\lim_{\phi \to +\infty} r_{\overset{\alpha}{\wedge}_\phi}(x) = \\ &= \lim_{\phi \to +\infty} \frac{\left(\phi r_1 + r_2 - \sqrt{(\phi r_1)^2 + r_2^2 - 2\alpha\phi r_1 r_2}\right)}{\phi + 1 - \sqrt{\phi^2 + 1 - 2\alpha\phi}} = \\ &= \lim_{\phi \to +\infty} \frac{\frac{2\phi r_1 r_2 (1+\alpha)}{\phi r_1 + r_2 + \sqrt{(\phi r_1)^2 + r_2^2 - 2\alpha\phi r_1 r_2}}}{\phi + 1 - \sqrt{\phi^2 + 1 - 2\alpha\phi}} = r_2(x). \end{aligned} \tag{9}$$

Analogously, $\lim_{\phi \to 0^+} r_{\overset{\alpha}{\wedge}_\phi}(x) = r_1(x)$. ■

Considering the case of study of Section II, the optimal QCLF for the unconstrained linearized system is described by the matrix P^* (5), while the state constraints (2) can be written in the form $\tilde{x} \in \mathcal{X} = \left\{x \in \mathbb{R}^2 :\ \|Fx\|_\infty \leq 1\right\}$, where

$$F = diag\left(\frac{1}{0.16}, \frac{1}{3}\right). \tag{10}$$

To compose the polyhedral function (*of the second order* [17]) $V_1(x) = \max\left\{x^\top F_1^\top F_1 x,\ x^\top F_2^\top F_2 x\right\}$, being F_i the i^{th} row of matrix F, and the quadratic function $V_2(x) = x^\top P^* x$, respectively in their 1-level sets $\mathcal{L}[V_1, 1]$ and $\mathcal{L}[V_2, 1]$, define functions $R_1(x) = 1 - V_1(x)$ and $R_2(x) = 1 - V_2(x)$. Without loss of generality, these functions are normalized so that their maximum value is 1. Then compute the *R-intersection* (AND rule $\overset{\alpha}{\wedge}_\phi$) $R_{\overset{\alpha}{\wedge}_\phi} = R_1 \overset{\alpha}{\wedge}_\phi R_2$, according to the equation of Table II, for arbitrary values of $\alpha \in [0, 1]$, $\phi \in \mathbb{R}^+$. The composed function $R_{\overset{\alpha}{\wedge}_\phi}$ is the (smoothed) *intersection* between the polyhedral function and the quadratic one in the sense that $R_{\overset{\alpha}{\wedge}_\phi}$ is positive inside the geometric intersection region $\mathcal{L}[V_1,1] \cap \mathcal{L}[V_2,1]$, it is zero on the boundary, negative outside, and its maximum value is 1 at the origin. The positive definite function associated to $R_{\overset{\alpha}{\wedge}_\phi}$ is $V_{\overset{\alpha}{\wedge}_\phi} = 1 - R_{\overset{\alpha}{\wedge}_\phi}$.

The sublevel sets of the function $V_{\overset{\alpha}{\wedge}_1}$ are shown in Figure 2, for the case of $\alpha = 1,\ \phi = 1$ (truncated ellipsoid [5], [6]), while in Figure 3 parameters $\alpha = 0,\ \phi = 20$ are used.

Remark 3: The intersection of a polyhedral region with an ellipsoid has been used as a candidate LF in [6]. The framework of R-functions generalizes TEs that are recovered as a special case ($\alpha = 1,\ \phi = 1$, see Figure 2).

Parameter α affects the smoothness of the inner sublevel sets of the composed function, while parameter ϕ can make the shape of the sublevel sets of the composed function closer to one of the two generating functions. The external shape of the overall region $\mathcal{L}[V_{\overset{\alpha}{\wedge}_\phi}, 1]$ is not affected by the choice of the parameters α, ϕ. In particular, for $\alpha \in [0,1)$ such smoothing technique yields non-homothetic sublevel sets and a differentiable function in the interior of the set $\mathcal{L}[V_{\overset{\alpha}{\wedge}_\phi}, 1]$.

The choice of $\alpha = 0$ yields the *smoothest* sublevel sets, usually providing the best control performances when an R-composed function is used as CLF [7], [8]. For this reason, in the rest of paper the smoothing is performed fixing $\alpha = 0$, with notation $\wedge_\phi$ in place of $\overset{0}{\wedge}_\phi$.

R-functions can be used to compose general functions and not just polyhedral or quadratic ones. Some examples of different control applications can be found in [16], [18].

IV. ON THE USE OF CONTROL LYAPUNOV R-FUNCTIONS FOR CONSTRAINED STABILIZATION

A. Stability analysis of nonlinear systems via R-composed Lyapunov functions

In this subsection, the intersection function $V_{\wedge_\phi}$ is used as candidate Lyapunov function for the stability analysis of nonlinear dynamical systems

$$\dot{x}(t) = f\left(x(t)\right):\ x(t) \in \mathcal{X} \subseteq \mathbb{R}^n,\ f: \mathcal{X} \to \mathcal{X} \text{ continuous} \tag{11}$$

Consider the R-composition of two LFs V_1, V_2 for the system (11) in the $\mathcal{C}$-set $\mathcal{L}[V_1,1] \cap \mathcal{L}[V_2,1]$. The external level set of the composed candidate LF $V_{\wedge_\phi}$ is not differentiable, in fact $\{x \in \mathcal{X} : V_{\wedge_\phi}(x) = 1\} = \{x \in \mathcal{X} : \max_i\{V_i(x)\} = 1\}$. Therefore the lack of differentiability can be avoided by considering the $\mathcal{C}$-set $\mathcal{L}\left[V_{\wedge_\phi}, 1-\epsilon\right]$, for any $\epsilon \in (0,1) \subset \mathbb{R}^+$.

Theorem 1 (from [19]): Assume that functions $V_i : \mathcal{L}[V_i,1] \subseteq \mathcal{X} \to \mathbb{R}$, $i = 1,2$, are two Lyapunov functions with time derivatives along the system trajectory (11) $\dot{V}_i\left(x(t)\right) \le -\eta V_i\left(x(t)\right)$, $i = 1,2$, in the 0-symmetric $\mathcal{C}$-set $\mathcal{L}[V_1,1] \cap \mathcal{L}[V_2,1]$. Then the R-composed function $V_{\wedge_\phi}$ is a Lyapunov function with decreasing rate $\eta \in \mathbb{R}^+$ for (11) in the intersection set $\mathcal{L}\left[V_{\wedge_\phi}, 1-\epsilon\right] = \mathcal{L}[V_1, 1-\epsilon] \cap \mathcal{L}[V_2, 1-\epsilon]$, $\forall \phi \in \mathbb{R}^+$, for any $\epsilon \in (0,1) \subset \mathbb{R}^+$.

Proof: Define the functions $R_i(x) = 1 - V_i(x),\ i = 1,2$ and the R-composition $R_{\wedge_\phi}$, according to the AND rule $\overset{0}{\wedge}_\phi$ of Table II:

$$R_{\wedge_\phi}(x) = \frac{\phi R_1(x) + R_2(x) - \sqrt{(\phi R_1(x))^2 + R_2(x)^2}}{\phi + 1 - \sqrt{\phi^2+1}}.$$

The candidate LF $V_{\wedge_\phi}$ is positive definite in the set $\mathcal{L}\left[V_{\wedge_\phi}, 1\right] = \mathcal{L}[V_1,1] \cap \mathcal{L}[V_2,1]$ because $R_{\wedge_\phi}(x) = 1 \Leftrightarrow R_1(x) = R_2(x) = 1 \Leftrightarrow x = 0$. Moreover, $V_{\wedge_\phi}$ is everywhere differentiable in the set $\mathcal{L}\left[V_{\wedge_\phi}, 1-\epsilon\right]$, for any $\epsilon \in (0,1) \subset \mathbb{R}^+$. The assumption on the decreasing rate is equivalent to $\dot{R}_i\left(x(t)\right) \ge \eta\left(1 - R_i\left(x(t)\right)\right),\ i = 1,2$, therefore, considering the time derivative

$$\dot{R}_{\wedge_\phi}(x(t)) = \frac{1}{\phi + 1 - \sqrt{\phi^2+1}} \cdot \left[\phi \dot{R}_1 \left(1 + \frac{-\phi R_1}{\sqrt{(\phi R_1)^2 + R_2^2}}\right) + \dot{R}_2 \left(1 + \frac{-R_2}{\sqrt{(\phi R_1)^2 + R_2^2}}\right)\right] \tag{12}$$

the following inequality for the R-intersection holds:

$$\begin{aligned}
&\dot{R}_{\wedge_\phi}(x(t)) \ge \\
&\ge \frac{\eta}{\phi+1-\sqrt{\phi^2+1}} \cdot \left[\phi(1-R_1)\left(1 + \frac{-\phi R_1}{\sqrt{(\phi R_1)^2 + R_2^2}}\right) + \right. \\
&\quad \left. +(1-R_2)\left(1 + \frac{-R_2}{\sqrt{(\phi R_1)^2 + R_2^2}}\right)\right] = \\
&= \frac{\eta}{\phi+1-\sqrt{\phi^2+1}} \left[\left(\frac{-\phi^2 R_1 - R_2}{\sqrt{(\phi R_1)^2 + R_2^2}} + \phi + 1\right) + \right. \\
&\quad \left. + \frac{\phi R_1\left(\phi R_1 - \sqrt{(\phi R_1)^2 + R_2^2}\right)}{\sqrt{(\phi R_1)^2 + R_2^2}} + \frac{R_2\left(R_2 - \sqrt{(\phi R_1)^2 + R_2^2}\right)}{\sqrt{(\phi R_1)^2 + R_2^2}}\right] \\
&= \eta \left[\frac{1}{\phi+1-\sqrt{\phi^2+1}}\left(\frac{-\phi^2 R_1 - R_2}{\sqrt{(\phi R_1)^2 + R_2^2}} + \phi + 1\right) - R_{\wedge_\phi}\right].
\end{aligned} \tag{13}$$

Finally, $\forall \phi \in \mathbb{R}^+$,

$$\frac{1}{\phi+1-\sqrt{\phi^2+1}}\left(\frac{-\phi^2 R_1 - R_2}{\sqrt{(\phi R_1)^2 + R_2^2}} + \phi + 1\right) \ge 1 \Leftrightarrow$$

$$\frac{\phi^2 R_1 + R_2}{\sqrt{(\phi R_1)^2 + R_2^2}} \le \sqrt{\phi^2+1} \ \Leftrightarrow\ 0 \le (R_1 - R_2)^2.$$

Therefore $\dot{R}_{\wedge_\phi}(x(t)) \ge \eta(1 - R_{\wedge_\phi}(x(t)))$ and

$$\dot{V}_{\wedge_\phi}(x(t)) \le -\eta V_{\wedge_\phi}(x(t)) \quad \forall x(t) \in \mathcal{L}\left[V_{\wedge_\phi}, 1-\epsilon\right]. \tag{14}$$

■

B. Control Lyapunov R-functions

Consider the constrained stabilization of nonlinear control affine systems

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t), \tag{15}$$

where f, g are continuous functions of $x \in \mathbb{R}^n$, with state and input constraints of the kind

$$\begin{aligned} x \in \mathcal{X} &= \left\{x \in \mathbb{R}^n :\ \|Lx\|_\infty \le 1\right\}, \\ u \in \mathcal{U} &= \left\{u \in \mathbb{R}^m :\ \|u\|_\infty \le 1\right\}, \end{aligned} \tag{16}$$

being $L \in \mathbb{R}^{s\times n}$ full column rank.

A candidate CLRF $V_{\wedge_\phi}$, corresponding to the smoothed intersection of a PCLF $\max_{i\in\mathbb{I}_s}\left\{x^\top F_i^\top F_i x\right\}$ and a QCLF (for the unconstrained system) $x^\top P x$ can be designed by defining functions

$$\begin{array}{l} R_1(x) = 1 - \max_{i\in\mathbb{I}_s}\left\{x^\top F_i^\top F_i x\right\}, \\ R_2(x) = 1 - x^\top P x, \end{array} \tag{17}$$

and the R-intersection

$$R_{\wedge_\phi} = R_1 \wedge_\phi\ R_2. \tag{18}$$

As previously remarked, $R_{\wedge_\phi}(x) \geq 0\ \forall x \in \mathcal{L}\left[V_{\wedge_\phi}, 1\right]$ and $\max_x\{R_{\wedge_\phi}(x)\} = R_{\wedge_\phi}(0) = 1$, therefore the candidate (positive definite) CLRF is

$$V_{\wedge_\phi}(x) = 1 - R_{\wedge_\phi}(x). \tag{19}$$

If a smoother composed function is desired, in (17) the polyhedral function of the second order $\max_{i\in\mathbb{I}_s}\left\{x^\top F_i^\top F_i x\right\}$ can be substituted by the corresponding everywhere differentiable $2p$-norm [10] (of the second order) $\|Fx\|_{2p}^2$. Using the above candidate CLRF (19), the result of Theorem 1 can be further exploited for the control design problem.

Corollary 1: Consider two control Lyapunov functions $V_i : \mathcal{L}[V_i, 1] \subseteq \mathcal{X} \rightarrow \mathbb{R}$, $i = 1, 2$, for the constrained nonlinear control affine system (15). Assume there exist a common continuous state feedback control law $K(x)$ and a decreasing rate $\eta \in \mathbb{R}^+$ such that, along the controlled trajectory,

$$\begin{aligned} \dot{V}_i(x(t)) = \nabla V_i(x(t))\left(f(x(t)) + g(x(t))K(x(t))\right) \leq \\ \leq -\eta V_i(x(t))\quad \forall i \in \mathbb{I}_2. \end{aligned} \tag{20}$$

Then the R-composed function $V_{\wedge_\phi}$ (19) is a control Lyapunov function with decreasing rate η in the set $\mathcal{L}[V_{\wedge_\phi}, 1] = \mathcal{L}[V_1, 1] \cap \mathcal{L}[V_2, 1]$ for the constrained system (15), $\forall \phi \in \mathbb{R}^+$.

Proof: Consider the closed-loop system $\dot{x}(t) = \bar{f}(x(t)) = f(x(t)) + g(x(t))K(x(t))$. Since both V_1 and V_2 are two LFs with decreasing rate η, according to Theorem 1, also $V_{\wedge_\phi}$ is a suitable LF for the closed-loop system, $\forall \phi \in \mathbb{R}^+$. Therefore in the set $\mathcal{L}[V_{\wedge_\phi}, 1] = \mathcal{L}[V_1, 1] \cap \mathcal{L}[V_2, 1]$ there exists an admissible continuous controller, at least $K(x)$, that makes $V_{\wedge_\phi}$ a valid CLF with decreasing rate η. ■

Remark 4: The problem of *piecing together* two CLFs has been recently addressed for unconstrained nonlinear control affine systems in [20], where, as in Corollary 1, the existence of a common continuous state feedback stabilizing control has been assumed. The benefit of using the CLRF approach is that also state and control input constraints are considered, and a unique CLF is designed. Moreover, by tuning the parameter ϕ, the inner sublevel sets of the CLRF can be shaped like the optimal ones, as shown in Figure 3.

Alternatively to the search of a common controller, to prove that the composed CLRF is a suitable CLF, a Petersen-like [14] condition has been derived in [8] for bounded controllers of the kind (16). Considering the control that minimize $\dot{V}_{\wedge_\phi}(x(t))$ pointwise, i.e. $u(x(t)) = -sign\left(g(x(t))^\top \nabla V_{\wedge_\phi}(x(t))^\top\right)$, the time derivative of $\dot{V}_{\wedge_\phi}(x(t))$ is

$$\nabla V_{\wedge_\phi}(x(t))f(x(t)) - \sum_{i=1}^{m} |\left(\nabla V_{\wedge_\phi}(x(t))g(x(t))\right)_i|,$$

where $\left(\nabla V_{\wedge_\phi}(x(t))g(x(t))\right)_i$ denotes the i^{th} component of the row vector $\nabla V_{\wedge_\phi}(x(t))g(x(t)) \in \mathbb{R}^{1\times m}$.

Therefore function $V_{\wedge_\phi}$ is a suitable CLF if and only if

$$\max_{x\in\mathcal{S}\setminus 0}\left\{\nabla V_{\wedge_\phi}(x)f(x) - \sum_{i=1}^{m} |\left(\nabla V_{\wedge_\phi}(x)g(x)\right)_i|\right\} < 0,$$

$$\mathcal{S} = \left\{x \in \mathbb{R}^n :\ \|Fx\|_\infty \leq 1,\ x^\top P x \leq 1\right\}. \tag{21}$$

C. Lyapunov-based control

A nonlinear gradient-based control can be associated to a smooth candidate CLRF $V_{\wedge_\phi}$:

$$\begin{aligned} u(x) = \arg\min_{v\in\mathcal{U}}\{\dot{V}_{\wedge_\phi}(x) + \dot{J}(x, v)\}\ \text{ sub. to} \\ \dot{V}_{\wedge_\phi}(x) \leq -\eta V_{\wedge_\phi}(x), \end{aligned} \tag{22}$$

where $J(x, u)$ (4) is the performance cost to be minimized. Control (22) follows from a *minimal selection* control [11, Section 2.4] and therefore it is continuous [11, Section 4.2].

In the above control, the state constraints are guaranteed because $V_{\wedge_\phi}$ is a CLF. Moreover, for large values of the shape parameter ϕ, the level sets of $V_{\wedge_\phi}$ can be made close to the optimal ones. For instance, considering the linearized system in a neighbor of the origin, the level sets of the optimal QCLF $x^\top P^* x$ can be obtained. With this choice, the control (22) actually converges to the optimal one, i.e. the control u such that

$$\min_u\{2x^\top P^*\left(Ax + Bu\right) + x^\top Q x + u^\top R u\} = 0.$$

V. SIMULATIONS

The constrained nonlinear model (1)-(3) of the CSTR, see Section II, is simulated together with the nonlinear controller (22). The CLRF $V_{\wedge_\phi}$ actually is the smoothed intersection between the optimal quadratic $\tilde{x}^\top P^* \tilde{x}$ (5) (for the unconstrained linearized system) and the polyhedral function of the second order $\max\{\tilde{x}^\top F_1^\top F_1 \tilde{x},\ \tilde{x}^\top F_2^\top F_2 \tilde{x}\}$ (10) that describes the state constraints (2). In the simulations, parameters $\alpha = 0, \phi = 20$ are chosen for the CLRF $V_{\wedge_\phi}$. In this particular case, the Petersen-like condition (21) is checked to prove that the candidate CLRF $V_{\wedge_\phi}$ (17)-(19) is a suitable CLF $\forall \phi \in \mathbb{R}^+$.

A direct comparison can be performed with respect to the Lyapunov-based receding horizon MPC controller proposed in [4], where the simulation corresponding to the initial condition $x_0 = [0.702\ kmol/m^3,\quad 392.6\ {}^\circ K]^\top$ is shown. The proposed control strategy successfully stabilizes the nonlinear system in accordance to both state and control constraints.

First note that the controlled invariant set is enlarged, as it is shown in Figures 1, 3. Secondly, the control performances are improved, as the state x definitively converges faster than [4] and the control signals u, see Figure 5, are much smoother. In fact, the chattering behavior of control $u_1 = \Delta C_{A0}$ (and partially also of $u_2 = Q$) in [4] is totally absent here.

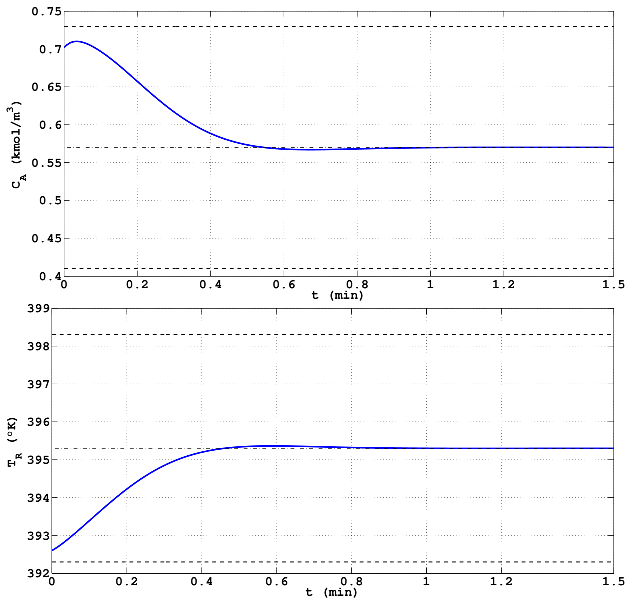

Fig. 4. Controlled state trajectory.

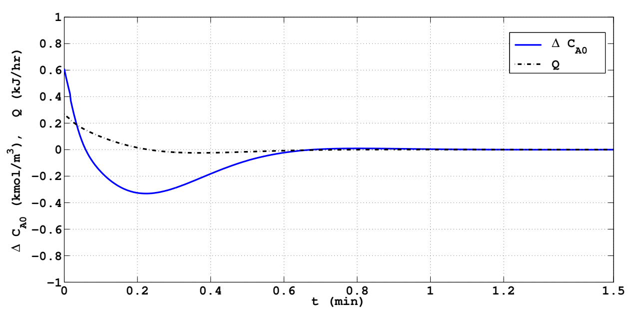

Fig. 5. Control inputs obtained by the control law (22) together with the proposed control Lyapunov R-function.

VI. CONCLUSIONS AND FUTURE WORK

In this paper, the constrained stabilization of a nonlinear control affine system, describing a chemical reaction taking place in a continuous stirred tank reactor, is addressed.

The novelty of the proposed approach stands in the use of the class of composite control Lyapunov R-functions. The definition of a novel composition rule allows the design of a control Lyapunov function with external level set in accordance to the state constraints and inner sublevel sets as desired, for instance the optimal ones. This approach is very convenient because, unlike standard switching control strategies, both constraints and optimality arguments can be handled by a unique smooth control Lyapunov function, together with a gradient-based time-continuous controller. For these reasons, the use of control Lyapunov R-functions is particularly effective for the feedback control of multi-input multi-output chemical processes.

In the opinion of the authors, the novel composition rule could be further exploited to remove the assumption of existence of a common controller and also in handling model uncertainties and disturbances.

REFERENCES

[1] N. El-Farra and P. Christofides, "Bounded robust control of constrained multivariabe nonlinear processes," *Chemical Engineering Science*, vol. 58, pp. 3025–3047, 2003.

[2] ——, "Integrating robustness, optimality, and constraints in control of nonlinear processes," *Chemical Engineering Science*, vol. 56, pp. 1841–1868, 2001.

[3] J. Primbs, V. Nevistic, and J. Doyle, "A receding horizon generalization of pointwise min-norm controllers," *IEEE Trans. on Automatic Control*, vol. 45, pp. 898–909, 2000.

[4] P. Mhaskar, N. El-Farra, and P. Christofides, "Stabilization on nonlinear systems with state and control constraints using Lyapunov-based predictive control," *Systems & Control Letters*, vol. 55, no. 8, pp. 650–659, 2006.

[5] B. O'Dell and E. Misawa, "Semi-ellipsoidal controlled invariant sets for constrained linear systems," *Journal of Dynamic Systems, Measurement and Control*, vol. 124, pp. 98–103, 2002.

[6] T. Thibodeau, W. Tong, and T. Hu, "Set invariance and performance analysis of linear systems via truncated ellipsoids," *Automatica*, vol. 45, pp. 2046–2051, 2009.

[7] A. Balestrino, A. Caiti, E. Crisostomi, and S. Grammatico, "Stabilizability of linear differential inclusions via R-functions," *IFAC Symposium on Nonlinear Control Systems, Bologna (Italy)*, 2010.

[8] A. Balestrino, E. Crisostomi, S. Grammatico, and A. Caiti, "Stabilization of constrained linear systems via smoothed truncated ellipsoids," *IFAC World Congress, Milan (Italy)*, 2011.

[9] N. El-Farra, P. Mhaskar, and P. Christofides, "Output feedback control of switched nonlinear systems using multiple Lyapunov functions," *Systems & Control Letters*, vol. 54, no. 12, pp. 1163–1182, 2005.

[10] F. Blanchini and S. Miani, "Constrained stabilization via smooth Lyapunov functions," *Systems & Control Letters*, vol. 35, pp. 155–163, 1998.

[11] R. Freeman and P. Kokotović, *Robust nonlinear control design*. Boston: Birkhäuser, 1996.

[12] N. El-Farra, P. Mhaskar, and P. Christofides, "Hybrid predictive control of nonlinear systems: method and applications to chemical processes," *International Journal of Robust and Nonlinear Control*, vol. 14, pp. 199–225, 2004.

[13] P. Mhaskar, N. El-Farra, and P. Christofides, "Robust hybrid predictive control of nonlinear systems," *Automatica*, vol. 41, pp. 209–217, 2005.

[14] I. R. Petersen and B. R. Barmish, "Control effort considerations in the stabilization of uncertain dynamical systems," *Systems and Control Letters*, vol. 9, pp. 417–422, 1987.

[15] V. Rvachev, "Geometric applications of logic algebra (in Russian)," *Naukova Dumka*, 1967.

[16] A. Balestrino, A. Caiti, E. Crisostomi, and S. Grammatico, "R-composition of Lyapunov functions," *IEEE Mediterranean Conference on Control and Automation, Thessaloniki (Greece)*, 2009.

[17] T. Hu and F. Blanchini, "Non-conservative matrix inequality conditions for stability/stabilizability of linear differential inclusions," *Automatica*, vol. 46, pp. 190–196, 2010.

[18] A. Balestrino, A. Caiti, E. Crisostomi, and S. Grammatico, "Stability analysis of dynamical systems via R-functions," *IEEE European Control Conference, Budapest (Hungary)*, 2009.

[19] A. Balestrino, A. Caiti, and S. Grammatico, "Stabilizability of constrained uncertain linear systems via smooth control Lyapunov R-functions," *IEEE Conference on Decision and Control, Orlando (Florida, USA)*, 2011.

[20] V. Andrieu and C. Prieur, "Uniting two Lyapunov functions for affine systems," *IEEE Trans. on Automatic Control*, vol. 55, no. 8, pp. 1923–1927, 2010.