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Scaling the Size of a Multiagent Formation via Distributed Feedback

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***Abstract*— We consider a multiagent coordination problem where the objective is to steer a group of agents into a formation that translates along a predefined trajectory velocity. Unlike previous control strategies that require a static desired formation or set of desired formations, we introduce a strategy in which one agent assigns a scale for the formation and the remaining agents adjust to the new scale. Thus, the formation can dynamically adapt to changes in the environment and in group objectives or respond to perceived threats. We introduce two strategies: one that requires agents to communicate estimates of the desired formation scale along edges of a communication network and one that only requires relative position sensing among agents. We show that the former strategy guarantees stability for any desired connected formation. For the latter strategy, we present a geometric constraint which can be used with the small gain theorem.**

I. Introduction

Group coordination of mobile agents using distributed feedback laws has been an active area of control theory research in recent years. The main emphasis of such coordination problems is to achieve a desired group behavior using only local feedback rules [1], [2], [3], [4]. In particular, formation control as a desired group behavior has received considerable attention, such as in [5], [6], [7]. Use of artificial potential fields for formation control is explored in [8], formation control strategies with communication topology switching are investigated in [9], and strategies allowing the desired formation to switch among various configurations is presented in [10].

In this paper, we consider the problem of formation control along a velocity trajectory when only one agent knows the desired formation size. The remaining follower agents estimate the desired formation size and proceed with a cooperative control law utilizing this estimate. By allowing the size of the formation to change, the group can dynamically adapt to changes in the environment such as unforeseen obstacles along the trajectory path, adapt to changes in group objectives, or respond to threats.

We present two strategies for updating the desired formation size estimates. In the first strategy, we assume a communication topology exists among the agents, allowing the agents to share formation size estimates. Rather than permitting the leader to simply dictate the desired formation scale, which requires a communication structure heavily dependent on the identity of the leader, we present an update rule utilizing the internal estimates of multiple neighbors while still guaranteeing stability for any desired formation. The advantage to this approach is that the follower agents do not need to know which agent is the leader. For example, the leader could be changed during operation without communicating this change to the remaining agents. In some scenarios, we are interested in the minimum amount of information required to allow formation scaling and control as it may be undesirable or impossible for the agents to actively communicate with one another. In the second strategy, we assume that only relative position information defined by a sensing topology is available to each agent. In this strategy, no interagent communication is needed.

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This paper is organized as follows: In Section II, we introduce the problem statement. In Section III we introduce a cooperative control strategy that requires interagent communication and guarantees stability for any connected formation. In Section IV we introduce a strategy that does not require interagent communication. In this case, we use the small gain theorem to derive a sufficient geometric condition for stability. In Section V, we present simulation results, and in Section VI, we summarize our results and provide directions for future research.

II. Problem Definition

Consider a team of n agents each represented by the vector $x_i \in \mathbb{R}^p, i = 1, \ldots, n$, and assume each agent is modeled with double integrator dynamics:

$$\ddot{x}_i = f_i. \tag{1}$$

We define $x = [x_1^T\ x_2^T\ x_3^T\ \ldots\ x_n^T]^T$ and $v = \dot{x}$ to denote the stacked vector containing the velocities of all of the agents.

We assume there is a position sensing topology by which agents are capable of inferring the relative position of other agents. We represent this topology with a *sensing* graph with n nodes representing the n agents. We assume relative position sensing is bidirectional, and if agent i and agent j have access to the quantity $x_i - x_j$, then the ith and jth nodes of the sensing graph are connected by an edge.

Suppose there are m edges in the sensing graph. We arbitrarily assign a direction to each edge of the sensing graph and define the $n \times m$ incidence matrix D as:

$$d_{ij} = \begin{cases} +1 & \text{if node } i \text{ is the head of edge } j \\ -1 & \text{if node } i \text{ is the tail of edge } j \\ 0 & \text{otherwise} \end{cases}$$

where d_{ij} is the ijth entry of D. Since the graph is bidirectional, this choice of direction does not affect any of the following results.

Suppose the agents are to maintain a certain formation by maintaining the relative interagent distances defined by the sensing topology. Suppose, in addition, it is allowable for the formation size to change, and one agent (for notational simplicity and without loss of generality, agent n) chooses a constant target formation scale λ^*. Each other agent possesses a local formation scalar λ_i representing its estimate of λ^*. We define $\lambda := [\lambda_1\ \lambda_2\ \dots\ \lambda_n]^T$.

Our goal is to achieve the following three desired group behaviors:

B1) Each agent reaches in the limit a common velocity vector $v(t)$ for the group, i.e.

$$\lim_{t\to\infty} |\dot{x}_i - v(t)| = 0, \qquad i = 1,\dots,n. \tag{2}$$

B2) Each agent's formation scaling factor reaches λ^* in the limit, i.e.

$$\lim_{t\to\infty} |\lambda_i - \lambda^*| = 0, \qquad i = 1,\dots,n. \tag{3}$$

B3) The relative position of two agents connected by an edge k

$$z_k := \sum_{i=1}^{n} d_{ik} x_i \tag{4}$$

converges to a prescribed target value $\lambda^* z_k^d$ for each $k = 1,\dots,m$.

Note that $z^d = [z_1^{d^T}\ \dots\ z_2^{d^T}]^T$ must be designed such that $z^d \in \mathcal{R}(D^T \otimes I_p)$ where $\mathcal{R}$ denotes range space, $\otimes$ is the Kronecker product operator, and I_p denotes the $p\times p$ identity matrix.

III. Formation Scaling with Communication Between Agents

We begin by defining a linear controller for the input f_i to each agent:

$$f_i = -\sum_{j=1}^{m} d_{ij}(z_j - \lambda_i z_j^d) - k_i(\dot{x}_i - v(t)) + \dot{v}(t) \tag{5}$$

where $k_i > 0$ is a damping coefficient. Let $\mathbf{1}_n$ denote the length n column vector of all ones and define

$$\tilde{v}(t) := \dot{x}(t) - \mathbf{1}_n \otimes v(t) \tag{6}$$

and

$$\Gamma := \begin{bmatrix} \sum_{j=1}^m d_{1j} z_j^d & & 0 \\ & \ddots & \\ 0 & & \sum_{j=1}^m d_{nj} z_j^d \end{bmatrix} \tag{7}$$

which can be written compactly as $\Gamma := \mathrm{diag}\{(D\otimes I_p)z^d\}\cdot(I_n \otimes \mathbf{1}_p)$ where the $\mathrm{diag}\{\cdot\}$ operation converts a vector into a diagonal matrix with the elements of the vector along the diagonal. Then (5) can be written in matrix form as

$$f = -(D\otimes I_p)z + \Gamma\lambda - (K\otimes I_p)\tilde{v} + \mathbf{1}_n \otimes \dot{v}(t) \tag{8}$$

where $K := \mathrm{diag}\{[k_1\ \dots\ k_n]\}$ and $f = [f_1^T\ \dots\ f_n^T]^T$.

Given that $\lambda_n = \lambda^*$ is constant, we now seek to design an update rule for $\lambda_1,\dots,\lambda_{n-1}$. Suppose agents are capable of communicating a scalar quantity to other agents. One obvious approach allows the leader agent to communicate the desired formation scale to its neighbors, who in turn communicate the desired scale to their neighbors, etc. This approach requires each agent to know *a priori* which neighboring agent forms a path to the leader, may not be robust to dropped communication links, and does not adapt well to a reassignment of the leader since a different communication tree is required for each possible leader agent. We now present an alternative approach in which a communication structure and update rule are designed to address these shortcomings.

Suppose each agent can share its formation scaling factor with neighboring agents as defined by a *communication* graph with n nodes labeled $\nu_1,\dots,\nu_n$ where an edge points from agent i to agent j if i has access to j's scaling factor, and this edge is denoted by (ν_i,ν_j). We will assume there exists a directed path from any node ν_i to the leader for all $i \neq n$ (in the undirected case, this condition is equivalent to connectedness). We denote the communication graph by $\mathcal{G} = (\mathcal{V},\mathcal{E})$ where $\mathcal{V} = \{\nu_1,\dots,\nu_n\}$ is the vertex set of the graph and $\mathcal{E} \subset \mathcal{V}\times\mathcal{V}$ is the edge set of the graph. Let $\Gamma_i = (D_i \otimes I_p)z^d$ be the $p\times 1$ block of Γ corresponding to agent i's control input where D_i indicates the ith row of the incidence matrix D, and define tunable parameters $\alpha_{ij} \geq 0$ for $i,j = 1,\dots,n$ such that $\alpha_{ij} > 0$ iff $(\nu_i,\nu_j) \in \mathcal{E}$. We propose the following control strategy:

$$\dot{\lambda}_i = -\sum_{j\in N_i^c} \alpha_{ij}(\lambda_i - \lambda_j) - \beta p_i^{-1}\Gamma_i^T(v_i - v(t)) \tag{9}$$

for $i = 1 \dots n-1$ where β is a tunable, scalar parameter such that $\beta \geq 0$, $N_i^c = \{j : (\nu_i,\nu_j) \in \mathcal{E}\}$, and p_i is a positive scalar designed below to ensure stability. Note that $\dot{\lambda}_n = 0$. This control strategy is intuitively appealing because it includes an agreement expression among the scaling parameters and an additional term dependent on the velocity error.

Define

$$\tilde{z} := z - \lambda^* z^d \tag{10}$$

and

$$\tilde{\lambda} := (\lambda - \lambda^* \mathbf{1}_n). \tag{11}$$

Let $\underline{\lambda}$ and $\underline{\tilde{\lambda}}$ be λ and $\tilde{\lambda}$ with the last element removed, i.e.

$$\underline{\lambda} = [\lambda_1\ \lambda_2\ \dots\ \lambda_{n-1}]^T \tag{12}$$

and

$$\underline{\tilde{\lambda}} = (\underline{\lambda} - \lambda^* \mathbf{1}_{n-1}). \tag{13}$$

Then objectives B1–B3 amount to asymptotic stability of the origin for the closed loop system with state

$$X = [\tilde{v}^T\ \tilde{z}^T\ \underline{\tilde{\lambda}}^T]^T. \tag{14}$$

Let L be an $n\times n$ matrix defined by

$$l_{ij} = \begin{cases} \sum_{k=1,k\neq i}^{n} \alpha_{ik}, & i = j \\ -\alpha_{ij}, & i \neq j. \end{cases} \tag{15}$$

Since agent n does not update its scaling factor via feedback, we set the last row of L to 0. L is called the graph Laplacian of the communication graph, and when we refer to the graph Laplacian of a given graph with edge weights, we imply the above construction.

Define $\bar{L}$ to be L with the last row removed, and define $\underline{L}$ to be L with the last row and the last column removed. Also, let $\underline{\Gamma}$ be equal to Γ with the last column removed, and let $P = \text{diag}\{\begin{bmatrix} p_1 & \dots & p_{n-1} \end{bmatrix}\}$. Then (9) can be written in matrix form as

$$\dot{\underline{\lambda}} = \dot{\underline{\tilde{\lambda}}} = -\bar{L}\lambda - \beta P^{-1}\underline{\Gamma}^T\tilde{v} \tag{16}$$

$$= -\bar{L}\tilde{\lambda} + \bar{L}\lambda^*\mathbf{1}_n - \beta P^{-1}\underline{\Gamma}^T\tilde{v} \tag{17}$$

$$= -\underline{L}\underline{\tilde{\lambda}} - \beta P^{-1}\underline{\Gamma}^T\tilde{v}, \tag{18}$$

where we use the fact that $L\mathbf{1}_n = \mathbf{0}_n$ and hence $\bar{L}\mathbf{1}_n = \mathbf{0}_{n-1}$ where $\mathbf{0}_n$ denotes the length n column vector of all zeros.

It is apparent from (8) that

$$f = -(D \otimes I_p)z + \Gamma\lambda - (K \otimes I_p)\tilde{v} + \mathbf{1}_n \otimes \dot{v}(t) \tag{19}$$

$$= -(D \otimes I_p)\tilde{z} - \lambda^*(D \otimes I_p)z^d + \Gamma\tilde{\lambda}$$
$$+ \lambda^*\Gamma\mathbf{1}_n - (K \otimes I_p)\tilde{v} + \mathbf{1}_n \otimes \dot{v}(t). \tag{20}$$

This gives

$$\dot{\tilde{v}} = -(D \otimes I_p)\tilde{z} + \Gamma\tilde{\lambda} - (K \otimes I_p)\tilde{v}. \tag{21}$$

Because $\tilde{\lambda}_n = 0$, (21) can be written as

$$\dot{\tilde{v}} = -(D \otimes I_p)\tilde{z} + \underline{\Gamma}\underline{\tilde{\lambda}} - (K \otimes I_p)\tilde{v}. \tag{22}$$

Let $0_{n\times m}$ denote the $n \times m$ matrix all zeros. When the dimensions are clear, the subscript is omitted. The dynamics of (8) and (18) can then be formulated as a homogenous linear system

$$\dot{X} = \begin{bmatrix} -K \otimes I_p & -D \otimes I_p & \underline{\Gamma} \\ D^T \otimes I_p & 0 & 0 \\ -\beta P^{-1}\underline{\Gamma}^T & 0 & -\underline{L} \end{bmatrix} X \tag{23}$$

evolving on the following invariant subspace of $\mathbb{R}^{(n+m)p+n-1}$:

$$S_X = \{(\tilde{v}, \tilde{z}, \tilde{\lambda}) : \tilde{v} \in \mathbb{R}^{np}, \tilde{z} \in \mathcal{R}(D^T \otimes I_p), \tilde{\lambda} \in \mathbb{R}^{n-1}\}. \tag{24}$$

To facilitate the proof of the stability of (23), we now introduce a generalization of Lemma 10.36 in [11] to the case of a directed communication topology.

Lemma 1. *All eigenvalues of $\underline{L}$ are in the open right half plane.*

Proof: Let $B(a, R)$ be the closed disk of radius R centered at a. The Gershgorin circle theorem states that all eigenvalues of $\underline{L}$ are located in the union of the following $n-1$ disks:

$$B_i\left(l_{ii}, \textstyle\sum_{j=1,j\neq i}^{n-1} |l_{ij}|\right), i = 1, \dots, n-1. \tag{25}$$

But $l_{ii} \geq -\sum_{j=1,j\neq i}^{n-1} |l_{ij}|$, so the union of the $n-1$ disks lie in the closed right half plane and may only intersect the $j\omega$-axis at the origin. It remains to show that $\underline{L}$ does not have a zero eigenvalue. As a proof by contradiction, assume $\underline{L}$ has a nontrivial nullspace such that $\underline{L}w = 0$ for some nontrivial w. Because the last row of L is $\mathbf{0}_n^T$, we have $\begin{bmatrix} w^T & 0 \end{bmatrix}^T \in \mathcal{N}(L)$ where $\mathcal{N}$ denotes null space. By our assumption that the communication graph contains a directed spanning tree, the rank of L is $n-1$ [2]. Therefore, $\{\mathbf{1}\}$ forms a basis for the nullspace of L, forming a contradiction. ■

We require the p_is in (9) to be chosen such that

$$\underline{L}^T P + P\underline{L} > 0 \tag{26}$$

where P is a diagonal matrix with the p_is along the diagonal. A diagonal solution $P > 0$ to (26) exists because $\underline{L}$ has nonnegative diagonal and nonpositive off-diagonal entries and eigenvalues in the open right half plane ([12], Chapter 6, Theorem 2.3).

In many practical cases, p_i can be taken to be 1 for all i. In particular, if the subgraph induced by removing the leader agent and its edges is *balanced* (i.e., $\sum_{j=1,j\neq i}^{n} \alpha_{ji} = \sum_{j=1,j\neq i}^{n-1} \alpha_{ij}$), then $\underline{L}^T + \underline{L}$ is positive definite and P can be taken to be the identity as shown in the following corollary. This encompasses any case in which the communication graph is bidirectional.

Corollary 1. *Let $\mathcal{G}_l$ be the subgraph of the communication graph induced by removing the leader agent. If $\mathcal{G}_l$ is balanced, then $\underline{L}^T + \underline{L} > 0$.*

Proof: Let E be the set of edges connected to node n in $\mathcal{G}$ with the same edge weighting but with the head and tail of these edges reversed. Define $\mathcal{G}^* := (\mathcal{V}, \mathcal{E} \cup E)$, i.e. $\mathcal{G}^*$ is the communication graph with additional edges pointing towards the leader node. By construction, $\mathcal{G}^*$ is balanced. Let L^* be the graph Laplacian corresponding to $\mathcal{G}^*$. Note that $\underline{L}$ is also equal to L^* with the last row and column removed. Because $\mathcal{G}^*$ is balanced, $L_s^* = L^* + (L^*)^T \geq 0$ [3]. Note that $\underline{L}_s = \underline{L} + \underline{L}^T$ is a principle subminor of L_s^*, hence $\underline{L}_s \geq 0$. Also note that, because $\mathcal{G}^*$ is balanced, $\mathbf{1}^T$ is a left eigenvector of L_s^*. As a proof by contradiction, suppose $\underline{L}_s$ has a nontrivial nullspace such that $\underline{L}w = 0$ for some nontrivial w. Because $\mathbf{1}^T L_s^* = \mathbf{0}$, we see that the last row of L_s^* is a linear combination of the remaining rows, and hence $y = \begin{bmatrix} w^T & 0 \end{bmatrix}^T \in \mathcal{N}(L_s^*)$. But $\{\mathbf{1}\}$ forms a basis for the nullspace of L_s^*, a contradiction.

■

Theorem 1. *The origin of* (23)–(24) *with P selected as in* (26) *is asymptotically stable.*

Proof:

Case 1. ($\beta > 0$) We construct a Lyapunov function

$$V = \frac{1}{2}(\tilde{v}^T\tilde{v} + \tilde{z}^T\tilde{z} + \beta^{-1}\underline{\tilde{\lambda}}^T P\underline{\tilde{\lambda}}) \geq 0. \tag{27}$$

Let $Q := \underline{L}^T P + P\underline{L}$.

Then

$$\dot{V} = -\tilde{v}^T(K \otimes I_p)\tilde{v} - \underline{\tilde{\lambda}}^T \left(\beta^{-1}\frac{1}{2}Q\right)\underline{\tilde{\lambda}} \leq 0. \tag{28}$$

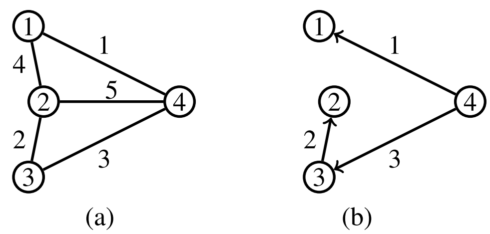

Fig. 1. (a) 4-agent, 5-edge sensing graph with edge indexing based on the monitoring subgraph. (b) One possible monitoring graph constructed from the sensing graph.

We now use LaSalle's invariance principle. Note that $\dot{V} \equiv 0$ means $\tilde{v} \equiv \mathbf{0}, \tilde{\lambda} \equiv \mathbf{0}$. From (23), this implies $\tilde{z} \in \mathcal{N}(D \otimes I_p)$ for all t. Recalling that $\tilde{z} \in \mathcal{R}(D^T \otimes I_p) = \mathcal{N}(D \otimes I_p)^{\perp}$, we see that $\tilde{z} \equiv \mathbf{0}$. Hence, $\dot{V} \equiv 0$ iff $\tilde{z} \equiv \mathbf{0}$, $\tilde{\lambda} \equiv \mathbf{0}$, and $\tilde{z} \equiv \mathbf{0}$. Therefore, $X = \mathbf{0}$ is asymptotically stable.

Case 2. ($\beta = 0$) The resulting transition matrix of (23) is then upper triangular, and therefore the system is asymptotically stable iff the subsystems

$$\begin{bmatrix} \dot{\tilde{v}} \\ \dot{\tilde{z}} \end{bmatrix} = \begin{bmatrix} -K \otimes I_p & -D \otimes I_p \\ D^T \otimes I_p & 0 \end{bmatrix} \begin{bmatrix} \tilde{v} \\ \tilde{z} \end{bmatrix} \tag{29}$$

and

$$\dot{\underline{\tilde{\lambda}}} = -\underline{L}\underline{\tilde{\lambda}} \tag{30}$$

are asymptotically stable. (30) is clearly asymptotically stable, and (29) can be seen to be asymptotically stable by a Lyapunov argument and again invoking LaSalle's invariance principle as in Case 1. ■

IV. Formation Scaling with No Communication

We again consider agent dynamics of the form in (5). However, we now present an update rule for λ_i that does not require agents to share local scaling estimates with other agents, i.e. only relative position information is shared along edges in the sensing graph. From the sensing graph, we construct a directed spanning tree (also known as an arborescence [13]) rooted at the leader with the following properties:

1) The arborescence contains all the vertices and a (directed) subset of the edges of the sensing graph.
2) Each vertex except the leader is the head of exactly one edge. The leader agent is the head of no edge.
3) A directed path exists from the leader node to all other nodes.

We call this graph the *monitoring* graph. Without loss of generality and for notational purposes, assume the leader is again agent n. To simplify notation and analysis, for each edge in the monitoring graph, we number the corresponding edge in the sensing graph to match the head node index of that edge in the monitoring graph, and we arbitrarily index the remaining edges of the sensing graph. Fig. 1 shows an example of a formation with the sensing graph, one possible construction of the monitoring graph, and the induced edge indexing scheme.

The proposed update rule for other agents is

$$\dot{\lambda}_i = -\left(\lambda_i - \frac{(z_i^d)^T z_i}{||z_i^d||^2}\right) \text{ for } i = 1, \dots, n-1. \tag{31}$$

This means that each agent monitors one edge to update its local formation scaling factor. For notational simplicity, we let

$$\underline{T} = \begin{bmatrix} \frac{(z_1^d)^T}{||z_1^d||^2} & & 0 \\ & \ddots & \\ 0 & & \frac{(z_{n-1}^d)^T}{||z_{n-1}^d||^2} \end{bmatrix} \tag{32}$$

and $\underline{\Psi} = \mathrm{diag}\{(\underline{D} \otimes I_p) z^d\} \cdot (I_{n-1} \otimes \mathbf{1}_p)$, i.e. $\underline{\Psi}$ is equal to Γ with the last row and column removed. We can then write the dynamics of (8) and (31) as a homogenous linear system

$$\begin{bmatrix} \dot{\tilde{v}} \\ \dot{\tilde{z}} \\ \dot{\tilde{\lambda}} \end{bmatrix} = \begin{bmatrix} -K \otimes I_p & -D \otimes I_p & \underline{\Gamma} \\ D^T \otimes I_p & 0 & 0 \\ 0 & \hat{\underline{T}} & -I_{n-1} \end{bmatrix} \begin{bmatrix} \tilde{v} \\ \tilde{z} \\ \tilde{\lambda} \end{bmatrix} \tag{33}$$

where $\hat{\underline{T}} = [\underline{T} \;\; 0_{(n-1)\times p(m-n+1)}]$. Objectives B1–B3 amount to asymptotic stability of the origin of (33). We can represent this cooperative control system as the interconnection of two subsystems as in Fig. 2:

- **$\tilde{v}, \tilde{z}$ subsystem:**

$$\begin{bmatrix} \dot{\tilde{v}} \\ \dot{\tilde{z}} \end{bmatrix} = \left[\begin{array}{c|c} -K \otimes I_p & -D \otimes I_p \\ \hline D^T \otimes I_p & 0 \end{array}\right] \begin{bmatrix} \tilde{v} \\ \tilde{z} \end{bmatrix} + \left[\begin{array}{c} I_{p(n-1)} \\ 0_{p\times p(n-1)} \\ \hline 0_{pm\times p(n-1)} \end{array}\right] u \tag{34}$$

$$y = \left[0_{p(n-1)\times pn} \middle| \; I_{p(n-1)} \;\; 0_{p(n-1)\times p(m-n+1)}\right] \begin{bmatrix} \tilde{v} \\ \tilde{z} \end{bmatrix}. \tag{35}$$

- **$\tilde{\lambda}$ subsystem:**

$$\dot{\underline{\tilde{\lambda}}} = -I_{n-1}\underline{\tilde{\lambda}} + \underline{T}y \tag{36}$$

$$u = \underline{\Psi}\underline{\tilde{\lambda}}. \tag{37}$$

Note that both subsystems are stable. The stability of the composite system can be checked using a number of techniques, including the small gain theorem. Despite its potential conservatism, the small gain theorem gives a stability condition that can be verified based on the geometry of the formation. Because both subsystems are linear and stable, asymptotic stability of the origin of (33) follows from the small gain theorem stability condition. We first present a geometric interpretation of the $\mathcal{L}_2$ gain of the $\tilde{\lambda}$ subsystem:

Theorem 2. *The $\mathcal{L}_2$ gain of the $\tilde{\lambda}$ subsystem is:*

$$\gamma_1 = \max_{i=1,\dots,n-1} \left\{ \frac{\left|\left|\sum_{j=1}^m d_{ij} z_j^d\right|\right|}{\left|\left|z_i^d\right|\right|} \right\}. \tag{38}$$

Proof: Let $H(s)$ be the transfer function of the $\tilde{\lambda}$ subsystem, i.e. $H(s) = \frac{1}{s+1}\underline{\Psi}\underline{T}$. To calculate $\gamma_1 = \sup_\omega ||H(j\omega)||_2$, we note that the supremum occurs when

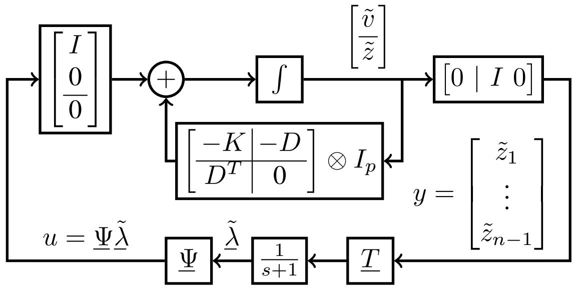

Fig. 2. Interconnection of subsystems (34)–(35) and (36)–(37). Dimension subscripts are removed for clarity.

$\omega = 0$ and is equal to $||\underline{\Psi}\underline{T}||_2$. Observe that $\underline{\Psi}$ and $\underline{T}$ are block diagonal. Therefore, it is clear that $\underline{\Psi}\underline{T}$ consists of $n-1$ blocks of dimension $p \times p$ along the diagonal. The singular values of $\underline{\Psi}\underline{T}$ are simply the singular values of each $p \times p$ block.

Let $g_i = \sum_{j=1}^{m} d_{ij} z_j^d$ for $i = 1, \ldots, n-1$. We have that γ_1 is the largest singular value of $\underline{\Psi}\underline{T}$, and the $p \times p$ blocks of $\underline{\Psi}\underline{T}$ take the form $\frac{1}{||z_i^d||^2} g_i (z_i^d)^T$. The singular values of $\frac{1}{||z_i^d||^2} g_i (z_i^d)^T$ are $\left\{ \frac{||g_i||}{||z_i^d||}, 0 \right\}$. ■

This leads to a geometric criterion ensuring stability when the small gain theorem is applied: if γ_2 is the $\mathcal{L}_2$ gain of the $\tilde{v}, \tilde{z}$ subsystem, then $\gamma_1 < \gamma_2^{-1}$ ensures stability of the composite system [14]. Note that γ_2 depends on the damping coefficients and on the incidence matrix D, but does not depend on the specifics of the formation geometry. We now present a bound on γ_2 in the case when the damping coefficients are identical for all agents:

Theorem 3. *Assume the agents have uniform velocity damping (i.e. $k := k_1 = \ldots = k_n$). Let $\mu_1, \ldots, \mu_{n-1}$ be the positive eigenvalues of the unweighted sensing graph Laplacian DD^T. Let*

$$\rho_i = \begin{cases} 1/\sqrt{\mu_i} & \text{if } k \geq \sqrt{2\mu_i} \\ \frac{2\sqrt{\mu_i}}{k\sqrt{4\mu_i - k^2}} & \text{if } k < \sqrt{2\mu_i} \end{cases} \tag{39}$$

for $i = 1, \ldots, n-1$. The $\mathcal{L}_2$ gain of the $\tilde{v}, \tilde{z}$ subsystem, denoted by γ_2, satisfies

$$\gamma_2 \leq \max_i \{\rho_i\}. \tag{40}$$

Proof: Define the following:

$$A := \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} -K \otimes I_p & -D \otimes I_p \\ D^T \otimes I_p & 0_{pm \times pm} \end{bmatrix} \tag{41}$$

$$B := \begin{bmatrix} I_{n-1} \\ 0_{(m+1)\times(n-1)} \end{bmatrix} \otimes I_p \tag{42}$$

$$C := \begin{bmatrix} 0_{(n-1)\times n} & I_{n-1} & 0_{(n-1)\times(m-n+1)} \end{bmatrix} \otimes I_p. \tag{43}$$

We have that the transfer matrix from u to y of the $\tilde{v}, \tilde{z}$ subsystem is

$$G(s) = C\left(sI_{(n+m)p} - A\right)^{-1} B. \tag{44}$$

If we view $(sI_{(n+m)p} - A)^{-1}$ as a 2×2 block matrix, we see from the structure of B and C that $G(s)$ is a submatrix of the lower left block of $(sI_{(n+m)p} - A)^{-1}$. We can compute the lower left block using block matrix inversion techniques and obtain an equivalent expression for $G(s)$:

$$G(s) = \begin{bmatrix} I_{p(n-1)} & 0_{p(n-1)\times p(m-n+1)} \end{bmatrix} F(s) \begin{bmatrix} I_{p(n-1)} \\ 0_{p \times p(n-1)} \end{bmatrix} \tag{45}$$

where

$$\begin{aligned} F(s) &= -((sI_{mp} - A_{22}) + A_{21}(sI_{np} - A_{11})^{-1})^{-1} \qquad (46) \\ &\quad \cdot (-A_{21})(sI_{np} - A_{11})^{-1} \\ &= \left[((s^2 + ks)I_m + D^T D)^{-1} D^T\right] \otimes I_p. \qquad (47) \end{aligned}$$

We are interested in the value of $||F(s)||_2$ evaluated along the $j\omega$-axis. First suppose that $s = j\omega$ where $\omega \neq 0$. Let $D = U\Sigma_1 V^T$ be a singular value decomposition of D with $U \in \mathbb{R}^{n \times n}, V \in \mathbb{R}^{m \times m}$ orthonormal, and $\Sigma_1 \in \mathbb{R}^{n \times m}$ diagonal. Note that the nonzero diagonal elements of Σ_1 are $\sqrt{\mu_i}$ for $i = 1, \ldots, n-1$. It follows that $(s^2 + ks)I_m + D^T D = V((s^2 + ks)I + \Sigma_1^T \Sigma_1)V^T$, and therefore we can write

$$((s^2 + ks)I_m + D^T D)^{-1} = V\Sigma_2 V^T \tag{48}$$

where Σ_2 is a diagonal matrix with the values $1/(s^2 + ks + \mu_i), i = 1, \ldots, n-1$ along the diagonal. Thus we see that for $s = j\omega$, $\omega \neq 0$,

$$F(s) = (V\Sigma_2 \Sigma_1^T U^T) \otimes I_p. \tag{49}$$

Since V and U are orthonormal, the nonzero singular values of $F(s)$ are the nonzero singular values of $\Sigma_2 \Sigma_1^T$, which are $\sqrt{\mu_i}/|s^2 + ks + \mu_i|, i = 1, \ldots, n-1$. In the case when $s = 0$, we have

$$\lim_{s \searrow 0} \left[((s^2 + ks)I_m + D^T D)^{-1} D^T\right] \otimes I_p = D^+ \otimes I_p \tag{50}$$

where $^+$ denotes the Moore-Penrose pseudoinverse, and the singular values of D^+ are $1/\sqrt{\mu_i}$ [15]. Thus the $p(n-1)$ nonzero singular values of $F(s)$ when $s = j\omega$, $\omega \in \mathbb{R}$ are the second order transfer functions

$$\sigma_i(s) = \frac{\sqrt{\mu_i}}{|s^2 + ks + \mu_i|}, \quad i = 1, \ldots, n-1, \tag{51}$$

each occurring with multiplicity p due to the Kronecker product with the identity in (47). We have thatsup$_\omega$ $\sigma_i(j\omega) = \rho_i$. We also have that $G(s)$ is a submatrix of $F(s)$. It follows that $||G(s)||_2 \leq ||F(s)||_2$ for all s, and hence $\gamma_2 \leq \max_i\{\rho_i\}$. ■

Corollary 2. *Assume the agents have uniform velocity damping k. Let μ_1 denote the smallest positive eigenvalue of DD^T, known as the Fiedler eigenvalue of the graph Laplacian. For sufficiently large k, we have*

$$\gamma_2 \leq \frac{1}{\sqrt{\mu_1}}. \tag{52}$$

Proof: Let μ_i be the nonzero eigenvalues of DD^T. If $k \geq \sqrt{2\mu_i}$ for all i, then $\rho_i = \frac{1}{\sqrt{\mu_i}}$ for all i, and the result follows. ■

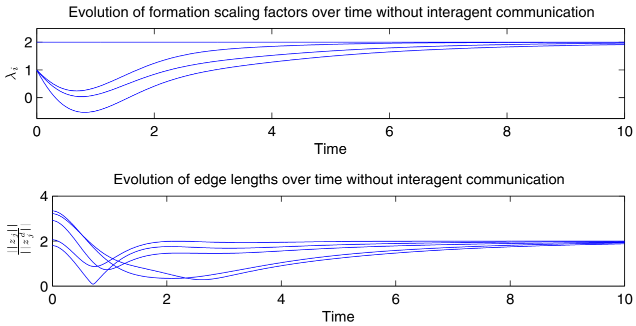

Fig. 3. Simulation results for a 4-agent, 5-edge formation with no interagent communication, random initial positions and a desired formation scaling factor of 2.

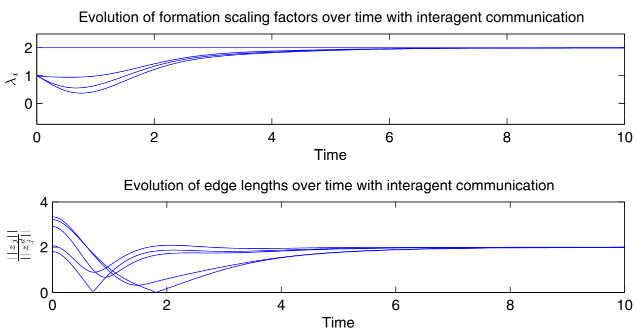

Fig. 4. Simulation results for a 4-agent, 5-edge formation with interagent communication, random initial positions and a desired formation scaling factor of 2.

V. SIMULATION RESULTS

Consider the formation in the plane depicted in Fig. 1(a) where each edge is assigned an arbitrary direction, D corresponds to the chosen node and edge numbering and direction assignment, agent 4 is the leader, and the monitoring subgraph is shown in Fig. 1(b). We let

$$z^d = \begin{bmatrix} -0.5 & -1 & -0.5 & 0.1 & 0.5 & -1 & 0.5 & 0.1 & 0 & 0.9 \end{bmatrix}^T, \tag{53}$$

which is consistent with the shape depicted in Fig. 1(a). We have that $\mu_1 = 2$ is the smallest positive eigenvalue of DD^T and $\mu_3 = 4$ is the largest eigenvalue of DD^T. Suppose $k := k_1 = k_2 = k_3 = k_4$. If $k \geq \sqrt{8}$, then, by Corollary 2,

$$\gamma_2 \leq \frac{1}{\sqrt{2}}. \tag{54}$$

Using the geometric condition presented in Theorem 2 we have that $\gamma_1 = 1.33$. Thus

$$\gamma_1\gamma_2 = \frac{1}{\sqrt{2}}1.33 = 0.94 < 1 \tag{55}$$

and therefore the system with no interagent communication is stable. Fig. 3 shows simulation results in the case of no interagent communication with $\lambda^* = 2$, $\lambda_i(0) = 1$ for $i = 1, \ldots, n-1$, and the agents' initial positions initialized randomly. Fig. 4 shows simulation results for the same formation and desired formation size, but interagent communication is allowed and the communication graph is equivalent to the unweighted sensing graph with edges pointing towards the leader agent removed. The plots demonstrate that the control strategy with communication converges to the desired formation more quickly, yet the improvement is modest.

VI. CONCLUSIONS

In this paper, we introduce increased flexibility and adaptivity to the standard formation maintenance problem by presenting cooperative control strategies that allow a multi-agent team to dynamically alter its formation size. We have presented a strategy that relies on interagent communication and produces a stable cooperative control system given any desired connected formation, and we have presented a strategy that relies only on relative position information with stability results that are dependent on the desired formation and velocity damping. In the latter case, we have introduced a simple geometric criterion which may be used with the small gain theorem to determine stability. In both cases, these strategies can be combined with other objectives such as collision avoidance using standard algorithms.

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