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Periodic Event-Triggered Control Based on State Feedback

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***Abstract*— In this paper, a novel event-triggered control (ETC) strategy is proposed by striking a balance between periodic sampled-data control and ETC. This leads to so-called periodic event-triggered control (PETC), in which the advantage of reduced resource utilisation is preserved on the one hand, while, on the other hand, the conditions that trigger the events still have a periodic character. The latter aspect has the advantage that the event-triggering condition has to be verified only at the periodic sampling times, instead of continuously, as in conventional ETC. To analyse the stability and the $\mathcal{L}_2$-gain properties of the resulting PETC systems, two different approaches will be presented based on (i) piecewise linear systems, and (ii) impulsive systems, respectively. Moreover, the advantages and disadvantages of each of the methods will be highlighted. The developed theory will be illustrated using a numerical example.**

I. Introduction

In many control applications nowadays, the controller is implemented on a digital platform. In such a digital implementation, the control task consists of sampling the outputs of the plant and computing and implementing new actuator signals. Typically, the control task is executed periodically, since this allows the closed-loop system to be analysed and the controller to be designed using the well-developed theory on sampled-data systems. Although periodic sampling is preferred from an analysis and design point of view, it is sometimes less preferable from a resource utlisation point of view. Namely, executing the control task at times when no disturbances are acting on the system and the system is operating desirably is clearly a waste of computation resources. Moreover, in case the measured outputs and/or the actuator signals have to be transmitted over a shared (and possibly wireless) network, unnecessary utilisation of the network (or power consumption of the wireless radios) is introduced. To mitigate the unnecessary waste of computation and communication resources as in periodic control, an alternative control paradigm, namely event-triggered control (ETC), has been proposed at the end of the nineties [1]–[4]. ETC is a control strategy in which the control task is executed after the occurrence of an event, generated by some well-designed event-triggering condition, rather than the elapse of a certain fixed period of time, as in conventional periodic sampled-data control. In this way, ETC is capable of significantly reducing the number of control task executions, while retaining a satisfactory closed-loop performance, as many simulation and experimental results show.

Although the advantages of ETC are well-motivated and practical applications show its potential, relatively few theoretical results exist that study ETC systems, see, e.g., [5]–[15], in which several different ETC strategies are proposed. The main difference between the aforecited papers and the ETC strategy that will be proposed in this paper is that in the former the event-triggering condition has to be monitored continuously, while in the latter the event-triggering condition is verified only periodically, and every sampling time it is decided whether or not to transmit new measurements and control signals. Only when necessary from a stability or performance point of view, the communication or computation resources are used. As a consequence, this control strategy aims at striking a balance between periodic sampled-data and event-triggered control and therefore we will propose to use the term *periodic event-triggered control* (PETC) for this class of ETC, while we will use the term continuous event-triggered control (CETC) to indicate the existing approaches [6]–[15]. By mixing ideas from ETC and periodic sampled-data control, the benefits of reduced resource utilisation are preserved in PETC as transmissions and controller computations are *not* performed periodically, while the event-triggering conditions still have a periodic character. The latter aspect leads to several benefits, including a guaranteed minimum inter-event time of (at least) the sampling interval of the event-triggering condition. Furthermore, as already mentioned, the event-triggering condition has to be verified only at periodic sampling times, making PETC better suited for practical implementations as it can be implemented in more standard time-sliced embedded software architectures, while CETC requires dedicated analogue hardware to detect the events. Initial attempts towards what we refer here to as PETC were taken in [2], [5], [7], however only for restricted classes of systems and/or controllers (PID, static state feedback, or simple impulse controllers), and for particular event-triggering conditions without providing a general analysis framework.

We will therefore provide a general framework for the introduced class of PETC that allows to carry out stability and performance analyses. In fact, we will provide two different analysis approaches, namely: (i) a discrete-time piecewise linear (PWL) system approach, and (ii) an impulsive system approach. The former approach adopts PWL models and piecewise quadratic (PWQ) Lyapunov functions, which lead to LMI-based stability conditions for the PETC system. The latter approach uses impulsive systems [16], [17], which explicitly include the intersample behaviour. Based on this modelling paradigm, we are able to provide guarantees on performance in terms of $\mathcal{L}_2$-gains. Although the focus in this paper will be on state-feedback controllers, the provided framework can be extended towards output-based dynamic controllers and decentralised event-triggering conditions. These extensions, a third analysis approach, and

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the full proofs of all the results in this paper can be found in [18].

A. Nomenclature

For a vector $x \in \mathbb{R}^n$, we denote by $\|x\| := \sqrt{x^\top x}$ its 2-norm. For a symmetric matrix $A \in \mathbb{R}^{n\times n}$, $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote the maximum and minimum eigenvalue of A, respectively. For a matrix $A \in \mathbb{R}^{n\times m}$, we denote by $A^\top \in \mathbb{R}^{m\times n}$ the transposed of A, and by $\|A\| := \sqrt{\lambda_{\max}(A^\top A)}$ its induced 2-norm. We sometimes write symmetric matrices of the form $\begin{bmatrix} A & B \\ B^\top & C \end{bmatrix}$ as $\begin{bmatrix} A & B \\ \star & C \end{bmatrix}$ or $\begin{bmatrix} A & \star \\ B^\top & C \end{bmatrix}$. We call a matrix $P \in \mathbb{R}^{n\times n}$ positive definite and write $P \succ 0$, if P is symmetric and $x^\top P x > 0$ for all $x \neq 0$. Similarly, we use $P \succeq 0$, $P \prec 0$ and $P \preceq$ to denote that P is positive semidefinite, negative definite and negative semidefinite, respectively. For a locally integrable signal $w : \mathbb{R}_+ \to \mathbb{R}^n$, where $\mathbb{R}_+$ denotes the set of nonnegative real numbers, we denote by $\|x\|_{\mathcal{L}_2} = (\int_0^\infty \|x(t)\|^2 \mathrm{d}t)^{1/2}$ its $\mathcal{L}_2$-norm, provided the integral is finite. Furthermore, we define the set of all locally integrable signals with a finite $\mathcal{L}_2$-norm as $\mathcal{L}_2$. For a signal $w : \mathbb{R}_+ \to \mathbb{R}^n$, we denote the limit from above at time $t \in \mathbb{R}_+$ by $w^+(t) = \lim_{s\downarrow t} w(s)$.

II. Periodic Event-Triggered Control

In this section, we introduce periodic event-triggered control (PETC) and give a precise formulation of the stability and performance analysis problems.

A. The Periodic Event-Triggered Control System

To introduce PETC, let us consider a linear time-invariant (LTI) plant, given by

$$\tfrac{\mathrm{d}}{\mathrm{d}t} x = A^p x + B^p \hat{u} + B^w w, \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ denotes the state of the plant, $\hat{u} \in \mathbb{R}^{n_u}$ is the input applied to the plant, and $w \in \mathbb{R}^{n_w}$ is an unknown disturbance. In a conventional sampled-data state-feedback setting, the plant is controlled using a controller

$$\hat{u}(t) = Kx(t_k), \quad \text{for} \quad t \in (t_k, t_{k+1}], \tag{2}$$

where t_k, $k \in \mathbb{N}$, are the sampling times, which are periodic in the sense that $t_k = kh$, $k \in \mathbb{N}$, for some properly chosen sampling interval $h > 0$.

Instead of using conventional periodic sampled-data control, we propose here to use PETC meaning that at each sampling time $t_k = kh$, $k \in \mathbb{N}$, state measurements are transmitted over a communication network and the control values are updated only when necessary from a stability or performance point of view. This modifies the controller from (2) to

$$\hat{u}(t) = K\hat{x}(t), \quad \text{for} \quad t \in \mathbb{R}_+, \tag{3}$$

where $\hat{x}$ is a left-continuous signal[1], given for $t \in (t_k, t_{k+1}]$, $k \in \mathbb{N}$, by

$$\hat{x}(t) = \begin{cases} x(t_k), & \text{when } \mathcal{C}(x(t_k), \hat{x}(t_k)) > 0 \\ \hat{x}(t_k), & \text{when } \mathcal{C}(x(t_k), \hat{x}(t_k)) \leqslant 0 \end{cases} \tag{4}$$

[1]A signal $x : \mathbb{R}_+ \to \mathbb{R}^n$ is called left-continuous, if for all $t > 0$, $\lim_{s\uparrow t} x(s) = x(t)$.

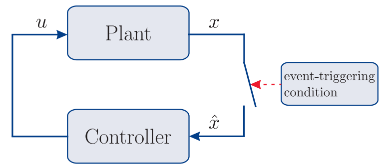


Fig. 1: Event-triggered control schematic.

and some initial value for $\hat{x}(0)$. Hence, considering the configuration in Fig. 1, the value $\hat{x}(t)$ can be interpreted as the most recently transmitted measurement of the state x to the controller at time t. Whether or not new state measurements are transmitted to the controller is determined by the event-triggering condition $\mathcal{C} : \mathbb{R}^{n_\xi} \to \mathbb{R}$ with $n_\xi := 2n_x$. In particular, if at time t_k it holds that $\mathcal{C}(x(t_k), \hat{x}(t_k)) > 0$, the state $x(t_k)$ is transmitted over the network to the controller and $\hat{x}$ and the control value $\hat{u}$ are updated accordingly at time t_k. In case $\mathcal{C}(x(t_k), \hat{x}(t_k)) \leqslant 0$, no new state information is sent to the controller, in which case the input $\hat{u}$ is not updated and kept the same for (at least) another sampling interval implying that no control computations are needed and no new state measurements and control values have to be transmitted.

B. Quadratic Event-Triggering Conditions

In this paper, we focus on *quadratic* event-triggering conditions, i.e., the function $\mathcal{C}$, as in (4), is given by

$$\mathcal{C}(\xi(t_k)) = \xi^\top(t_k) Q \xi(t_k) > 0, \tag{5}$$

where $\xi := [x^\top \ \hat{x}^\top]^\top \in \mathbb{R}^{n_\xi}$, for some symmetric matrix $Q \in \mathbb{R}^{n_\xi \times n_\xi}$. To show that these event-triggering conditions form a relevant class, we will review some existing event-triggering conditions that have been applied in the context of *continuous* event-triggered control (CETC), and show how they can be written as quadratic event-triggering conditions for PETC as in (5).

1) Event-Triggering Conditions Based on the State Error: An important class of event-triggering conditions, which has been applied to CETC in [10], [11], are given by

$$\|\hat{x}(t_k) - x(t_k)\| > \sigma \|x(t_k)\|, \tag{6}$$

for $k \in \mathbb{N}$, where $\sigma > 0$. Clearly, (6) is of the form (5) with

$$Q = \begin{bmatrix} (1-\sigma^2)I & -I \\ -I & I \end{bmatrix}. \tag{7}$$

2) Event-Triggering Conditions Based on the Input Error: In [15], where the objective was to develop *output-based* CETC, an event-triggering condition was proposed that would translate for state-feedback-based PETC systems to

$$\|K\hat{x}(t_k) - Kx(t_k)\| > \sigma \|Kx(t_k)\|, \tag{8}$$

where $\sigma > 0$. Condition (8) is equivalent to $\|\hat{u}(t_k) - u(t_k)\| > \sigma \|u(t_k)\|$ in which $u(t_k) = Kx(t_k)$ is the control value determined on the basis of $x(t_k)$ as in standard periodic state-feedback (see (2)). The event-triggering condition (8) is equivalent to (5), in which

$$Q = \begin{bmatrix} (1-\sigma^2)K^\top K & -K^\top K \\ -K^\top K & K^\top K \end{bmatrix}. \tag{9}$$

3) Event-Triggering Conditions as in [19]: A PETC version of the condition used in [19] is

$$\|\hat{u}(t_k) - u(t_k)\|^2 > (1-\beta^2)\|x(t_k)\|^2 + \|\hat{u}(t_k)\|^2, \quad (10)$$

where $0 < \beta \leqslant 1$ and, again, $u(t_k) = Kx(t_k)$, which results in an event-triggering condition (5) with $Q = \begin{bmatrix} (\beta^2-1)I + K^\top K & -K^\top K \\ -K^\top K & 0 \end{bmatrix}$, as $\hat{u}(t_k) = K\hat{x}(t_k)$, $k \in \mathbb{N}$.

Remark II.1 In [20], [21], in the context of CETC, and in [22], in the context of self-triggered control [23], Lyapunov-based event-triggering conditions have been proposed. Using quadratic Lyapunov functions, one can extend these ideas also towards PETC leading also to quadratic event-triggering conditions (5).

The examples show the relevance of the class of *quadratic* event-triggering conditions (5), as their CETC counterparts have been considered in the literature extensively.

C. Problem Formulation

To obtain a complete model of the PETC system, we combine (1), (3), (4) and (5), and define $\xi := [x^\top \ \hat{x}^\top]^\top$,

$$\bar{A} := \begin{bmatrix} A^p & B^pK \\ 0 & 0 \end{bmatrix},\ \bar{B} := \begin{bmatrix} B^w \\ 0 \end{bmatrix},\ J_1 := \begin{bmatrix} I & 0 \\ I & 0 \end{bmatrix},\ J_2 := \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \quad (11)$$

to arrive at an impulsive system [16], [17] given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \xi \\ \tau \end{bmatrix} = \begin{bmatrix} \bar{A}\xi + \bar{B}w \\ 1 \end{bmatrix}, \ \text{when } \tau \in [0,h], \quad (12a)$$

$$\begin{bmatrix} \xi^+ \\ \tau^+ \end{bmatrix} = \begin{cases} \begin{bmatrix} J_1\xi \\ 0 \end{bmatrix}, & \text{when } \xi^\top Q\xi > 0,\ \tau = h \\ \begin{bmatrix} J_2\xi \\ 0 \end{bmatrix}, & \text{when } \xi^\top Q\xi \leqslant 0,\ \tau = h \end{cases} \quad (12b)$$

$$z = \bar{C}\xi + \bar{D}w, \quad (12c)$$

where $z \in \mathbb{R}^{n_z}$ is a performance output with $\bar{C}$ and $\bar{D}$ appropriately chosen matrices, and the state τ keeps track of the time elapsed since the last sampling time.

Besides the introduction of PETC, the main objective of this paper is to analyse and design event-triggering conditions of the form (5) such that the corresponding closed-loop system (1), (3), (4) and (5) is stable and has a certain closed-loop performance, both defined in an appropriate sense, while the number of transmissions between the plant and the controller is minimised. To make precise what we mean by stability and performance, let us define the notion of global exponential stability and $\mathcal{L}_2$-performance.

Definition II.2 The PETC system (1), (3) (4) and (5) is said to be globally exponentially stable (GES), if there exist $c > 0$ and $\rho > 0$ such that for all solutions to the impulsive system (12) with $\tau(0) \in [0,h]$ and $w = 0$, it holds that $\|\xi(t)\| \leqslant ce^{-\rho t}\|\xi(0)\|$ for all $t \in \mathbb{R}_+$. In this case, we call ρ an (upper bound on the) decay rate.

Definition II.3 The PETC system (1), (3) (4) and (5), with (12c), is said to have an $\mathcal{L}_2$-gain from w to z smaller than or equal to γ, if there is a function $\beta : \mathbb{R}^{n_\xi} \to \mathbb{R}_+$ such that for any $w \in \mathcal{L}_2$, any initial state $\xi(0) = \xi_0 \in \mathbb{R}^{n_\xi}$ and $\tau(0) \in [0,h]$, the corresponding solution satisfies

$$\|z\|_{\mathcal{L}_2} \leqslant \beta(\xi_0) + \gamma\|w\|_{\mathcal{L}_2}. \quad (13)$$

III. Stability and $\mathcal{L}_2$-Gain Analysis

In this section, we analyse stability and performance of the PETC system given by (1), (3), (4), (5) and (12c) using two different approaches, namely: (i) a discrete-time piecewise linear (PWL) system approach, and (ii) an impulsive system approach. In particular, in the former approach we will focus on GES only and, thus, take $w = 0$, while in the latter we also include an $\mathcal{L}_2$-gain analysis.

A. A Piecewise Linear System Approach

To obtain a discrete-time PWL model, we discretise the impulsive system (12), with $\tau(0) = h$ and $w = 0$, at the sampling times $t_k = kh$, $k \in \mathbb{N}$. By defining the state variable $\xi_k := \xi(t_k)$ (and assuming ξ to be left-continuous), we obtain the bimodal PWL model

$$\xi_{k+1} = \begin{cases} A_1\xi_k, & \text{when } \xi_k^\top Q\xi_k > 0, \\ A_2\xi_k, & \text{when } \xi_k^\top Q\xi_k \leqslant 0, \ \text{where} \end{cases} \quad (14)$$

$$A_1 = e^{\bar{A}h}J_1 = \begin{bmatrix} A+BK & 0 \\ I & 0 \end{bmatrix},\ A_2 = e^{\bar{A}h}J_2 = \begin{bmatrix} A & BK \\ 0 & I \end{bmatrix},$$
$$A := e^{A^ph} \text{ and } B := \int_0^h e^{A^ps}B^p. \quad (15)$$

Using the PWL model (14) and a piecewise quadratic (PWQ) Lyapunov function of the form

$$V(\xi) = \begin{cases} \xi^\top P_1\xi, & \text{when } \xi^\top Q\xi > 0, \\ \xi^\top P_2\xi, & \text{when } \xi^\top Q\xi \leqslant 0, \end{cases} \quad (16)$$

we can guarantee GES of the PETC system.

Theorem III.1 *The PETC system given by* (1), (3) (4) *and* (5) *is GES with decay rate ρ, if there exist matrices P_1, P_2 and scalars $\alpha_{ij} \geqslant 0$, $\beta_{ij} \geqslant 0$ and $\kappa_i \geqslant 0$, $i,j \in \{1,2\}$, satisfying for all $i,j \in \{1,2\}$,*

$$e^{-2\rho h}P_i - A_i^\top P_jA_i + (-1)^i\alpha_{ij}Q + (-1)^j\beta_{ij}A_i^\top QA_i \succeq 0, \quad (17a)$$
$$P_i + (-1)^i\kappa_iQ \succ 0. \quad (17b)$$

B. An Impulsive System Approach

In this section, we will directly apply stability and performance analysis techniques to the impulsive system (12). The analysis is based on a Lyapunov/storage function of the form

$$V(\xi,\tau) = \xi^\top P(\tau)\xi, \quad (18)$$

for $\xi \in \mathbb{R}^{n_\xi}$ and $\tau \in [0,h]$, where $P : [0,h] \to \mathbb{R}^{n_\xi \times n_\xi}$ with $P(\tau) \succ 0$, for $\tau \in [0,h]$. The choice of Lyapunov function is inspired by the developments in [16], [24]. The function $P : [0,h] \to \mathbb{R}^{n_\xi \times n_\xi}$ will be chosen such that it becomes a candidate storage function for the system (12) with the supply rate $\gamma^{-2}z^\top z - w^\top w$. In particular, we will select the matrix function P to satisfy the Riccati differential equation

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}\tau}P = -\bar{A}^\top P - P\bar{A} - 2\rho P - \gamma^{-2}\bar{C}^\top\bar{C} - \\ (P\bar{B} + \gamma^{-2}\bar{C}^\top\bar{D})M(\bar{B}^\top P + \gamma^{-2}\bar{D}^\top\bar{C}),\end{aligned} \quad (19)$$

provided the solution exists on $[0, h]$ for a desired convergence rate $\rho > 0$, in which $M := (I - \gamma^{-2}\bar{D}^\top\bar{D})^{-1}$ is assumed to exist and to be positive definite, which means that $\gamma^2 > \lambda_{\max}(\bar{D}^\top\bar{D})$. This choice for the matrix function P yields

$$\tfrac{\mathrm{d}}{\mathrm{d}t}V \leqslant -2\rho V - \gamma^{-2}z^\top z + w^\top w, \quad (20)$$

during the flow (12a). Combining inequality (20) with the conditions

$$V(J_1\xi, 0) \leqslant V(\xi, h), \text{ for all } \xi \text{ with } \xi^\top Q\xi > 0, \quad (21a)$$

$$V(J_2\xi, 0) \leqslant V(\xi, h), \text{ for all } \xi \text{ with } \xi^\top Q\xi \leqslant 0, \quad (21b)$$

which imply that the storage function does not increase during the jumps (12b) of the impulsive system (12), we can guarantee that the $\mathcal{L}_2$-gain from w to z is smaller than or equal to γ, see, e.g., [25]. The result that we present below, is based on verifying the satisfaction of (21) by relating $P_0 := P(0)$ to $P_h := P(h)$. To do so, we introduce the Hamiltonian matrix

$$H := \begin{bmatrix} \bar{A} + \rho I + \gamma^{-2}\bar{B}M\bar{D}^\top\bar{C} & \bar{B}M\bar{B}^\top \\ -\bar{C}^\top L\bar{C} & -(\bar{A} + \rho I + \gamma^{-2}\bar{B}M\bar{D}^\top\bar{C})^\top \end{bmatrix}, \quad (22)$$

with $L := (\gamma^2 I - \bar{D}\bar{D}^\top)^{-1}$, which is positive definite again if $\gamma^2 > \lambda_{\max}(\bar{D}^\top\bar{D}) = \lambda_{\max}(\bar{D}\bar{D}^\top)$, and the matrix exponential

$$F(\tau) := e^{-H\tau} = \begin{bmatrix} F_{11}(\tau) & F_{12}(\tau) \\ F_{21}(\tau) & F_{22}(\tau) \end{bmatrix}, \quad (23)$$

allowing us to provide the explicit solution to the Riccati differential equation (19), yielding

$$P_0 = (F_{21}(h) + F_{22}(h)P_h)\big(F_{11}(h) + F_{12}(h)P_h\big)^{-1}, \quad (24)$$

provided that the solution (24) is well defined on $[0, h]$. To guarantee this, we will use the following assumption.

Assumption III.2 $F_{11}(\tau)$ is invertible for all $\tau \in [0, h]$.

Before presenting the main result, observe that Assumption III.2 is always satisfied for sufficiently small h. Namely, $F(\tau) = e^{-H\tau}$ is a continuous function and we have that $F_{11}(0) = I$. Let us also introduce the notation $\bar{F}_{11} := F_{11}(h)$, $\bar{F}_{12} := F_{12}(h)$, $\bar{F}_{21} := F_{21}(h)$ and $\bar{F}_{22} := F_{22}(h)$, and a matrix $\bar{S}$ that satisfies $\bar{S}\bar{S}^\top := -\bar{F}_{11}^{-1}\bar{F}_{12}$. A matrix $\bar{S}$ exists under Assumption III.2, because this assumption will guarantee that the matrix $-\bar{F}_{11}^{-1}\bar{F}_{12}$ is positive semidefinite, see [18] for the details.

Theorem III.3 *Consider the impulsive system* (12) *and let $\rho > 0$, $\gamma > \sqrt{\lambda_{max}(\bar{D}^\top\bar{D})}$, and Assumption III.2 hold. Suppose that there exist a matrix $P_h \succ 0$, and scalars $\mu_i \geqslant 0$, $i \in \{1, 2\}$, such that for $i \in \{1, 2\}$*

$$\begin{bmatrix} P_h + (-1)^i\mu_i Q & J_i^\top\bar{F}_{11}^{-\top}P_h\bar{S} & J_i^\top(\bar{F}_{11}^{-\top}P_h\bar{F}_{11}^{-1} + \bar{F}_{21}\bar{F}_{11}^{-1}) \\ \star & I - \bar{S}^\top P_h\bar{S} & 0 \\ \star & \star & \bar{F}_{11}^{-\top}P_h\bar{F}_{11}^{-1} + \bar{F}_{21}\bar{F}_{11}^{-1} \end{bmatrix} \succ 0 \quad (25)$$

hold. Then, the PETC system given by (1), (3) (4) *and* (5) *is GES with convergence rate ρ (when $w = 0$) and has an $\mathcal{L}_2$-gain from w to z smaller than or equal to γ.*

The results of Theorem III.3 guarantee both GES (for $w = 0$) and an upper bound on the $\mathcal{L}_2$-gain. In case disturbances are absent (i.e., $w = 0$), the conditions of Theorem III.3 simplify and GES can be guaranteed using the following corollary.

Corollary III.4 *Consider the impulsive system* (12) *and let $\rho > 0$ be given. Assume there exist a matrix $P_h \succ 0$ and scalars $\mu_i \geqslant 0$, $i \in \{1, 2\}$, such that*

$$\begin{bmatrix} e^{-2\rho h}P_h + (-1)^i\mu_i Q & J_i^\top e^{\bar{A}^\top h}P_h \\ \star & P_h \end{bmatrix} \succ 0, \; i \in \{1, 2\}. \quad (26)$$

Then, the PETC system given by (1), (3) (4) *and* (5) *is GES (for $w = 0$) with decay rate ρ.*

IV. Comparison of the Approaches

When comparing the two analysis approaches, several observations can be made. The first observation is that the impulsive system approach is the only approach of the two that, at present, allows the $\mathcal{L}_2$-gain from w to z to be studied, which makes this approach important for PETC. The second observation is that the PWL system approach never yields more conservative results than the impulsive system approach.

To formally prove this statement, we substitute (15) into (26), and apply a Schur complement to (26), yielding that $e^{-2\rho h}P_h + (-1)^i\mu_i Q - A_i^\top P_h A_i \succ 0$, $i \in \{1, 2\}$ and $P_h \succ 0$ and $\mu_i \geqslant 0$, $i \in \{1, 2\}$. As these conditions are equivalent to the LMIs (17a), with $P_1 = P_2 = P_h$, $\alpha_{ij} = \mu_i$ and $\beta_{ij} = 0$, $i, j \in \{1, 2\}$, this shows that if the LMIs (26) are feasible, then the LMIs (17a) are feasible. In addition, since $P_h \succ 0$ the LMIs (17b) hold with $\kappa_1 = \kappa_2 = 0$. Hence, we have proven the following result.

Theorem IV.1 *In case the impulsive system approach in Section III-B guarantees GES with convergence rate ρ of the PETC system, given by* (1), (3), (4) *and* (5), *using Corollary III.4, then the PWL system approach of Section III-A, using Theorem III.1, proves GES with convergence rate ρ of the PETC system as well. In other words, for given $\rho > 0$ satisfaction of* (26) *for some $P_h \succ 0$, $\mu_1 \geqslant 0$ and $\mu_2 \geqslant 0$ implies satisfaction of* (17a) *and* (17b), *for some P_1, P_2 and constants $\alpha_{ij} \geqslant 0$, $\beta_{ij} \geqslant 0$, and κ_i, $i, j \in \{1, 2\}$.*

V. Minimum Inter-Event Times

Due to the periodic sampled-data nature of PETC, the sampling interval h is always a lower bound on the time difference between two consecutive updates of the control signal in the PETC system (1), (3), (4) and (5). In fact, this is one of the main advantages of PETC over CETC. The largest lower bound on the time differences between two consecutive control updates is called the *minimum inter-event time*, which might actually be larger than h. Below we will outline how the exact minimum inter-event time can be computed, where, for ease of exposition, we restrict ourselves to the case $w = 0$. Using bounds on the disturbances, one can also obtain lower bounds on the minimum inter-event time for the case with disturbances, i.e., $w \neq 0$, by applying similar reasoning as in [15].

Let us consider the PWL model (14). Now given that the current state $\xi_{\tilde{k}} = \xi$ and the assumption that, at time $t_{\tilde{k}}$, $\tilde{k} \in \mathbb{N}$, an update of the control signal has occurred, the next control update time is given by $t_{\tilde{k}} + ht(\xi)$, where

$$t(\xi) := \inf\left\{ l \in \mathbb{N}_{\geqslant 1} \,\middle|\, \xi^\top (A_2^{l-1}A_1)^\top Q A_2^{l-1} A_1 \xi > 0 \right\}. \tag{27}$$

The expression in (27) follows from the facts that the control signal is updated when $\xi_{\tilde{k}+l}^\top Q \xi_{\tilde{k}+l} > 0$, and as long as there is no update of the control signal $\xi_{\tilde{k}+l} = A_2^{l-1}A_1\xi_{\tilde{k}}$. Based on (27), it is now immediate that the minimum inter-event time for the PETC system (1), (3) and (4) with event-triggering condition (5) is given by $t^r_{\min,h} = ht^*_{\min,h} \geqslant h$, with $t^*_{\min,h} := \inf\{t(\xi) \,|\, \xi \in \mathbb{R}^{n_\xi}\}$ and h is the sampling interval of the event-triggering condition. Interestingly, $t^*_{\min,h}$ can equivalently be characterised as the computationally friendly expression

$$t^*_{\min,h} = \inf\{ l \in \mathbb{N}_{\geqslant 1} \,|\, \lambda_{\max}((A_2^{l-1}A_1)^\top Q A_2^{l-1}A_1) > 0\}. \tag{28}$$

VI. Illustrative Example

Let us consider the example taken from [11] with plant (1) given by

$$\tfrac{\mathrm{d}}{\mathrm{d}t}x = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w, \tag{29}$$

and state-feedback controller (3), where we take $K = [1\ -4]$ and $t_k = kh$, $k \in \mathbb{N}$, with sampling interval $h = 0.05$. In this example, we first consider the situation where the event-triggering condition is given by (6) and, later, by (8). For this PETC system, we will apply all the two developed approaches for stability analysis (for $w = 0$), and the impulsive system approach for performance analysis. For all methods, we aim at constructing the largest value of σ in (6) and (8) such that GES or a certain $\mathcal{L}_2$-gain can be guaranteed. The reason for striving for large values of σ is that then large (minimum) inter-event times are obtained, due to the forms of (6) and (8).

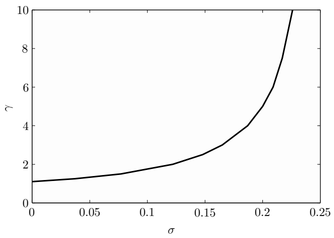
For the case that the event-triggering condition is given by (6), the PWL system approach using Theorem III.1 yields the maximum value of σ, while still guaranteeing GES, equal to $\sigma_{\mathrm{PWL}} := 0.2425$. Using the impulsive system approach (Corollary III.4) we obtain $\sigma_{\mathrm{IS}} = 0.2425$. Note that $\sigma_{\mathrm{IS}} \leqslant \sigma_{\mathrm{PWL}}$ (note that the latter inequality holds with equality in this case) is in accordance with Theorem IV.1. Obviously, for these values of σ a lower bound on the minimum inter-event time of $h = 0.05$ is guaranteed. However, in absence of disturbances we can use the expression in (28) to obtain the exact minimum inter-event times for these two cases, which result for $\sigma = \sigma_{\mathrm{PWL}} = \sigma_{\mathrm{IS}} = 0.2425$ in a minimum inter-event time of $3h = 0.15$.

Let us now consider the event-triggering condition given by (8). In this case, the PWL system approach (using Theorem III.1) yields a maximum value for σ of $\sigma_{\mathrm{PWL}} = 0.2550$, while still guaranteeing stability of the PETC system. The impulsive system approach results in the maximum $\sigma_{\mathrm{IS}} = 0.2532$ in this case. Hence, as expected, we again see that $\sigma_{\mathrm{IS}} \leqslant \sigma_{\mathrm{PWL}}$, although the values are rather close. In fact, the minimum inter-event time according to (28) is equal to $h = 0.05$ for both values σ_{PWL} and σ_{IS} in the event-triggering condition (8). When analysing the $\mathcal{L}_2$-gain from the disturbance w to the output variable z as in (12c) where $z = [0\ 1\ 0\ 0]\xi$, we obtain Fig. 2a, in which the smallest upper bound on the $\mathcal{L}_2$-gain that can be guaranteed on the basis of Theorem III.3 is given as function of σ. This figure clearly demonstrates that better control performance (i.e., smaller γ), necessitates more updates (i.e., smaller σ), allowing us to make tradeoffs between these two competing objectives. Note that for $\gamma \to \infty$ (meaning no performance requirements), the value of σ approaches the value obtained using Corollary III.4 equal to $\sigma_{\mathrm{IS}} = 0.2532$. On the other hand, for $\sigma \to 0$, we recover an upper bound on the $\mathcal{L}_2$-gain for the periodic sampled-data system, given by (1) of the controller (2) with sampling interval $h = 0.05$ and $t_k = kh$, $k \in \mathbb{N}$.

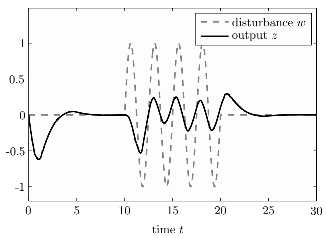
Fig. 2b shows the response of the performance output z of the PETC system with $\sigma = 0.2$ subject to a disturbance w, which is also depicted in Fig. 2b. For the same situation, Fig. 2c shows the evolution of the inter-event times. We see inter-event times ranging from $h = 0.05$, up to 0.85 (17 times the sampling interval h). Hence, this figure illustrates that using PETC instead of periodic sampled-data control, a significant reduction in the number of transmissions/controller computations can be achieved.

VII. Conclusions

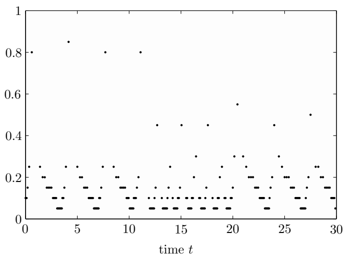
In this paper, we proposed a novel event-triggered control (ETC) strategy, which aims at combining the benefits that periodic sampled-data control and ETC offer. In particular, the ETC strategy is based on the idea of having an event-triggering condition that is verified only periodically, and at every time it is decided whether or not to transmit new measurements and control signals. Only when necessary from a stability or performance point of view, the communication or computation resources are used. This control strategy, for which we coined the term *periodic event-triggered control* (PETC), preserves the benefits of reduced resource utilisation as transmissions and controller computations are not



(a) $\mathcal{L}_2$-gain versus event-triggering condition.



(b) The evolution of the disturbances and the output z as a function of time



(c) The inter-event times as a function of time.

Fig. 2: Example 1, with $\sigma = 0.2$.

performed periodically, while the event-triggering condition still has a periodic character. The latter aspect leads to several benefits as the event-triggering condition has to be verified only at the periodic sampling times, instead of continuously, which makes it suitable for implementation in standard time-sliced embedded system architectures. Moreover, the strategy has an inherently guaranteed minimum inter-event time of (at least) one sampling interval of the event-triggering condition.

In this paper, PETC was developed focusing on static state-feedback controllers, although extensions towards dynamical output-based controllers and decentralised event-triggering conditions can be obtained as well, see [18]. To analyse the stability and $\mathcal{L}_2$-gain properties of the PETC systems, we used two approaches: (i) a discrete-time piecewise linear (PWL) system approach, and (ii) an impulsive system approach. The PWL system approach provides the least conservative LMI-based results in case of stability analysis only, while the impulsive system approach provides a direct $\mathcal{L}_2$-gain analysis of the system. Besides presenting the two analysis methodologies, we also provided techniques to compute (tight) lower bounds on the minimum inter-event times. We illustrated the theory using a numerical example and showed that PETC is able to reduce the utilisation of communication and computation resources significantly.

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