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An Extension of the Invariance Principle for Switched Nonlinear Systems

Michele C. Valentino, Vilma A. Oliveira, Luis F. C. Alberto, Douglas A. Sant'Anna

***Abstract*— In this paper, we propose an extension of the invariance principle for switched systems under dwell-time switched solutions. Our approach allows the derivative of an auxiliary function V along the solutions of the switched system to be positive on some bounded sets. The auxiliary function V, which plays the role of a Lyapunov function, is called a Lyapunov-like function in this paper. Our results are useful to estimate attractors of switched systems and basins of attraction. Results for a common Lyapunov-like function and multiple Lyapunov-like functions are given. Illustrative examples show the potential of the theoretical results in providing concrete information on the asymptotic behavior of nonlinear dynamical switched systems.**

I. INTRODUCTION

There has been an increasing interest in studying dynamics of switched systems. Switched systems arise in practice when modeling the operation of many engineering systems [1]–[4]. The existence of a common Lyapunov function is a sufficient condition for asymptotic stability of equilibrium of switched systems under arbitrary switching [5]. However, a common Lyapunov function may be difficult to find or may not exist. To overcome this difficulty, a multiple Lyapunov function approach has been considered (see for example [6]). On the other hand, the attractor of many switched systems is not an equilibrium. A classical example is the on-off control of temperature. For this class of problems we are not interested in studying the stability of a particular equilibrium but the asymptotic behavior of solutions.

LaSalle's invariance principle [7] is useful to analyze the asymptotic behavior of dynamical system solutions. Various extensions of LaSalle's invariance principle have been proposed. An extension of LaSalle's invariance principle, which allows the derivative of a Lyapunov-like function to be positive on some bounded sets, was proposed for continuous systems in [8], for discrete systems in [9] and for delayed systems in [10]. These results were applied to the problem of synchronization [8] and to obtain estimates of attractors of uncertain dynamical systems [11].

Various invariance principles for switched systems have been proposed [12], [13], [14], [15]. A version of the invariance principle is given in [13] for dwell-time switched systems composed of a finite number of continuous nonlinear vector fields.

We propose an extension of the results of [8] for switched systems that is also an extension of the invariance principle for switched systems given in [13]. The advantage of this extension is that the derivative of the Lyapunov-like function along the solutions of the switched system can be positive on some bounded sets. Also, the assumptions on the Lyapunov-like function along the solution are less restrictive than those in [13].

II. PRELIMINARIES

In this paper, we consider the following class of continuous-time switched systems

$$\dot{x}(t) = f_{\sigma(t)}(x(t)), \quad x(0) = x_0, \tag{1}$$

where $\sigma(t) : [0, \infty) \to \mathcal{P} = \{1, 2, ..., N\}$ is a piecewise constant function, continuous from the right, called a switching signal and $f_\sigma(x)$ is a smooth vector field of $\mathbb{R}^n$. Let $\{\tau_k\}$ be a sequence of consecutive switching times associated to σ and $I_p = \{t \in [\tau_k, \tau_{k+1}) : \sigma(\tau_k) = p\}$ be the union of intervals where system p is active. A continuous piecewise-smooth function $x_{\sigma(t)}(t) : [0, \infty) \to \mathbb{R}^n$ is a solution of the switched system (1) if $x_{\sigma(t)}(t)$ satisfies $\dot{x}_{\sigma(t)}(t) = f_p(x_{\sigma(t)}(t))$ for every $t \in I_p$ for all $p \in \mathcal{P}$. We assume that the switching sequence τ_k is divergent and, without loss of generality, that each system p is active infinite times. In other words, we assume for every $T > 0$ and $p \in \mathcal{P}$ the existence of a k such that $\sigma(\tau_k) = p$ and $\tau_k > T$. The set of all switched solutions will be denoted by $\mathcal{S}$. We denote $\varphi_{\sigma(t)}(t, x_0)$, or simply $\varphi(t, x_0)$, the solution of (1) starting at x_o at time $t = 0$ under the switching signal $\sigma(t)$.

The following definitions were taken from [13] (see also [16] and [5]).

Definition 1: The solution $\varphi(t, x_0) \in \mathcal{S}$ of (1) has a non vanishing dwell time if there exists $h > 0$ such that

$$\inf_k(\tau_{k+1} - \tau_k) \geq h. \tag{2}$$

where $\{\tau_k\}$ is the sequence of switching times associated with $\varphi(t, x_0)$. The number h is called a dwell time for $\varphi(t, x_0)$ and the set of all switched solutions possessing a nonvanishing dwell time is denoted by $\mathcal{S}_{dwell} \subset \mathcal{S}$.

Definition 2: Let U be an open subset of $\mathbb{R}^n$ containing the origin. We say that $V : U \to [0, +\infty)$ is a common weak Lyapunov function for (1) if it is a smooth function, positive definite, and the following holds

$$\nabla V(x) f_p(x) \leq 0 \tag{3}$$

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M. C. Valentino is with the Department of Electrical Engineering, University of São Paulo, michele.valentino@usp.br

V. A. Oliveira is with the Department of Electrical Engineering, University of São Paulo, vilma@sc.usp.br

L. F. C. Alberto is with the Department of Electrical Engineering, University of São Paulo, lfalberto@usp.br

D. A. Sant'Anna is with the Institute of Mathematical Sciences and Computation , University of São Paulo, douglass@icmc.usp.br

for each $x \in U$ and each $p \in \mathcal{P}$.

Definition 3: A compact set $\mathcal{M}$ is weakly invariant with respect to the switched system (1) if for each $x_0 \in \mathcal{M}$ there exist an index $p \in \mathcal{P}$, a solution $\varphi(t, x_0)$ of the vector field $f_p(x)$ and a real number $b > 0$ such that $\varphi(t, x_0) \in \mathcal{M}$ for either $t \in [-b, 0]$ or $t \in [0, b]$.

Definition 4: A switched solution $\varphi(t, x_0)$ of (1) is attracted to a compact set $\mathcal{M}$ if for each $\epsilon > 0$ there exists a time $T > 0$ such that for each $t \geq T$ one has

$$\varphi(t, x_0) \in B(\mathcal{M}, \epsilon) \tag{4}$$

where $B(\mathcal{M}, \epsilon) = \cup_{a \in \mathcal{M}} B(a, \epsilon)$. Clearly $\varphi(t, x_0)$ is attracted to $\mathcal{M}$ if and only if

$$\lim_{t \to \infty} dist(\varphi(t, x_0), \mathcal{M}) = 0. \tag{5}$$

Definition 5: Let $\varphi(t, x_0) : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous curve. A point p is a limit point of $\varphi(t, x_0)$ if there exists a sequence $\{t_k\}_{k \in N}$,with $t_k \to \infty$, as $k \to \infty$ such that $\lim_{k \to \infty} \varphi(t_k, x_0) = p$. The set of all limit points of $\varphi(t, x_0)$ will be denoted by $\omega^+(x_0)$. If $\varphi(t, x_0)$ is bounded, $\omega^+(x_0)$ is nonempty and compact (see [17]).

Proposition 1: Let $\varphi(t, x_0) \in \mathcal{S}_{dwell}$ be a bounded switched solution of (1) for $t \geq 0$. Then, $\omega^+(x_0)$ is weakly invariant.

Proof: See [13]. ■

The following results are the basis for the development of the extension of the invariance principle for switched systems proposed here. Next theorem is an invariance principle for switched systems.

Theorem 1: [13] Let $V : U \to [0, +\infty)$ be a weak common Lyapunov function for (1), with $f_p(0) = 0$. Let $\ell > 0$ and Ω_ℓ be a connected component of the level set $\{x \in U : V(x) \leq \ell\}$ such that $0 \in \Omega_\ell$. Suppose that Ω_ℓ is bounded and $E_p = \{x \in U : \nabla V(x) f_p(x(t)) = 0\}$. Finally, let $\mathcal{M}$ be the union of all the weakly invariant sets which are contained in $\cup E_p \cap \Omega_\ell$. Then every solution $\varphi(t, x) \in \mathcal{S}_{dwell}$ starting in Ω_ℓ is attracted to $\mathcal{M}$.

Next theorem is an extension of LaSalle's invariance principle for continuous systems, which allows the derivative of the Lyapunov-like function V be positive in some bounded sets.

Theorem 2: [8] Let $V : \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{C}^1$-function and $f : \mathbb{R}^n \to \mathbb{R}^n$ a smooth vector field. Let $Z = \{x \in \mathbb{R}^n : \nabla V(x) f(x) > 0\}$. Suppose $\ell = \sup_{x \in Z} V(x)$ and that $\Omega_\ell = \{x \in \mathbb{R}^n : V(x) \leq \ell\}$ is bounded. Let $\mathcal{M}$ be the largest invariant set contained in $\{x \in \mathbb{R}^n : \nabla V(x) f(x) = 0\} \cup \Omega_\ell$. Then, every solution, $\varphi(t, x_0)$, of $\dot{x} = f(x)$ that is bounded for $t \geq 0$ is attracted to the invariant set $\mathcal{M}$. Moreover, if $x_0 \in \Omega_\ell$, then $\varphi(t, x_0)$ is defined for every $t \geq 0$, $\varphi(t, x_0) \in \Omega_\ell$ for every $t \geq 0$ and $\varphi(t, x_0)$ is attracted to the largest invariant set contained in Ω_ℓ.

III. Main Results

In this section, extensions of the invariance principle for switched systems will be developed. The main feature of these extensions is that the derivative of the Lyapunov-like function V can be positive on bounded sets. By relaxing the signs of the derivatives of the Lyapunov-like function V on some bounded set, it becomes easier to find V satisfying the assumptions of these extensions. As a consequence, the asymptotic behavior of a larger class of switched dynamical systems can be studied with this theory.

Two extensions of the invariance principle are developed in this paper. One considers a common Lyapunov-like function for all subsystems and another considers multiple Lyapunov-like functions.

A. Common Lyapunov-Like Function

In this subsection, we consider the existence of a single Lyapunov-like function V for all subsystems of the switched system (1). Let $C_p = \{x \in \mathbb{R}^n : \nabla V(x) f_p(x) > 0\}$ be the set where the derivative of function V along trajectories of system p is positive and $E_p = \{x \in R^n : \nabla V(x) f_p(x) = 0\}$.

Theorem 3: Consider the switched system (1) and let $V(x) : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. Suppose that $\ell = \sup_{x \in \cup C_p} V(x) < \infty$ and $\Omega_\ell = \{x \in \mathbb{R}^n : V(x) \leq \ell\}$ is bounded. Finally, let $\mathcal{M}$ be the union of all weakly invariant sets that are contained in $\cup E_p \cup \Omega_\ell$. Then,

(i) If $x_0 \in \Omega_\ell$ and $\varphi(t, x_0) \in \mathcal{S}_{dwell}$, then $\varphi(t, x_o) \in \Omega_\ell$ for all $t \geq 0$ and $\varphi(t, x_o)$ is attracted to the largest weakly invariant set contained in Ω_ℓ.

(ii) Every bounded solution $\varphi(t, x_0) \in \mathcal{S}_{dwell}$ is attracted to $\mathcal{M}$.

Proof: (i): Let $x_0 \in \Omega_\ell$. Suppose there exists $\tau > 0$ such that $\varphi(\tau, x_0) \notin \Omega_\ell$. Then, there exist $\bar{\tau} \in (0, \tau)$ such that $V(\varphi(\bar{\tau}, x_0)) = \ell$ (by continuity of V and $\varphi(t, x_o)$) and $V(\varphi(t, x_0)) > \ell, \forall t \in (\bar{\tau}, \tau]$, but this is a contradiction, since $\nabla V(x) f_p(x(t)) \leq 0$ out of $\Omega_\ell \supset \cup C_p$, $\forall p \in \mathcal{P}$. Therefore, $\varphi(t, x_0) \in \Omega_\ell, \forall t \geq 0$. Hence $\omega^+(x_0) \neq \emptyset$ is compact and $\omega^+(x_0) \subset \Omega_\ell$. Moreover, $\varphi(t, x_0)$ tends to $\omega^+(x_0)$, as $t \to \infty$. Since $\omega^+(x_0)$ is a weakly invariant set, then the solution is attracted to the largest weakly invariant set of Ω_ℓ.

(ii): Let $x_0 \notin \Omega_\ell$. If $\varphi(t, x_0) \in \mathcal{S}_{dwell}$ enters Ω_ℓ at some t, then the result follows from the first part of this proof. Suppose the bounded solution $\varphi(t, x_0) \notin \Omega_\ell, \forall t \geq 0$. We have $V(\varphi(t, x_0)) > \ell$ and $\nabla V(\varphi(t, x_0)) f_p(\varphi(t, x_0)) \leq 0, \forall t \geq 0, \forall p \in \mathcal{P}$. Since $\ell = sup_{x \in \cup C_p} V(x)$, $\varphi(t, x_0) \notin \cup C_p \subset \Omega_\ell, \forall t \geq 0$. We conclude that $V(\varphi(t, x_0))$ is a bounded non-increasing function of t. Then, there exists $L \in \mathbb{R}$ such that $L = \lim_{t \to \infty} V(\varphi(t, x_0))$.

Since the solution is bounded, $\omega^+(x_0) \neq \emptyset$. Let $a \in \omega^+(x_0)$, then $\exists \{t_k\}$ with $t_k \to \infty$ as $k \to \infty$ such that $\varphi(t_k, x_0) \to a$. The continuity of V ensures that $V(\varphi(t_k, x_0)) \to V(a)$ as $k \to \infty$, then $V(a) = L$, $\forall a \in \omega^+(x_0)$.

By Proposition 1, $\omega^+(x_0)$ is a weakly invariant set, thus there exists an interval $[\alpha, \beta]$ containing the origin and a function $\upsilon(t)$ such that $\upsilon(0) = a$, $\upsilon(t) \in \omega^+(x_0), \forall t \in [\alpha, \beta]$ and $\exists j \in \mathcal{P}$ such that $\dot{\upsilon}(t) = f_j(\upsilon(t)), \forall t \in [\alpha, \beta]$. Then, $V(\upsilon(t)) = V(a) = L$, $\forall t \in [\alpha, \beta]$ and $\nabla V(\upsilon(t)) f_j(\upsilon(t)) = 0$ $\forall t \in [\alpha, \beta]$. Particularly, for $t = 0$ we have $\nabla V(\upsilon(0)) f_j(\upsilon(0)) = \nabla V(a) f_j(a) = 0$. Hence $a \in \cup E_p$. Then $\omega^+(x_0) \subset \cup E_p$. Since $\omega^+(x_0)$ is a weakly invariant set, the solution is attracted to the largest weakly invariant set $\mathcal{M}$ in $\cup E_p \cup \Omega_\ell$.

■

Example 1: Consider the switched system (1) with $\mathcal{P} = \{1, 2\}$ and

$$f_1(x) = \begin{bmatrix} x_2 \\ -x_1 + x_2(1 - x_1^2 - x_2^2) \end{bmatrix},$$

$$f_2(x) = \begin{bmatrix} x_2 \\ -x_1 - x_2 \end{bmatrix}.$$

Let $V(x) = (x_1^2 + x_2^2)/2$. Then $C_1 = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\} \setminus \{x : x_2 = 0\}$, $C_2 = \emptyset$, $E_1 = \{x_1^2 + x_2^2 = 1\} \cup \{x : x_2 = 0\}$ and $E_2 = \{x \in \mathbb{R}^2 : x_2 = 0\}$. Therefore, $\ell = \sup_{x \in \cup C_p} V(x) = 1/2$ and $\Omega_\ell = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$. Then, by Theorem 3, every bounded solution $\varphi(t, x_0) \in \mathcal{S}_{dwell}$ is attracted to the largest weakly invariant set of $\cup E_p \cup \Omega_\ell = \{(x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\} \cup \{x : x_2 = 0\}$.

Figure 1 illustrates the time-domain simulation for $x_0 = [1\ 1.2]'$ and $\tau_{k+1} = \tau_k + 1$, $k = 1, \cdots, 50$.

B. Multiple Lyapunov-Like Functions

In what follows, we consider the use of multiple Lyapunov-like functions, that is, we consider the existence of smooth functions $V_p : \mathbb{R}^n \to \mathbb{R}, \forall p \in \mathcal{P}$. Let $C = \{x \in \mathbb{R}^n : \exists p \in P \ \ such \ \ that \ \ \nabla V_p(x) f_p(x) > 0\}$ be the set where the derivative of function V_p along trajectories of system p is positive and $E = \{x \in \mathbb{R}^n : \exists p \in \mathcal{P} \ \ such \ \ that \ \ \nabla V_p(x) f_p(x) = 0\}$.

Assumption 1: There exist continuous functions $a, b : \mathbb{R}^n \to \mathbb{R}$ satisfying $a(x) \leq \inf_p V_p(x)$ and $b(x) \geq \sup_p V_p(x)$ for all $x \in \mathbb{R}^n$ [11].
Under Assumption 1, we define $\Omega_{\ell_j} = \{x \in \mathbb{R}^n : a(x) \leq \ell_j\}$ with $\ell_0 = \sup_{x \in C} b(x)$ and $\ell_j = \sup_{x \in \Omega_{\ell_{j-1}}} b(x)$, $j \in P$. It is clear by construction that

$$C \subseteq \Theta \subset \Omega_{\ell_0} \subset \ldots \subset \Omega_{\ell_j} \subset \Omega_{\ell_{j+1}} \subset \ldots \subset \Omega_{\ell_N}.$$

with $\theta = \{x \in \mathbb{R}^n : b(x) \leq \ell_0\}$.

Assumption 2: The set $\Omega_{\ell_N} = \{x \in \mathbb{R}^n : a(x) \leq \ell_N\}$ is bounded.

Assumption 3: For every pair of consecutive switching times $\tau_h < \tau_j$ such that $\sigma(\tau_h) = \sigma(\tau_j) = p$, the following holds:
$V_p(\varphi(\tau_h, x_0)) > V_p(\varphi(\tau_j, x_0))$ for $\varphi(\tau_h, x_0) \notin \Theta$ and $\varphi(\tau_j, x_0) \notin \Theta$.

Before stating the main result on multiple Lyapunov-like functions, the following lemma is needed to guarantee that a switched solution starting in $\Omega_{\ell_{j-1}}$ never leaves Ω_{ℓ_j} while a fixed $p \in P$ is active.

Lemma 1: Suppose, $\varphi(\tau_k, x_0) \in \Omega_{\ell_{j-1}}$ at the switching time τ_k with $\sigma(\tau_k) = p \in P$, then $\varphi(t, x_0) \in \Omega_{\ell_j}$ for all $t \in [\tau_k, \tau_{k+1})$.

Proof: Suppose by contradiction the existence of $T \in [\tau_k, \tau_{k+1})$, such that $\varphi(T, x_0) \notin \Omega_{\ell_j}$. Then $V_p(\varphi(\tau_k, x_0)) \leq b(\varphi(\tau_k, x_0)) \leq \ell_j$ and $V_p(\varphi(T, x_0)) \geq a(\varphi(T, x_0)) \geq \ell_j$, which is a contradiction because $\nabla V_p(x) f_p(x) \leq 0$ for x outside Θ and for all $p \in \mathcal{P}$.

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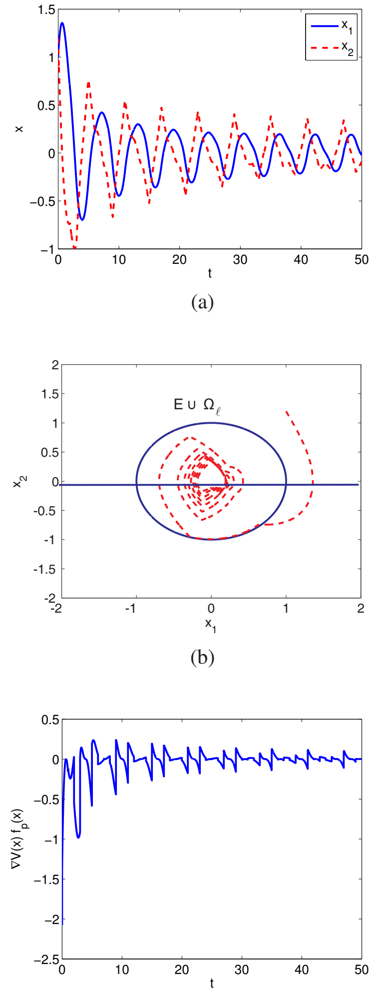


(a)

(b)

(c)

Fig. 1: (a) Switching solution with initial condition $x_0 = [1\ 1.2]' \notin \Omega_\ell$ for Example 1, (b) corresponding phase portrait and (c) corresponding $\dot{V}(\varphi(t, x))$.

Theorem 4: Consider the switched system (1) and let $V_p : \mathbb{R}^n \to \mathbb{R}$ be a smooth function for all $p \in \mathcal{P}$. Under Assumptions 1 - 3 we have:
(i) For all $x_0 \in \Theta$ and all $\sigma(t)$ such that $\varphi(t, x_0) \in \mathcal{S}_{dwell}$, $\varphi(t, x_0) \in \Omega_{\ell_N}$ for $t \geq 0$ and is attracted to the largest weakly invariant set in Ω_{ℓ_N}.
(ii) Every bounded solution of (1) $\varphi(t, x_0) \in S_{dwell}$ is attracted to the largest weakly invariant set of $E \cup \Omega_{\ell_N}$.

Proof: (i): The proof of part (i) explores the same ideas of induction on the number N of vector fields used in the proof of Lemma 2 in [13].

For $N = 1$, that is, $\mathcal{P} = \{1\}$, it is straightforward to show that solutions starting in Θ do not leave Ω_{ℓ_0}. Indeed, let $x_0 \in \Theta$ and suppose the existence of $T > 0$ such that $\varphi(T, x_0) \notin \Omega_{\ell_0}$. Then $V_1(x_0) \leq \ell_0$ and $V_1(\varphi(T, x_0)) >$

ℓ_0. The continuity ensures that $V_1(\varphi(t,x_0))$ must increase over some subintervals of $[0,T)$ outside Θ, but this is a contradiction. Therefore, $\varphi(t,x_0) \in \Omega_{\ell_0}, \forall t \geq 0$.

Next we assume (i) holds for $N = j-1$ vector fields, $\mathcal{P} = \{1,\cdots,j-1\}$, that is, if $x_0 \in \Theta$ and $\varphi(t,x_0) \in \mathcal{S}_{dwell}$ then $\varphi(t,x_0) \in \Omega_{\ell_{j-1}}$, for all $t \geq 0$.

Now, we show that (i) holds for $N = j$. Let $x_0 \in \Theta$. While the first $j-1$ systems are active, the trajectory does not leave $\Omega_{\ell_{j-1}}$, then at the switching time τ_k when the jth system becomes active for the first time, we have that $\varphi(\tau_k,x_0) \in \Omega_{\ell_{j-1}}$. Then by Lemma 1, $\varphi(t,x_0) \in \Omega_{\ell_j}$ for all $t \in [\tau_k, \tau_{k+1})$. Moreover, for any τ_r such that $r \geq k+1$, $\sigma(\tau_r)$ has to be equal to any of the systems $p \in \mathcal{P}$. Assumption 3 implies that $\varphi(\tau_r,x_0)$ has to belong to some Ω_{ℓ_i} with $i \leq j-1$. Therefore $\varphi(t,x_0) \in \Omega_{\ell_j}$ for all $t \geq 0$.

Hence $\varphi(t,x_0) \in \Omega_{\ell_N}$ for all $t \geq 0$. By assumption 2, Ω_{ℓ_N} is bounded and hence $\omega^+(x_0) \neq \emptyset$, compact and $\omega_+(x_0) \subset \Omega_{\ell_N}$. Since $\omega^+(x_0)$ is a weakly invariant set, then the solution is attracted to the largest weakly invariant set in Ω_{ℓ_N}.

(ii) Let $x_0 \notin \Theta$ and $\varphi(t,x_0) \in \mathcal{S}_{dwell}$ be a bounded solution. If there exists $T > 0$ such that $\varphi(T,x_0) \in \Theta$ then the proof follows directly from (i). Now, we suppose that $\varphi(t,x_0) \notin \Theta, \forall t \geq 0$. Then, $\nabla V_p(\varphi(t,x_0)) f_p(\varphi(t,x_0)) \leq 0, \forall t \in I_p$ and $\forall p \in \mathcal{P}$, since $C \subset \Theta$. For all $p \in \mathcal{P}$, consider the subsequence of switching times $\{\tau_{k_p}\}$ at which the system p becomes active, that is, $\sigma(\tau_{k_p}) = p$ for all k_p. From assumption 3, we have that the sequence $V_p(\varphi(\tau_{k_p},x_0))$ is a decreasing sequence of real numbers bounded from below. Then $V_p(\varphi(\tau_k,x_0)) \to L_p$ as $k \to +\infty$.

Since $\varphi(t,x_0)$ is bounded then $\omega^+(x_0)$ is nonempty. Let $c \in \omega^+(x_0)$, then there exists a sequence $\{t_j\}$ such that $\varphi(t_j,x_0) \to c$ as $j \to \infty$. Since the set $\mathcal{P}$ is finite, there exists at least one index $\overline{p} \in \mathcal{P}$ and a subsequence $\{t_{j_k}\}$ such that $\{t_{j_k}\} \in I_{\overline{p}}$. Then, $V_{\overline{p}}(\varphi(t_{k_j},x_0)) \to V_{\overline{p}}(c) = L_{\overline{p}}$, $\forall c \in \omega^+(x_0)$, for $\overline{p} \in \mathcal{P}$.

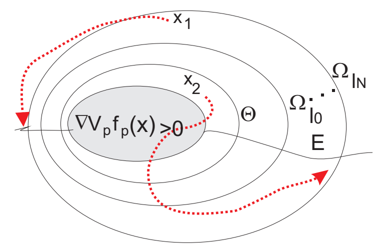
By Proposition 1, $\omega^+(x_0)$ is a weakly invariant set, thus there exist an interval $[\lambda,\gamma]$ containing the origin, a function $\upsilon(t)$ such that $\upsilon(0) = c$, $\upsilon(t) \in \omega^+(x_0), \forall t \in [\lambda,\gamma]$ and $\exists \overline{p} \in \mathcal{P}$ such that $\dot{\upsilon}(t) = f_{\overline{p}}(\upsilon(t)), \forall t \in [\lambda,\gamma]$. We showed that for $\overline{p} \in \mathcal{P}$, $V_{\overline{p}}(c) = L_{\overline{p}}, \forall c \in \omega^+(x_0)$ thus $V_{\overline{p}}(\upsilon(t)) = L_{\overline{p}}$ $\forall t \in [\lambda,\gamma]$, then $\nabla V_{\overline{p}}(\upsilon(t)) f_{\overline{p}}(\upsilon(t)) = 0, \forall t \in [\lambda,\gamma]$. Particularly, for $t = 0$, $\nabla V_{\overline{p}}(\upsilon(0)) f_{\overline{p}}(\upsilon(0)) = \nabla V_{\overline{p}}(c) f_{\overline{p}}(c) = 0$, then $c \in E$. Therefore $\omega^+(x_0) \subset E$, but $\omega^+(x_0)$ is a weakly invariant set, then the solution is attracted to the largest weakly invariant set of $\Omega_{\ell_N} \cup E$.

■

Observation 1: If the switching times are finite then the proof of Theorem 4 (ii) follows from Theorem 2.

Observation 2: In Theorem 4, if we assume that $V_p : \mathbb{R}^n \to \mathbb{R}$ is a radially unbounded function for all $p \in \mathcal{P}$, that is, $V_p(x) \to \infty$, as $||x|| \to \infty$, then every solution $\varphi(t,x_0)$ is bounded for $t \geq 0$ and the conclusions of Theorem 4 hold for any solution $\varphi(t,x_0) \in \mathcal{S}_{dwell}$.

Figure 2 illustrates the behavior of the switched solution inside the sets θ and Ω_{l_j} for two initial conditions. In Figure 2, the trajectory $\varphi(t,x_1)$ is attracted to the largest weakly invariant set in $E \cup \Omega_{\ell_N}$ when $x_1 \notin \Theta$ and $\varphi(t,x_1) \notin \Theta$ for all $t \geq 0$, on the other hand, trajectories are attracted to the largest weakly invariant set in Ω_{ℓ_N} when $x_2 \in \Theta$.



(a)

Fig. 2: Geometric interpretation of Theorem 4.

Example 2: Consider the switched system (1) with $\mathcal{P} = \{1,2\}$ and

$$f_1(x) = \begin{bmatrix} -x_1(-1+x_1^2+x_2^2) \\ -2x_2(-1+x_1^2+x_2^2) \end{bmatrix},$$

$$f_2(x) = \begin{bmatrix} x_2 \\ -x_1 - x_2 \end{bmatrix}.$$

Let $V_1(x) = 2x_1^2 + x_2^2$ and $V_2(x) = (x_1^2+x_2^2)/2$. We have that $C = \{x \in \mathbb{R}^2 : x_1^2+x_2^2 < 1\} \setminus \{(0,0)\}$; $E = \{x_1^2+x_2^2 = 1\} \cup \{x \in \mathbb{R}^2 : x_2 = 0\}$ Choose

$$\begin{aligned} a(x) &= (x_1^2+x_2^2)/2 \\ b(x) &= 2x_1^2+x_2^2; \\ \ell_0 &= \sup_{x \in C} b(x) = 2; \\ \Omega_{\ell_0} = \Omega_{\ell_2} &= \{x \in \mathbb{R}^2 : x_1^2+x_2^2 \leq 4\}; \\ \Theta &= \{x \in \mathbb{R}^2 : 2x_1^2+x_2^2 \leq 2\}; \end{aligned}$$

then, every bounded solution $\in \mathcal{S}_{dwell}$ is attracted to the largest weakly invariant set of $E \cup \Omega_{\ell_N} = \{x \in \mathbb{R}^2 : x_1^2+x_2^2 \leq 4\} \cup \{x \in \mathbb{R}^2 : x_2 = 0\}$.

Figure 3 illustrates the time-domain simulation for $x_0 = [1\ 1.2]'$ and $\tau_{k+1} = \tau_k + 1$, $k = 1,\cdots,51$.

IV. CONCLUSION

In this paper, a more general version of the invariance principle for switched systems was presented. Results for a common and multiple Lyapunov-like functions were given. The invariant sets of switched systems were obtained with less restrictive conditions on the Lyapunov-like functions.The proposed theorems are useful to estimate the basin of attraction of switched systems.

V. ACKNOWLEDGMENTS

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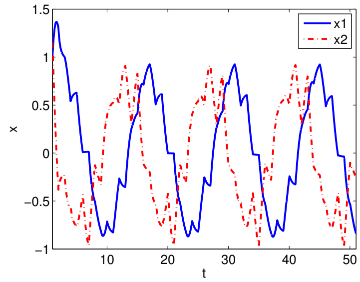
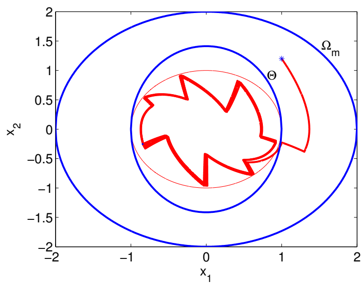
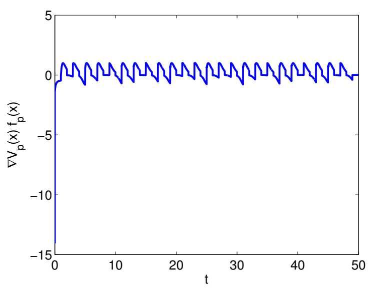


(a)



(b)



(c)

Fig. 3: (a) Switching solution with initial condition $x_0 = [1\ 1.2]' \notin \Theta$ for Example 2, (b) corresponding phase portraits and (c) corresponding $\nabla V_p(x) f_p(x)$.

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