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Near Optimal LQR Control in the Decentralized Setting

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Abstract— **In this paper we consider the use of a linear periodic controller (LPC) for the control of linear time-invariant (LTI) plants in the decentralized setting. If a plant has an unstable decentralized fixed mode (DFM), it is well known that no LTI controller can stabilize it, let alone provide good performance. Here we show that, if the plant is centrally controllable and observable and the graph associated with the plant is strongly connected, even if the plant has an unstable decentralized fixed mode we can still design an LPC to provide LQR performance as close to optimality as desired. The controller in each channel consists of a sampler, a zero-order-hold, and a discrete-time linear periodic compensator, which makes it easy to implement.**

Keywords: Decentralized control, decentralized fixed modes, linear periodic control, LQR optimal control.

I. Introduction

In this paper we are interested in control in the decentralized setting. It is well known that a linear time invariant (LTI) system can be stabilized using a decentralized LTI controller if and only if the system does not possess any unstable DFMs [6]. The use of time-varying controllers in the decentralized setting has been investigated for some time. It was shown in [3] that some DFMs can be moved using time-varying feedback; at about the same time, [17] argued that, in some cases, a generalized hold based controller can achieve a similar result. This lead to further work on this topic, including that of [18], [8], [1] and [2]. It also lead to an attempt to classify DFMs into those which are truly fixed and those which can be moved using a sufficiently sophisticated controller; the key work is that of [14] and [7].

Here we are interested not only in stability but also in performance. Carrying out optimal LTI controller design in the decentralized setting is difficult. In [5] the authors show constructively that, under a technical condition, if the plant has no unstable decentralized fixed modes and is minimum phase, then an LTI decentralized controller can do almost as well as an LTI centralized controller. For more general cost functions, the results which yield tractible design algorithms are limited; [15] provides a detailed historical account of this work and argues that the underlying concept in many of these approaches is that of 'quadratic invariance', which yields a structural constraint on the plant. An explicit state-space solution for a decentralized two channel problem with a block triangular structure is provided in the H_2 context [16]. However, a tractible design approach for finding the optimal LTI decentralized controller in the general case remains open.

In this paper our objective is to construct an optimal (or near optimal) controller, even if it is not LTI. Indeed, if the LTI plant has an unstable decentralized fixed mode, an LTI controller cannot stabilize it, let alone provide good performance, so more complicated controllers must be used. The paper [7] provides a necessary and sufficient condition for the existence of a stabilizing nonlinear time-varying (NLTV) controller. However, the performance provided by the controller given there, as well as those provided in the aforementioned papers on linear time-varying controllers ([3], [14], [17], [18], [8], [1], and [2]), is not discussed in detail, and it is not claimed to be optimal in any way. In this paper we adopt the classical LQR measure of performance, and demonstrate that, if the plant is centrally controllable and observable and the graph associated with the system is strongly connected, then we can design a linear periodic sampled-data decentralized controller which provides performance as close to the optimal as desired. Under our plant hypothesis, the system may have DFMs, possibly unstable, but it follows from [7] that all are moveable using a suitable NLTV controller. Our approach is motivated by our earlier work on the control of linear (possibly time-varying) systems using periodic control, especially that of [12], [10], [11] and [13].

II. The Setup

Here we consider the strictly proper plant

$$\left.\begin{array}{rcl} \dot{x} &=& Ax + \sum_{i=1}^{p} B_i u_i, \;\; x(t_0) = x_0 \\ y_i &=& C_i x, \;\; i = 1, ..., p \end{array}\right\} \qquad (1)$$

with $x(t) \in \mathbf{R}^n$, $u_i(t) \in \mathbf{R}^{m_i}$ and $y_i(t) \in \mathbf{R}^{r_i}$ for $i = 1, ..., p$; we set $m = \sum_{i=1}^{p} m_i$ and $r = \sum_{i=1}^{p} r_i$; we represent this model by $(A; B_1, ..., B_p; C_1, ..., C_p)$. Associated with this model are

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix}, \;\; u = \begin{bmatrix} u_1 \\ \vdots \\ u_p \end{bmatrix}, \;\; B := \begin{bmatrix} B_1 & \cdots & B_p \end{bmatrix},$$

$$C := \begin{bmatrix} C_1 \\ \vdots \\ C_p \end{bmatrix}, \;\; \mathcal{O}(C_i, A) := \begin{bmatrix} C_i \\ C_i A \\ \vdots \\ C_i A^{n-1} \end{bmatrix}.$$

In the decentralized context we require u_i to depend solely on y_i, whether the controller is LTI, linear time-varying (LTV), or NLTV.

Assumption 1: (A, B) is controllable and (C, A) is observable.

The notion of a decentralized fixed mode was introduced in [6]; the goal is to capture which eigenvalues are immovable using LTI feedback which respects the information flow constraints. To this end, we define the set of feedback gains $\mathcal{K}_{dec}$ by

$$\{K \in \mathbf{R}^{r\times r}: \quad K = \text{diag}\{K_1, ..., K_p\} \text{ with } K_i \in \mathbf{R}^{r_i \times r_i}\}.$$

Definition 1: The DFMs of $(A; B_1, ..., B_p; C_1, ..., C_p)$ are given by $\bigcap_{K\in\mathcal{K}_{dec}} \text{sp}(A + BKC)$.

Following [4], it turns out that graph theory can be used to study decentralized systems. One can envision building a directed graph of the plant (1) as follows: there are p nodes representing the p control agents and p sensor agents, with an arc from node i to node j iff $C_j(sI - A)^{-1}B_i \neq 0$. A directed graph is said to be strongly connected if there is a path from every node to every other node along the arcs.

Proposition 1: [3] The directed graph corresponding to $(A; B_1, ..., B_p; C_1, ..., C_p)$ is strongly connected iff for every partition of the set of indices $\{1, ..., p\}$ into nonempty sets $S_1 = \{i_1, ..., i_q\}$ and $S_2 = \{i_{q+1}, ..., i_p\}$ satisfies

$$\begin{bmatrix} C_{i_{q+1}} \\ \vdots \\ C_{i_p} \end{bmatrix} (sI - A)^{-1} \begin{bmatrix} B_{i_1} & \cdots & B_{i_q} \end{bmatrix} \neq 0.$$

Assumption 2: The directed graph corresponding to $(A; B_1, ..., B_p; C_1, ..., C_p)$ is strongly connected.

III. The Problem

Here our goal is to design a linear time-varying controller which not only provides closed loop stability but also provides near optimal LQR performance. To this end, choose positive definite symmetric matrices $Q \in \mathbf{R}^{n\times n}$ and $R \in \mathbf{R}^{m\times m}$ and consider the quadratic performance index

$$J(x_0) = \int_0^\infty [x^T(t)Qx(t) + u^T(t)Ru(t)]dt.$$

With $t_0 = 0$, the LQR problem is to find, for each $x_0 \in \mathbf{R}^n$, the control signal u which minimizes this cost. As is well-known, the optimal controller is state-feedback:

$$u = Fx,$$

which gives rise to an optimal cost of the form $J^0(x_0) = x_0^T P x_0$ with P the positive definite solution of an associated Riccati equation. For a given initial condition x_0, we label the optimal state trajectory by $x^0(t)$ and the optimal control signal by u^0, i.e.

$$x^0(t) = e^{(A+BF)t}x_0, \;\; u^0(t) = Fe^{(A+BF)t}x_0, \;\; t \geq 0.$$

Here we consider sampled-data controllers of the form

$$\left.\begin{aligned} z_i[k+1] &= G_i(k)z_i[k] + H_i(k)y_i(kh), \\ &\qquad z_i[0] = z_{i_0} \in \mathbf{R}^{l_i}, \\ u_i(kh+\tau) &= J_i(k)z_i[k] + K_i(k)y_i(kh), \\ &\qquad \tau \in [0, h) \end{aligned}\right\} \quad (2)$$

with the controller gains G_i, H_i, J_i, and K_i periodic of period $\ell \in \mathbf{N}$ for every $i \in \{1, 2, ..., p\}$; the period of the overall controller is $T := \ell h$, and we associate this system with $((G_i, H_i, J_i, K_i), i = 1, ..., \ell; T; \ell)$. Note that for each i, (2) can be implemented with a sampler, a zero-order-hold, and an l^{th} order periodically time-varying discrete-time system of period ℓ. We define the augmented controller state as

$$z[k] := \begin{bmatrix} z_1[k] \\ \vdots \\ z_p[k] \end{bmatrix}, \;\; z[0] := z_0.$$

The state of the closed loop-system is a combination of discrete and continuous states, defined by

$$x_{sd}(t) := \begin{bmatrix} x(t) \\ z[k] \end{bmatrix}, \quad t \in [kh, (k+1)h).$$

Now we make precise our notion of stability.

Definition 2: The sampled-data controller (2) exponentially stabilizes (1) if there exist constants $\gamma > 0$ and $\lambda < 0$ so that, with $t_0 = 0$, for every $x_0 \in \mathbf{R}^n$ and $z_0 \in \mathbf{R}^{l_1+\cdots+l_p}$, we have

$$\|x_{sd}(t)\| \leq \gamma e^{\lambda t}\|x_{sd}(0)\|, \;\; t \geq 0.$$

With $t_0 = 0$, if the sampled-data controller (2) exponentially stabilizes the plant (1), then we label the corresponding cost by $J(x_0, z_0)$. The goal of this paper is to design (2), parametrized by $\varepsilon > 0$, so that (i) it exponentially stabilizes (1) and (ii) for every $x_0 \in \mathbf{R}^n$ and $z_0 \in \mathbf{R}^l$, we have

$$|J(x_0, z_0) - J^0(x_0)| \leq \varepsilon(\|x_0\|^2 + \|z_0\|^2).$$

Before presenting any results, let us first provide some motivation for the approach adopted in this paper. First consider the control law

$$u(t) = Fx(kT), \;\; t \in [kT, (k+1)T).$$

It is intuitively reasonable that this will be near optimal if T is small enough, with the difference in cost tending to zero as $T \to 0$. Of course, $Fx(kT)$ is not measureable in each channel in most cases, so it needs to be estimated. First, make the natural partition of F:

$$F := \begin{bmatrix} F_1 \\ \vdots \\ F_p \end{bmatrix}.$$

If $u = 0$, then

$$x(t) = e^{At}x(0), \;\; y_i(t) = C_i e^{At}x(0), \;\; t \geq 0.$$

Hence, by measuring $y_i(t)$ only, we can easily obtain an estimate of $\mathcal{O}(C_i, A)x(0)$. In most situations (C_i, A) is not observable, so $x(0)$ cannot be obtained. However, since (C, A) is observable by assumption, there always exists n linearly independent rows, say $C_{i_j}A^{n_j}, \quad j = 1, 2, ..., n$; indeed, we can choose $W \in \mathbf{R}^{n\times nr}$ to pick off those rows, so that

$$\mathcal{O}_{est} := \begin{bmatrix} C_{i_1}A^{n_1} \\ \vdots \\ C_{i_n}A^{n_n} \end{bmatrix} = W \begin{bmatrix} \mathcal{O}(C_1, A) \\ \vdots \\ \mathcal{O}(C_n, A) \end{bmatrix}.$$

Hence, while $x(0)$ is not obtainable from every channel, the quantity

$$\zeta(0) := \mathcal{O}_{est}x(0) \tag{3}$$

has sufficent information from which to identify $x(0)$, and hence $Fx(0)$.

The above motivates the following idea. On each period $[kT,(k+1)T)$, we apply an estimate $\hat{u}^0(kT)$ of $u^0(kT)$, at the same time doing a small amount of probing to obtain a better estimate of this quantity for use during the next period; we make use of the fact that the graph associated with the system is strongly connected in order to pass information among the channels. Specifically, for each $j = 1,2,...,n$, we first estimate the quantity $C_{i_j}A^{n_j}x(kT)$ by examining $y_{i_j}(t)$. Second, we then apply probing on top of the nominal control signal $\hat{u}^0(kT)$ to pass this information to all of the other channels. It turns out that we can do this in a linear (though time-varying) fashion.

IV. REGULARIZITION

At this point it is convenient to put our system into a form which is amenable to analysis. To proceed, it will be convenient to impose

Assumption 3: For every $i,j,\in\{1,...,p\}$, the transfer function $C_i(sI-A)^{-1}B_j$ is not identically zero.

While this seems restrictive at first glance, it turns out that we can always regularize our system by first applying some feedback. To proceed, we first need some notation. We will use a slightly modified version of the notion of a generic property given in [19]. Let $x=\{x_{ij}\}\in\mathbf{R}^{n\times m}$, and consider polynomials $\phi(x)=\phi(x_{11},...,x_{nm})$ with coefficients in $\mathbf{R}$. A *property* π is a function $\pi:\mathbf{R}^{n\times m}\to\{0,1\}$, where $\pi(x)=1$ (or 0) means that π holds (or fails) at x. A property π holds for *almost all* $x\in\mathbf{R}^{n\times m}$ if there exists a *nonzero* polynomial ϕ for which π fails at x iff $\phi(x)=0$.

Proposition 2: [9] For almost all $K\in\mathcal{K}_{dec}$, for every $i,j,\in\{1,...,p\}$ the transfer function $C_i(sI-A-BKC)^{-1}B_j$ is not identically zero.

After regularization, our cost function must be adjusted. To see this, write

$$u = Ky + u^{new}.$$

If we apply this to (1), then we end up with

$$\begin{aligned}\dot{x} &= (A+BKC)x + Bu^{new}\\ y &= Cx\\ J(x_0) &= \int_0^\infty [x(t)^TQx(t) +\\ &\quad (u^{new}(t)+Ky(t))^TR(u^{new}(t)+Ky(t))]\,dt\end{aligned}$$

with an optimal control law of

$$u^{new} = \tilde{F}x = (F-KC)x.$$

Rather than bog down the ensuing analysis and design procedure with undue notation, *henceforth we shall assume that Assumptions 1, 2 and 3 all hold but that the cost function is of the form*

$$\begin{aligned}J(x_0) &= \int_0^\infty [x(t)^TQx(t) +\\ &\quad (u(t)+Ky(t))^TR(u(t)+Ky(t))]\,dt.\end{aligned}$$

In order to implement the idea of the previous section, we will be passing information from one channel to the rest. In order to so, it is particularly convenient to convert the system to one with single-input single-output (siso) channels. To this end, consider vectors $v\in\mathbf{R}^r$ and $w\in\mathbf{R}^m$ partitioned in a natural way as

$$v=\begin{bmatrix}v_1\\ \vdots\\ v_p\end{bmatrix},\ \ v_i\in\mathbf{R}^{r_i},\ \ w=\begin{bmatrix}w_1\\ \vdots\\ w_p\end{bmatrix},\ \ w_i\in\mathbf{R}^{m_i}.$$

Proposition 3: For almost all $(v,w)\in\mathbf{R}^r\times\mathbf{R}^m$, we have that for every $i,j\in\{1,2,...,p\}$, the transfer function $v_i^TC_i(sI-A)^{-1}B_jw_j$ is not identically zero.

Now freeze $(v,w)\in\mathbf{R}^r\times\mathbf{R}^m$ so that for every $i,j\in\{1,2,...,p\}$, the transfer function $v_i^TC_i(sI-A)^{-1}B_jw_j$ is not identically zero. We now introduce the natural notation

$$\bar{C}_i := v_i^TC_i,\ \ \bar{y}_i=\bar{C}_ix=v_i^TC_ix,\ \ i=1,...,p,$$

$$\bar{B}_j := B_jw_j,\ \ j=1,...,p.$$

On occasion, we shall need to apply an input only through the j^{th} channel; to this end, we define $\bar{w}_j := \begin{bmatrix}0\\ w_j\\ 0\end{bmatrix}\in\mathbf{R}^m$.

V. ESTIMATION OF THE CONTROL SIGNAL

To construct the proposed control law, the idea is to periodically apply an estimate $\hat{u}^0(kT)$ of $u^0(kT)$, at the same time constructing a better estimate.[1] The approach adopted here is motivated by that used in [10] to solve the optimal centralized LQR problem in the face of significant plant uncertainty. To this end, we first choose $\bar{n}\in\{1,...,n\}$ as well as a number of matrices, starting with two $(\bar{n}+1)\times(\bar{n}+1)$ matrices:

$$S=\begin{bmatrix}1&0&0&\cdots&0\\ 1&1&1&\cdots&1\\ 1&2&2^2&\cdots&2^{\bar{n}}\\ &&\vdots&&\\ 1&\bar{n}&\bar{n}^2&\cdots&\bar{n}^{\bar{n}}\end{bmatrix},$$

$$H(h)=\mathrm{diag}\{1,h,\frac{h^2}{2!},...,\frac{h^{\bar{n}}}{\bar{n}!}\}.$$

With $I_j\in\mathbf{R}^{r_j\times r_j}$ the identity matrix, we also define

$$S_j := S\otimes I_j,\ \ H_j := H\otimes I_j,\ \ j\in\mathbf{N}.$$

[1] We partition $\hat{u}^0(kT)$ in a natural way as $\begin{bmatrix}\hat{u}_1^0(kT)\\ \vdots\\ \hat{u}_p^0(kT)\end{bmatrix}$.

We now define the observability-like matrices

$$\mathcal{O}_i := \begin{bmatrix} C_i \\ C_i A \\ \vdots \\ C_i A^{\bar{n}} \end{bmatrix} \in \mathbf{R}^{r_i(\bar{n}+1)\times n},$$

as well as two versions of the sampled output:

$$\mathcal{Y}_i(t) := \begin{bmatrix} y_i(t) \\ y_i(t+h) \\ \vdots \\ y_i(t+\bar{n}h) \end{bmatrix}, \quad \bar{\mathcal{Y}}_i(t) := \begin{bmatrix} \bar{y}_i(t) \\ \bar{y}_i(t+h) \\ \vdots \\ \bar{y}_i(t+\bar{n}h) \end{bmatrix}.$$

The following result has two parts: the first explains how to estimate $x(kT)$; the second explains how to pass this information amongst the channels. Here we carry out probing with a test signal, scaled by $\rho \in (0,1)$.

Lemma 1: For every $\bar{h} \in (0,1)$ there exists a constant $\gamma > 0$ so that for every $t_0 \in \mathbf{R}$, $x_0 \in \mathbf{R}^n$, $h \in (0,\bar{h})$, $\bar{u} \in \mathbf{R}^m$ and $\delta \in \mathbf{R}$:
(i) The solution of (1) with

$$u(t) = \begin{cases} (1+\rho)\bar{u} & t \in [t_0, t_0+\bar{n}h) \\ (1-\rho)\bar{u} & t \in [t_0+\bar{n}h, t_0+2\bar{n}h) \end{cases}$$

satisfies, for $i = 1,...,p$:

$$\|H_i(h)^{-1} S_i^{-1}[\frac{\rho-1}{2\rho}\mathcal{Y}_i(t_0) + \frac{\rho+1}{2\rho}\mathcal{Y}_i(t_0+\bar{n}h)] - \mathcal{O}_i x_0\| \leq \gamma h(\|x_0\| + \|\bar{u}\|),$$

and

$$\|x(t) - x_0\| \leq \gamma h(\|x_0\| + \|\bar{u}\|), \quad t \in [t_0, t_0+2\bar{n}h].$$

(ii) The solution of (1) with

$$u(t) = \begin{cases} \bar{u} + \bar{w}_j\delta & t \in [t_0, t_0+\bar{n}h) \\ \bar{u} - \bar{w}_j\delta & t \in [t_0+\bar{n}h, t_0+2\bar{n}h) \end{cases}$$

satisfies, for $i = 1,...,p$:

$$\|H(h)^{-1}S^{-1}[\bar{\mathcal{Y}}_i(t_0) - \bar{\mathcal{Y}}_i(t_0+\bar{n}h)] - 2\begin{bmatrix} 0 \\ \bar{C}_i\bar{B}_j \\ \vdots \\ \bar{C}_i A^{\bar{n}-1}\bar{B}_j \end{bmatrix}\delta\| \leq \gamma h(\|x_0\| + \|\bar{u}\| + \|\delta\|),$$

and

$$\|x(t) - x_0\| \leq \gamma h(\|x_0\| + \|\bar{u}\| + \|\delta\|), \quad t \in [t_0, t_0+2\bar{n}h].$$

To see how this result can be applied, assume that

$$u(t) = \begin{cases} (1+\rho)\hat{u}^0(kT) & t \in [kT, kT+\bar{n}h) \\ (1-\rho)\hat{u}^0(kT) & t \in [kT+\bar{n}h, kT+2\bar{n}h); \end{cases}$$

part (i) says that

$$\|H_i(h)^{-1}S_i^{-1}[\frac{\rho-1}{2\rho}\mathcal{Y}_i(t_0) + \frac{\rho+1}{2\rho}\mathcal{Y}_i(t_0+\bar{n}h)] - \mathcal{O}_i x(kT)\| \leq \gamma h(\|x(t_0)\| + \|\bar{u}\|).$$

Hence, we can obtain a good estimate Est$[\mathcal{O}_i x(kT)]$ of $\mathcal{O}_i x(kT)$ for $i = 1,...,p$. **Now assume that** $\bar{n} \geq \max\{n_1,...,n_n\}$;[2] clearly we can form an estimate of $\zeta(t)$ given by (3):

$$\hat{\zeta}(kT) := W \begin{bmatrix} \text{Est}[\mathcal{O}_1 x(kT)] \\ \vdots \\ \text{Est}[\mathcal{O}_p x(kT)] \end{bmatrix};$$

notice that the j^{th} element of $\hat{\zeta}(kT)$, namely $\hat{\zeta}_j(kT)$, is simply an estimate of $C_{i_j}A^{n_j}x(kT)$, which is simply an element of Est$[\mathcal{O}_{i_j}x(kT)]$, so it obtainable from channel i_j. The idea is to now pass the information about $\hat{\zeta}(kT)$ through the system so that it is available at each channel. Specifically, $\hat{\zeta}_1(kT)$ is available at channel i_1; with $\rho > 0$, if we set

$$u(t) = \hat{u}^0(kT) + \begin{cases} \rho\bar{w}_{i_1}\hat{\zeta}_1(kT) & t \in [kT+2\bar{h}, kT+3\bar{h}) \\ -\rho\bar{w}_{i_1}\hat{\zeta}_1(kT) & t \in [kT+3\bar{h}, kT+4\bar{h}), \end{cases}$$

then the probing takes place only on input i_1 using information available from channel i_1. We can then form an estimate of $\hat{\zeta}_1(kT)$ on every other channel using Lemma 1 (ii):

$$\frac{1}{2\rho}H(h)^{-1}S^{-1}[\bar{\mathcal{Y}}_i(t_0+2\bar{h}) - \bar{\mathcal{Y}}_i(t_0+3\bar{h})] \approx \underbrace{\begin{bmatrix} 0 \\ \bar{C}_i\bar{B}_{i_1} \\ \vdots \\ \bar{C}_i A^{\bar{n}-1}\bar{B}_{i_1} \end{bmatrix}}_{M_{i,i_1}} \hat{\zeta}_1(kT), \quad i = 1,...,p.$$

Now assume that $\bar{n} \in \{1,...,n\}$ **is sufficiently large that** $M_{i,j} \neq 0$ **for every** $i,j \in \{1,...,p\}$**.**[3] This now means that

$$\underbrace{\frac{1}{2\rho}(M_{i,i_1}^T M_{i,i_1})^{-1}M_{i,i_1}^T H(h)^{-1}S^{-1}}_{=:\bar{M}_{i,i_1}}[\bar{\mathcal{Y}}_i(t_0+2\bar{h}) - \bar{\mathcal{Y}}_i(t_0+3\bar{h})] \approx \hat{\zeta}_1(kT), \quad i = 1,...,p.$$

Of course, we can do the same for $\hat{\zeta}_2(kT)$ as well: set

$$u(t) = \hat{u}^0(kT) + \begin{cases} \rho\bar{w}_{i_2}\hat{\zeta}_2(kT) & t \in [kT+4\bar{h}, kT+5\bar{h}) \\ -\rho\bar{w}_{i_2}\hat{\zeta}_2(kT) & t \in [kT+5\bar{h}, kT+6\bar{h}), \end{cases}$$

which means that the probing takes place only on channel i_2 using information obtainable from channel i_2, yielding

$$\bar{M}_{i,i_2}[\bar{\mathcal{Y}}_i(t_0+4\bar{h}) - \bar{\mathcal{Y}}_i(t_0+5\bar{h})] \approx \hat{\zeta}_2(kT), \ i = 1,...,p.$$

We can carry out the same procedure for the remaining elements of $\hat{\zeta}(kT)$ as well.

[2] Choosing $\bar{n} = n$ will do.
[3] Choosing $\bar{n} = n$ will do.

Since our objective is to form an estimate of $\zeta(kT)$ at each channel, it is convenient to define some new block diagonal matrices containing associated $\bar{M}_{i,j}$ terms:

$$\tilde{M}_i := \mathrm{diag}\{\bar{M}_{i,i_1}, ..., \bar{M}_{i,i_n}\}, \quad i = 1, ..., n.$$

We shall label the estimate of $\zeta(kT)$ on channel i by $\hat{\zeta}^i(kT)$. At this point we are ready to provide a precise definition of the controller.

VI. The Controller

Now we fix $\bar{n} \in \{1, .., n\}$ to satisfy the two constraints discussed in the previous section, and let $h > 0$ and $\rho > 0$. We use an interval of length $\bar{h} := \bar{n}h$ often; we set $\ell := (2n + 3)\bar{n}$ and the controller period to be $T := \ell h$.

With $\hat{u}^0(0) \in \mathbf{R}^m$, we define the controller via three parts - for $k \in \mathbf{Z}^+$:

(i) **Construct** $\hat{\xi}(kT)$ **on** $[kT, kT + 2\bar{h})$.

Set

$$u(t) = \begin{cases} (1+\rho)\hat{u}^0(kT) & t \in [kT, kT + \bar{h}) \\ (1-\rho)\hat{u}^0(kT) & t \in [kT + \bar{h}, kT + 2\bar{h}), \end{cases} \quad (4)$$

and define

$$\mathrm{Est}[\mathcal{O}_i x(kT)] :=$$

$$H_i(h)^{-1} S_i^{-1}[\frac{\rho - 1}{2\rho}\mathcal{Y}_i(kT) + \frac{\rho + 1}{2\rho}\mathcal{Y}_i(kT + \bar{h})]. \quad (5)$$

Now define

$$\hat{\zeta}(kT) := W \begin{bmatrix} \mathrm{Est}[\mathcal{O}_1 x(kT)] \\ \vdots \\ \mathrm{Est}[\mathcal{O}_n x(kT)] \end{bmatrix}.$$

(ii) **Estimate** $\hat{\zeta}^i(kT)$ on $[kT + 2\bar{h}, kT + (2n + 2)\bar{h})$.

We first probe in sequence using the n elements of $\hat{\zeta}(kT)$:

$$u(t) = \hat{u}^0(kT) +$$

$$\begin{cases} \rho\bar{w}_{i_1}\hat{\zeta}_1(kT) & t \in [kT + 2\bar{h}, kT + 3\bar{h}) \\ -\rho\bar{w}_{i_1}\hat{\zeta}_1(kT) & t \in [kT + 3\bar{h}, kT + 4\bar{h}) \\ \vdots & \\ \rho\bar{w}_{i_n}\hat{\zeta}_n(kT) & t \in [kT + 2n\bar{h}, kT + (2n + 1)\bar{h}) \\ -\rho\bar{w}_{i_n}\hat{\zeta}_n(kT) & t \in [kT + (2n + 1)\bar{h}, \\ & \quad kT + (2n + 2)\bar{h}) \end{cases} \quad (6)$$

and then form an estimate of $\hat{\zeta}^i(kT)$ at the i^{th} channel:

$$\hat{\zeta}^i(kT) := \tilde{M}_i \begin{bmatrix} \bar{\mathcal{Y}}_i(kT + 2\bar{h}) - \bar{\mathcal{Y}}_i(kT + 3\bar{h}) \\ \vdots \\ \bar{\mathcal{Y}}_i(kT + 2n\bar{h}) - \bar{\mathcal{Y}}_i(kT + (2n + 1)\bar{h}) \end{bmatrix},$$

for $i = 1, ..., p$.

(iii) **Form the updated control law** on $[kT+(2n+2)\bar{h}, kT+(2n + 3)\bar{h})$.

Set

$$u(t) = \hat{u}^0(kT), \quad t \in [kT + (2n + 2)\bar{h}, kT + (2n + 3)\bar{h}),$$

and define the estimate of $x(kT)$ on the i^{th} channel by

$$\hat{x}^i(kT) := \mathcal{O}_{est}^{-1}\hat{\zeta}^i(kT),$$

as well as the updated control signal:

$$\hat{u}^0((k + 1)T) = \begin{bmatrix} F_1\hat{x}^1(kT) \\ \vdots \\ F_p\hat{x}^p(kT) \end{bmatrix}. \quad (7)$$

It turns out that the above controller has a state-space representation of the form under consideration. Also, the performance provided by the controller (4)-(7) tends toward the optimal performance as $\rho \to 0$ and $T \to 0$, yielding:

Theorem 1: For every $\varepsilon > 0$ there exists a decentralized controller of the form (2) which exponentially stabilizes the plant (1) and ensures that, for every $x_0 \in R^n$ and $z_0 \in \mathbf{R}^l$, the closed-loop system satisfies

$$J(x_0, z_0) - J^0(x_0) \le \varepsilon(\|x_0\|^2 + \|z_0\|^2).$$

Remark 1: Notice that if the plant has unstable decentralized fixed modes then no LTI controller can stabilize the plant let alone provide near optimal performance. Also, if we set $z_0 = 0$ then the cost depends solely on x_0. Last of all, since the controller is linear periodic, it is easy to prove that if we inject noise and each plant controller interface, the map from the noise to all internal signals has a finite induced gain in the ∞-norm sense.

VII. An Example

Consider the following system

$$\dot{x} = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u_2$$

$$y_1 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} x$$

$$y_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x.$$

Suppose that the goal is to design a controller to minimize the optimal cost with $Q = I$ and $R = I$. This yields an optimal control law of

$$u = \begin{bmatrix} -12.7082 & 0 & 7.4721 \\ 0 & -2.4142 & 0 \end{bmatrix} x.$$

It can be easily checked that this system is controllable and observable, so it satisfies Assumption 1. The transfer function is $\begin{bmatrix} 0 & \frac{1}{s-1} \\ \frac{s+1}{s(s+2)} & 0 \end{bmatrix}$, so it is strongly connected - it satisfies Assumption 2. It can be easily checked that the system has a decentralized fixed mode at 1, which means that no LTI controller can stabilize it, let alone provide good performance.

Since some elements of the blocks of $G(s)$ are zero, we need to regularize the system by applying some output feedback: we choose

$$u = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} y + u^{new},$$

so the optimal feedback for the regularized system is

$$u^{new} = \begin{bmatrix} -12.7082 & -1.0000 & 7.4721 \\ -1.0000 & -2.4142 & 0 \end{bmatrix} x.$$

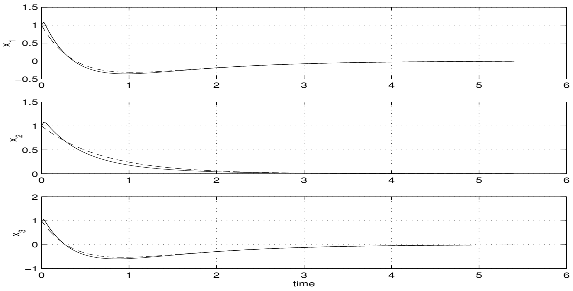

Fig. 1. The closed loop state response (x is solid while x^0 is dashed).

The first step is to choose $\bar{n}$ - we set $\bar{n} = n = 3$. Since (C_1, A_{new}) is observable, we shall set

$$\zeta(t) = \begin{bmatrix} C_1 \\ C_1 A_{new} \\ C_1 A_{new}^2 \end{bmatrix} x(t).$$

Since $\bar{n} = 3$, we first carry out estimation of $\zeta(kT)$ for $2\bar{n}h = 6h$. We then probe $n = 3$ times, each of $6h$ units of time. We then pause for $n = 3$ units of time, yielding a period of $T = 27h$ units of time. For large values of T our controller may not provide closed loop stability; as T decreases, stability is obtained, and the closed loop performance improves as $T \to 0$; however, to get near optimal performance we need ρ to be small.

We have carried out a simulation for the case of $\rho = 0.1$, $h = 0.001$, $T = 0.027$, $x(0) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$, and $\hat{u}^0(0) = 1$, and displayed the results in Figures 1 and 2; in each plot we have also placed the optimal response for comparison and for simplicity we have presented the signals for the regularized system. We see that the response of x is nearly optimal, while the control signal looks like the optimal control signal with some dither added.

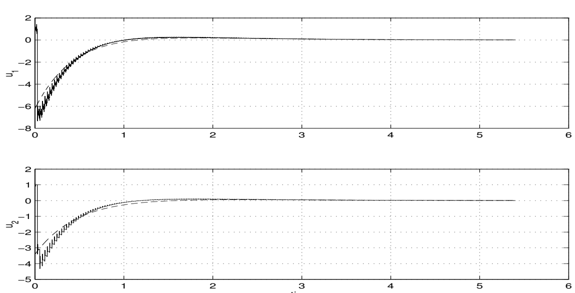

Fig. 2. The closed-loop control signal response (u is solid while u^0 is dashed.).

VIII. Summary and Conclusions

In this paper we considered the use of a linear periodic controller (LPC) for the control of linear time-invariant (LTI) plants in the decentralized setting. If a plant has an unstable decentralized fixed mode, it is well known that no LTI controller can stabilize it, let alone provide good performance. Here we show that, if the plant is centrally controllable and observable and the graph associated with the plant is strongly connected, even if the plant has unstable decentralized fixed modes we can still design an LPC to provide LQR performance as close to optimality as desired. The controller in each channel consists of a sampler, a zero-order-hold, and a discrete-time linear periodic compensator, which makes it easy to implement. However, to obtain good performance we need to sample quickly, which may result in poor performance in the presence of noise.

We would like to extend this work to the H_∞ and/or L_1 paradigm. Furthermore, we would like to investigate the case in which the graph associated with the plant is not strongly connected.

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