

Robust L_2 -Gain Observation for structured uncertainties: an LMI approach

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Abstract—The robust L_2 -gain estimation is investigated for general uncertain systems with structured uncertainties. A new estimation structure is introduced: the Augmented-Gain Observer which encompasses both filters and observers and allows robust estimation even for some classes of unstable systems. Our approach is based on a separation of graphs theorem using frequency dependent Integral Quadratic Constraints. We prove that the design of an Augmented-Gain Observer ensuring a robust L_2 -gain performance can be expressed as a convex optimization problem. This problem involves Linear Matrix Inequalities constraints and can be solved using an efficient algorithm. A numerical example illustrates the interest of the method.

I. INTRODUCTION

In many applications, it is often useful to measure some state vector components of a system for diagnosis, control, or supervision. If the necessary state vector components are not measured, they have to be estimated from the measure of other components of this vector, based on a model of the system. If this model is assumed to be linear time invariant and to perfectly represent the real system, computing an estimator has been largely investigated [8].

Two structures are mainly designed for the estimation. In this paper, these structures are referred to as the observer structure and the filter structure. An observer explicitly contains a model of the system whose dynamical evolution is driven through a gain by the error between the measured output and the estimated one [13]. A filter is defined as a (matrix of) transfer function(s) between the measured output and the estimation of the state components. Many approaches were proposed, for different purposes: unknown initial conditions (Luenberger Observer [13]), stochastic perturbations (Kalman observer [10]), minimization of a norm of the estimation error (H_2 or H_∞ [23], [12]) The approaches proposed for the direct synthesis of a filter cannot be applied in the case of unstable systems.

These approaches do not take into account the modeling error, which can have a strong impact on the performance level achieved on the real system [2]. To deal with this difference the modeling error is represented with uncertainties [27]. The robust estimation problem is to find a robust estimator which ensures a guaranteed performance for all the systems represented by the uncertain model.

In the case of linear systems, two classes of uncertainties are in general represented: uncertainties on the model parameters (parametric uncertainties) and uncertainties on

the model dynamics (dynamical uncertainties). Results were proposed for a non structured parametric uncertainty: the state space matrices are assumed to be affine [5], [18] or rational [12], [3] functions of a unique uncertain matrix. Unfortunately, in general, there are several (scalar) parameter uncertainties in a model. A first simple case is when the state space matrices are affine in the uncertain parameters. In this case, referred to as polytopic systems [9], [26], [1], [7], [11], [17], the matrices of state-space representations lie in polytopes.

A more general interesting case is the case when matrices of state-space representations are rational functions of several parametric or dynamic uncertainties [25]. These systems can be represented as Linear Fractional Transformations (LFT) of a structured uncertainty Δ block by a nominal system G [27]. The advantageous feature is that general linear systems with both parameter and dynamical uncertainties (including e.g. delays) can be represented as LFT [22], [21], [3]. This representation is actually one of the most general kind of representation [27].

In the LFT framework, the uncertain block is related to the nominal system using Integral Quadratic Constraints (IQCs) [14]. These IQCs allow to characterize a system using its input signals and its output signals. These IQCs provide us general analysis tools for the uncertain systems, such as stability tools, or performance analysis tools, using a Separation of Graph argument [20]. In [12], [25], the H_2 and H_∞ filtering problems are considered for parametric time-varying uncertainties. The proposed approach can be interpreted in terms of static (constant) IQCs. In [22], [21], the filtering problem is tackled using dynamic (frequency-dependent) IQCs in order to reduce the conservatism for time invariant parametric uncertainties with respect to the use of static IQCs. The problem is recast as an optimization problem involving LMIs using the Kalman-Yakubovitch-Popov Lemma [19]. The discrete time H_2 case has been solved with FIR filters [3]. In addition to be a more general approach of the robust filtering (due to rational parametric uncertainty dependence and the dynamical uncertainties), less conservative conditions are possibly obtained in the case of time-invariant uncertainties.

The major issue with all the previous approaches to the robust estimation problem is that the designed estimator is a filter. The proposed approach cannot be applied in the case of unstable systems. In some cases, this assumption is not satisfied as in [24]: it is then necessary to design an observer.

In this paper, we propose a new method for designing an L_2 -gain sub-optimal observer in the case of uncertain systems in the LFT framework using the IQC/Separation of

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Graph approach. The proposed results pertain to our long-standing effort for investigating the robust filtering problem: in [16] with the desensitivity problem, in [22], [3] where this approach is applied to the design of some robust filters and in [15] where the benefits of new observer structures were emphasized. To our best knowledge, results on robust estimation using observers for such general uncertain models do not exist.

To this purpose, we propose a new structure for robust estimation with structured uncertainties. This new structure, referred to as the Augmented-Gain Observer (AGO), can be interpreted as the combination of an observer structure and a filter one. It contains the nominal model of the system, and a frequency-dependent gain to tackle the sensitivity problem. With this structure, it is then possible to make robust estimation for some classes of unstable systems.

The structured representation of uncertainties is very general and we use dynamics IQCs used only in [22], [21] to reduce the conservatism. Restricting ourselves to parametric/dynamic Linear Time Variant/Invariant uncertainties, we present a theorem to synthesize a L_2 -gain sub-optimal Augmented-Gain Observer as a finite dimensional, convex optimization problem. Nevertheless, the result can be obtained for more general uncertainties (such as nonlinear ones).

Notations

$\|\Delta\|_{i_2}$ is the induced norm by the L_2 space of signals. A^* denotes the transpose conjugate of the complex matrix function A . A^T is its transpose. The \star denotes the Redheffer star product. $T_{w \rightarrow e}$ denotes the operator from w to e . The Kronecker product is noted \otimes . The identity matrix is written I . The dimensions are appropriate in the equations. $G = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ denotes the state space representation of $G(s) = C(sI - A)^{-1}B + D$. Finally, $\text{diag}(A, B)$ is a diagonal block matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. \bullet is a term of no importance, and \diamond denotes a symmetric element in a symmetric matrix. We also define the following expression $W(A, B, C, D, P, X, T) = \begin{bmatrix} A^T P + PA & PB \\ B^T P & T \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} X \begin{bmatrix} C^T \\ D^T \end{bmatrix}^T$.

II. THE ROBUST L_2 -GAIN OBSERVATION PROBLEM

A. Problem Definition

The uncertainty set is defined by :

$$\Gamma_\Delta = \left\{ \Delta \left| \begin{cases} \|\Delta\|_{i_2} < 1 \\ \Delta = \text{diag}(\text{diag}(\delta_i^{TV}), \text{diag}(\Delta_j^{TV}), \dots) \\ \dots, \text{diag}(\delta_k^{TI}), \text{diag}(\Delta_l^{TI}) \end{cases} \right. \right\} \quad (1)$$

- δ_i^{TV} is a time-variant real scalar
- Δ_j^{TV} is a time-variant Multi-Inputs Multi-Outputs (MIMO) operator
- δ_k^{TI} is a time-invariant real scalar
- Δ_l^{TI} is a time-invariant MIMO transfer function

The structure of the problem is presented in figure 1. The plant G is interconnected with the Δ operator. The output to

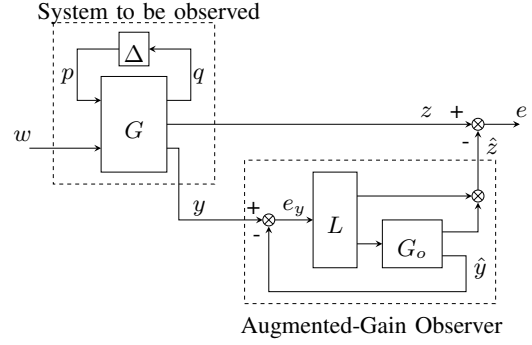


Fig. 1. The robust observation problem

be estimated is $z \in \mathbb{R}^{n_z}$, the measured output is $y \in \mathbb{R}^{n_y}$ and $p \in \mathbb{R}^{n_p}$ and $q \in \mathbb{R}^{n_q}$ are respectively the outputs and the inputs of the uncertainty bloc. The state space representation of the plant is:

$$G = \begin{bmatrix} \tilde{A} & \tilde{B}_p & \tilde{B}_w \\ \tilde{C}_q & \tilde{D}_{qp} & \tilde{D}_{qw} \\ \tilde{C}_z & \tilde{D}_{zp} & \tilde{D}_{zw} \\ \tilde{C}_y & \tilde{D}_{yp} & \tilde{D}_{yw} \end{bmatrix}. \quad (2)$$

The purpose of the observer is to estimate the output z from the measure of the input y . The minimal representation of

the nominal system (for $\Delta = 0$) is $G_n = \begin{bmatrix} A & B_w \\ C_z & D_{zw} \\ C_y & D_{yw} \end{bmatrix}$. The

state-space representations of the estimator blocks are given by $G_o = \begin{bmatrix} A & I \\ C_z & 0 \\ C_y & 0 \end{bmatrix}$ and $L = \begin{bmatrix} A_L & B_L \\ C_L & D_L \end{bmatrix}$.

The observer contains the nominal dynamics G_o of the plant and a comparison between the measured output and its estimation. This innovation term e_y is transmitted through the Augmented-Gain L . The original structure of Kalman [10] or Luenberger [13] can be retrieved by choosing A_L, B_L, C_L, D_L as empty matrices. The filter structure can be recovered by choosing C_L, D_L as zeros matrices and by withdrawing G_o the central model from the estimator and the corresponding output of the augmented gain [22].

The L_2 -gain observation problem is defined as follows: For a given $\gamma > 0$, find if there exist an augmented gain L such that

$$\forall \Delta \in \Gamma_\Delta, \|T_{w \rightarrow e}\|_{i_2} < \gamma \quad (3)$$

and compute it.

B. About the Augmented-Gain Observer

The choice of the estimation structure is developed in this part, to present how the AGO allows to take into account some classes of uncertain systems. Basically, the observer structure allows us to work on a minimal system for a observation problem. The states of this system represent the error between the states of the nominal plant and the states simulated in the observer. This form is obtained using the following lemma:

Lemma 1 (System reduction Lemma for observers):

Given G a matrix of transfer functions defined by the following state space representation:

$$G = \left[\begin{array}{cccc|c} A_1 & 0 & 0 & 0 & B_1 \\ 0 & A_2 & 0 & 0 & B_2 \\ 0 & C_2 & A_2 - C_2 & C_3 & B_3 \\ 0 & C_4 & -C_4 & A_3 & B_4 \\ \hline C_1 & 0 & 0 & 0 & D_1 \\ 0 & C_5 & -C_5 & C_6 & D_2 \end{array} \right].$$

Then G also admits the following state space representation:

$$G_{min} = \left[\begin{array}{ccc|c} A_1 & 0 & 0 & B_1 \\ 0 & A_2 - C_2 & C_3 & B_2 - B_3 \\ 0 & C_4 & A_3 & B_4 \\ \hline C_1 & 0 & 0 & D_1 \\ 0 & C_5 & C_6 & D_2 \end{array} \right].$$

Proof: The state-space transformation T is used:

$$T = \text{diag}(I, \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}, I).$$

The state space representation obtained is a Kalman form of observability, and can therefore be reduced from an observability point of view. The complete proof can be found in [13].

This lemma is applied on the robust observation problem (figure 1). The plant G defined equation (2) is separated in two minimal parts H and G_N ,

such that
$$\begin{bmatrix} q(j\omega) \\ z(j\omega) \\ y(j\omega) \end{bmatrix} = \begin{bmatrix} H(j\omega) \\ G_N(j\omega) \end{bmatrix} \begin{bmatrix} p(j\omega) \\ w(j\omega) \end{bmatrix}$$

The state-space representations are as follows:

$$H = \left[\begin{array}{ccc|c} A_H & B_{Hp} & B_{Hw} & \\ \hline C_H & D_{Hp} & D_{Hw} & \end{array} \right] \quad G_N = \left[\begin{array}{ccc|c} A & B_p & B_w & \\ \hline C_z & D_{zp} & D_{zw} & \\ C_y & D_{yp} & D_{yw} & \end{array} \right]$$

H is the part that acts on the uncertainty bloc. When $H = 0$ or $\Delta = 0$ there is no uncertainty on the model. G_N represents the nominal model augmented with the uncertainty entry p . This case recovers the nominal case, without uncertainties. A complete representation G_{tot} of the system withdrawing the uncertain block is:

$$\left[\begin{array}{cccc|c} A_H & 0 & 0 & 0 & B_H \\ 0 & A & 0 & 0 & B_G \\ 0 & D_L C_y & A - D_L C_y & C_L & D_L D_y \\ 0 & B_L C_y & -B_L C_y & A_L & B_L D_y \\ \hline C_H & 0 & 0 & 0 & D_H \\ 0 & C_z - D_d C_y & -C_z + D_d C_y & -C_d & D_z - D_d D_y \\ \hline B_H & B_{Hp} & B_{Hw} & & \\ B_G & B_p & B_w & & \\ D_L D_y & D_L D_{yp} & D_L D_{yw} & & \\ B_L D_y & B_L D_{yp} & B_L D_{yw} & & \\ \hline D_H & D_{Hp} & D_{Hw} & & \\ D_z - D_d D_y & D_{zp} - D_d D_{yp} & D_{zw} - D_d D_{yw} & & \end{array} \right] = \left[\begin{array}{ccc|c} B_H & B_{Hp} & B_{Hw} & \\ B_G & B_p & B_w & \\ D_L D_y & D_L D_{yp} & D_L D_{yw} & \\ B_L D_y & B_L D_{yp} & B_L D_{yw} & \\ \hline D_H & D_{Hp} & D_{Hw} & \\ D_{zp} - D_d D_{yp} & D_{zw} - D_d D_{yw} & & \end{array} \right]$$

Applying lemma 1, a minimal state space representation of G_r is:

$$\begin{bmatrix} H(j\omega) \\ G_m(j\omega) \end{bmatrix} = \left[\begin{array}{ccc|c} A_H & 0 & 0 & B_H \\ 0 & A - D_L C_y & C_L & B_G - D_L D_y \\ 0 & B_L C_y & A_L & B_L D_y \\ \hline C_H & 0 & 0 & D_H \\ 0 & C_z - D_d C_y & -C_d & D_z - D_d D_y \end{array} \right] \quad (4)$$

This minimal system is used to compute the augmented-gain.

III. UNCERTAIN SYSTEMS ANALYSIS

In order to synthesize the observer gain, analysis tools for the uncertain systems are needed. Those are given by the IQCs framework which allows to characterize a system through constraints over its inputs and its outputs. The constraints for LTV and LTI uncertainties are now presented. Associated to the set of uncertainties, the set of hermitian bounded matrices of transfer functions $\bar{\Pi}(\Gamma_\Delta)$ is presented.

$$\Pi(j\omega) = \begin{bmatrix} \Pi_s(j\omega) & \Pi_{as}(j\omega) \\ \Pi_{as}(j\omega)^* & -\Pi_s(j\omega) \end{bmatrix}$$

with the following definitions:

$\Pi_s(j\omega) = \text{diag}(\text{diag}(S_i), \text{diag}(d_j), \text{diag}(S_k(j\omega))\text{diag}(s_l(j\omega)))$
 $\Pi_{as}(j\omega) = \text{diag}(\text{diag}(Y_i), 0, \text{diag}[Y_k(j\omega)], 0)$ where $S_i = S_i^T > 0$, $Y_i = -Y_i^T$, $d_j > 0$, $S_k(j\omega) = S_k(j\omega)^* > 0$, $Y_k(j\omega) = -Y_k(j\omega)^*$, $s_l(j\omega) = s_l(j\omega)^* > 0$.

From [14], we get the following analysis result:

Theorem 2: Robust L_2 -Gain analysis

Let $M(j\omega)$ be LTI, for all $\Delta \in \Gamma_\Delta$, $\Delta \star M$ is stable with an L_2 -Gain less than γ if there exists $\Pi \in \bar{\Pi}(\Gamma_\Delta)$ and $\epsilon > 0$ such that for all ω :

$$\left[\frac{M(j\omega)}{I} \right]^* \Phi \left[\frac{M(j\omega)}{I} \right] \leq -\epsilon I \quad (5)$$

$$\Phi = \left[\begin{array}{cc|cc} \Pi_s(j\omega) & 0 & \Pi_{as}(j\omega) & 0 \\ 0 & I & 0 & 0 \\ \hline \Pi_{as}(j\omega)^* & 0 & -\Pi_s(j\omega) & 0 \\ 0 & 0 & 0 & -\gamma^2 I \end{array} \right]$$

For a given system and a given γ , to find $\Pi(j\omega)$ such that the condition (5) is satisfied is an optimization problem. But it is an infinite dimensional one, as all the possible transfer functions of every order are candidates. In order to get a finite-dimension optimization problem, the matrix of transfer function $\Pi(j\omega)$ is restricted to rational transfer function matrices with a fixed order in numerator and a fixed denominator $f(j\omega)$, a Hurwitz polynomial.

$$\Pi(j\omega) = \text{diag}(K(j\omega), K(j\omega))^* X \text{diag}(K(j\omega), K(j\omega))$$

The quadratic condition (5) can be recast as

$$\left[\begin{array}{cc|cc} K^*(j\omega) & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hline 0 & 0 & K^*(j\omega) & 0 \\ 0 & 0 & 0 & I \end{array} \right] \Phi_X \left[\begin{array}{cc|cc} K(j\omega) & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hline 0 & 0 & K(j\omega) & 0 \\ 0 & 0 & 0 & I \end{array} \right] \cdot (6)$$

with

$$K(j\omega) = \text{diag}(\text{diag}(I_i), \text{diag}(I_j), \dots \dots \text{diag}\left(\frac{E(j\omega)}{f(j\omega)} \otimes I_k\right), \text{diag}\left(\frac{E(j\omega)}{f(j\omega)} \otimes I_l\right)) \quad (7)$$

defined as :

$$\Phi_X = \left[\begin{array}{cc|cc} X_s & 0 & X_{as} & 0 \\ 0 & I & 0 & 0 \\ \hline X_{as}^T & 0 & -X_s & 0 \\ 0 & 0 & 0 & -\gamma^2 I \end{array} \right].$$

$E(j\omega)$ is a basis for the possible transfer functions:

$$E(j\omega) = [(j\omega)^n \quad (j\omega)^{n-1} \quad \dots \quad 1]^T.$$

We define the set of matrices $X \in \bar{X}(\bar{\Pi})$, which generates the transfer functions

- $X = \begin{bmatrix} X_s & X_{as} \\ X_{as}^T & -X_s \end{bmatrix}$
- $X_s = \text{diag}[\text{diag}(S_i), \text{diag}(d_j), \dots, \text{diag}(X_k^S), \text{diag}(I \otimes x_l)]$
- $X_{as} = \text{diag}[\text{diag}(G_i), 0_j, \text{diag}(X_k^G), 0]$

With this representation, the problem has a semi-finite dimension: the decision variables are no longer the $\Pi(j\omega)$ transfer functions, but the finite-dimensional matrices X_k^S and X_k^G . The number of constraints is still infinite as all the frequencies have to be tested. For the particular structure of X_k^S and X_k^G , an economical parametrization can be chosen, see [22].

As the quadratic condition is now defined, the analysis problem can be recast as a finite dimensional optimization problem under LMIs constraints using the Kalman-Yakubovitch-Popov Lemma [19].

In this part the fundamental analysis tools for uncertain systems have been presented with the steps to transform the analysis condition in a convex finite-dimensional optimization problem.

IV. MAIN RESULT

In this section, we present the theorem which allows to test the existence of an Augmented-Gain Observer (AGO) solution of the L_2 -gain observation problem presented section II-A. The proof is developed using the tools presented in section II-B and section III.

Theorem 3: For a given $\gamma > 0$, the robust observation problem has a solution if there exists $P_K = P_K^T$, $R = R^T$, $Z = Z^T > 0$, $X \in \bar{X}$ of appropriate dimensions such that the following conditions are satisfied.

- 1) $R - \text{diag}(Z, 0) > 0$
- 2)

$$U_{R\perp}^T \left[\begin{array}{c|c} W(A_R, B_R, C_\Delta, \dots) & \begin{bmatrix} C_R^T \\ D_R^T \end{bmatrix} \\ \hline \dots, D_\Delta, R, X, T(\gamma) & \begin{bmatrix} C_R \\ D_R \end{bmatrix} \\ \hline & -I \end{array} \right] U_{R\perp} < 0 \quad (8)$$

$$\text{with } U_{R\perp}^T = \text{diag}(I, I, I, [N_1^T \ N_2^T \ N_3^T], I).$$

$$[N_1^T \ N_2^T \ N_3^T] = [C_y \ D_{yp} \ D_y]^\perp \quad (9)$$

and $T(\gamma) = \text{diag}(0, -\gamma^2 I)$

- 3) $W(A_Z, B_Z, C_{\Delta Z}, D_\Delta, Z, X, T(\gamma)) < 0$
- 4) $W(A_K, B_K, C_K, D_K, P_K, X_s, 0) > 0$

with the following matrices:

$$\left[\begin{array}{c|c} A_Z & B_Z \\ \hline C_{\Delta Z} & D_\Delta \end{array} \right] = \left[\begin{array}{ccc|cc} A_H & 0 & 0 & B_{Hp} & B_{Hw} \\ B_K C_H & A_K & 0 & B_K D_{Hp} & B_K D_{Hw} \\ 0 & 0 & A_K & B_K & 0 \\ \hline D_K C_H & C_K & 0 & D_K D_{Hp} & D_K D_{Hw} \\ 0 & 0 & C_K & D_K & 0 \end{array} \right]$$

$$\left[\begin{array}{c|c} A_R & B_R \\ \hline C_\Delta & D_\Delta \\ \hline C_R & D_R \end{array} \right] = \left[\begin{array}{cc|cc} A_Z & 0 & B_Z & \\ 0 & A & B_p & B_w \\ \hline C_{\Delta Z} & 0 & D_\Delta & \\ 0 & C_z & D_{zp} & D_{zw} \end{array} \right]$$

$K = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array} \right]$ is the state space representation used for the IQC factorization presented equation (7). ■

These 4 conditions define a convex feasibility problem under LMI constraints. If they are satisfied, an observer with an Augmented-Gain exists such that the L_2 gain of the estimation error is less than γ . Minimizing over γ under these conditions gives an upper bound on the best performance reachable for the estimation error. Basically, the second condition can be interpreted as a generalization of the observability condition on the system. The third condition of the theorem ensures the stability of the block $H \star \Delta$ which is necessary for the whole system to be stable. This can be proven by applying the KYP lemma on this condition. The first condition with $Z > 0$ ensures that the P matrix introduced with the KYP lemma is defined positive, which is a condition for the system to be stable [6]. The augmented gain has yet to be reconstructed.

Proof: Applying theorem 2 to system defined equation (4), we have:

$$\left[\begin{array}{c|c} H(j\omega) \\ \hline I \ 0 \\ \hline G_m(j\omega) \\ \hline 0 \ I \end{array} \right]^* \left[\begin{array}{c|c} \Pi(j\omega) \ 0 \ 0 \\ \hline 0 \ I \ 0 \\ \hline 0 \ 0 \ -\gamma^2 I \end{array} \right] \left[\begin{array}{c|c} H(j\omega) \\ \hline I \ 0 \\ \hline G_m(j\omega) \\ \hline 0 \ I \end{array} \right] < -\epsilon I \quad (10)$$

This constraint can be recast with the factorization presented section III, equation (6) as:

$$\left[\begin{array}{c|c} K(j\omega)H(j\omega) \\ \hline K(j\omega) \ 0 \\ \hline G_m(j\omega) \\ \hline 0 \ I \end{array} \right]^* \left[\begin{array}{c|c} X \ 0 \ 0 \\ \hline 0 \ I \ 0 \\ \hline 0 \ 0 \ -\gamma^2 I \end{array} \right] \left[\begin{array}{c|c} K(j\omega)H(j\omega) \\ \hline K(j\omega) \ 0 \\ \hline G_m(j\omega) \\ \hline 0 \ I \end{array} \right] < -\epsilon I \quad (11)$$

The state space representation of the system is:

$$\left[\begin{array}{c|c} K(j\omega)H(j\omega) \\ \hline K(j\omega) \ 0 \\ \hline G_m(j\omega) \\ \hline 0 \ I \end{array} \right] = \left[\begin{array}{c|c} A_T & B_T \\ \hline C_T & D_T \end{array} \right] \quad (12)$$

$$\left[\begin{array}{c|c} A_T \\ \hline C_T \end{array} \right] = \left[\begin{array}{cccc|ccc} A_H & 0 & 0 & 0 & 0 & 0 & 0 \\ B_K C_H & A_K & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_K & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A - D_L C_y & C_L & & \\ 0 & 0 & 0 & B_L C_y & A_L & & \\ \hline D_K C_H & C_K & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_K & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_z - D_d C_y & -C_d & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$\left[\begin{array}{c|c} B_T \\ \hline D_T \end{array} \right] = \left[\begin{array}{cc|cc} B_{Hp} & B_{Hw} & & \\ B_K D_{Hp} & B_K D_{Hw} & & \\ B_K & 0 & & \\ B_p - D_L D_{yp} & B_w - D_L D_{yw} & & \\ B_L D_{yp} & B_L D_{yw} & & \\ \hline D_K D_{Hp} & D_K D_{Hw} & & \\ D_K & 0 & & \\ D_{zp} - D_d D_{yp} & D_{zw} - D_d D_y & & \\ 0 & 0 & & I \end{array} \right]$$

Using the KYP lemma [19], the condition (11) is equivalent to the existence of $P = P^T > 0$ such that the condition $W(A_T, B_T, C_T, D_T, P, \Phi_T, 0) < 0$ holds, with

$$\Phi_T = \text{diag} \left(\left[\begin{array}{c|c} X_s & X_{as} \\ \hline X_{as}^T & -X_s \end{array} \right], I, -\gamma^2 I \right)$$

Using a Schur complement ([4, page 7]), we have:

$$\begin{bmatrix} W(A_T, B_T, \begin{bmatrix} C_\Delta \\ 0 \end{bmatrix}, \dots) & \begin{bmatrix} (C_R - D_d C_y)^T \\ -C_d^T \\ (D_{zp} - D_d D_{yp})^T \\ (D_{zw} - D_d D_y)^T \end{bmatrix} \\ \hline \dots, D_\Delta, P, X, T(\gamma) & -I \end{bmatrix} < 0$$

This condition is a Bilinear Matrix Inequality, and has to be linearized. It could be written as:

$$\begin{aligned} & \overbrace{\begin{bmatrix} W \left(\begin{bmatrix} A_R^T & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} B_R \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} C_R^T \\ 0 \\ D_R^T \end{bmatrix}^T \right) \\ \hline \dots, D_\Delta, P, X, T(\gamma) & -I \end{bmatrix}}^{\Phi_G} + \dots \\ & + V \overbrace{\begin{bmatrix} A_L & B_L \\ C_L & D_L \\ C_d & D_d \end{bmatrix}}^{\Phi_L} \overbrace{\begin{bmatrix} U_1 & U_2 & 0 \end{bmatrix}}^U + \dots \\ & \dots + U^T \Phi_L^T \overbrace{\begin{bmatrix} V_1 P & 0 & V_2 \end{bmatrix}}^V < 0 \end{aligned} \quad (13)$$

$$\text{with } [V_1 | V_2] = \begin{bmatrix} 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & -I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -I \end{bmatrix} \text{ and}$$

$$[U_1 | U_2] = \begin{bmatrix} 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & C_y & 0 & D_{yp} & D_y \end{bmatrix}.$$

We can apply the elimination lemma [4, page 22]. The basis of the nullspaces of the matrices V and U presented equation (13) are:

- $V_\perp = \text{diag}(P^{-1}, I, I) \begin{bmatrix} \text{diag}(I, I, I, 0) \\ 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 \end{bmatrix}$
- $U_\perp^T = \text{diag}(I, I, I, [N_1^T \ 0 \ N_2^T \ N_3^T], I)$

The partitions of the P matrix and P^{-1} are introduced.

$$P = \begin{bmatrix} R & \bullet^T \\ \bullet & \bullet \end{bmatrix}, P^{-1} = \begin{bmatrix} \hat{R} & \bullet^T \\ \bullet & \bullet \end{bmatrix}, \hat{R} = \begin{bmatrix} Z^{-1} & \hat{R}_2 \\ \hat{R}_2^T & \hat{R}_3 \end{bmatrix}$$

The first constraint $U_\perp^T \Phi_G U_\perp < 0$ gives by applying the elimination lemma and due of the line of zeros in U_\perp the second condition of the theorem.

The second constraint $V_\perp^T \Phi_G V_\perp < 0$ could be simplified due to the zeros on the external terms to:

$$W(A_Z Z^{-1}, B_Z, Z^{-1} C_{\Delta Z}, D_\Delta, I, X, T(\gamma)) < 0$$

Then, using a congruent multiplication by $J = \text{diag}(Z, I)$ we get the third condition of the theorem.

Finally, the use of the elimination lemma introduces a rank constraint over the partitions of the Lyapunov matrix and its inverse [6]:

$$\begin{bmatrix} R & I \\ I & Z^{-1} \hat{R}_2 \\ & \hat{R}_2^T \hat{R}_3 \end{bmatrix} > 0 \quad (14)$$

but as the partition \hat{R}_2 and \hat{R}_3 only appears in this condition, \hat{R}_3 can be eliminated using the elimination lemma once again, and then a Schur complement give us the first condition of theorem $R - \text{diag}(Z, 0) > 0$.

The transfer function Π_s is defined positive:

$$K(j\omega)^* X_s K(j\omega) > \epsilon I$$

The application of the KYP lemma gives the fourth condition [19]. ■

V. OBSERVER RECONSTRUCTION

Reconstructing the augmented-gain of the observer from the equation (13) is an optimization problem under LMI constraints. But the original P matrix introduced by the KYP lemma of this problem has to be reconstructed first. As Z^{-1} is a partition of \hat{R} , itself a partition of P^{-1} , find \hat{R} is a feasibility problem under LMI constraints, and the constraint is given by the equation (14). P is obtained using the following parametrization:

$$P = \begin{bmatrix} I & 0 \\ 0 & M^T \end{bmatrix} \begin{bmatrix} R & I \\ I & (R - \hat{R}^{-1})^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}$$

Once P is obtained, to find the state-space representation of the augmented gain can be done solving an optimization problem given by the constraint (13). As P, X are known, this constraint is an LMI constraint and the reconstruction can be made as a feasibility problem.

VI. EXAMPLE

In this section, we present two numerical applications of the AGO. We first compare it with the other methods using dynamic IQCs, on an example of the literature. We then present an example of random walk [24], where the previous methods do not lead to feasible conditions.

We first consider the application presented in [21], for the synthesis of an L_2 -Gain estimator. We compare the methods presented in [22], [21], which gives equivalent results with different conditions, and our method, with the same basis for the dynamic IQCs. The denominator is chosen as $f = (s + 0.01)^n$, with n varying from 0 (static IQCs) to 5. The results are presented table I. The results proves that the AGO, recovers the methods presented in the same framework.

Method	$n = 0$	$n = 1$	$n = 3$	$n = 5$
[22], [21]	4.54	3.08	2.64	2.64
AGO	4.54	3.08	2.64	2.64

TABLE I
COMPARISON OF METHODS

We present an example of uncertain random walk. The system is defined ad follows:

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} \left(\frac{a_\delta}{s^2 + b a_\delta s + a_\delta} \right) \\ \left(\frac{1}{s} \right) \left(\frac{a_\delta}{s^2 + b a_\delta s + a_\delta} \right) \end{bmatrix} w.$$

We want to estimate z , from the output y . This is a first differentiation of y . A white noise $v(t)$ is added on the output, with a factor 0.1. The second order transfer function is dependent from the parameter a_δ which nominal value a_n is set to 1. This parameter present a variation of $a_{gab}\%$. This variation acts on the natural frequency of the second order filter. Note that as the systems contains a non-strictly stable pole, methods for synthesis of robust filters do not lead to feasible conditions.

The state space representation of the uncertain random walk, as an LFT representation is presented equation (15).

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ q(t) \\ z(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -ba_n & -1 & 0 & -b & 1 & 0 \\ a_n & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ a_{gab} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & d \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ p(t) \\ w(t) \\ v(t) \end{bmatrix} \quad (15)$$

This example is tractable with our method. In order to illustrate the interest of dynamic IQCs, we apply theorem 3 on this system with static and dynamic IQCs. For the dynamic IQCs, the denominator is chosen as $f(s) = (s+0.1)^n$, for $n = 1, 2, 3$. We compare the upper-bounds on the L_2 -gain performance reached with those four cases (static IQCs, dynamic IQCs with $n = 1, 2, 3$) versus a_{gab} .

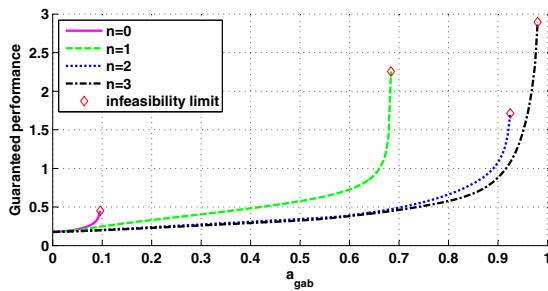


Fig. 2. Upper-bound on the performance versus r_{gab}

Without uncertainties, we recover performance reached by the nominal H_∞ observer. When the uncertainty grows, the constraints become infeasible depending on the conservatism of the method. The use of static IQC ($n = 0$) allows to take into account uncertainties on the parameter about 10%. The use of dynamic IQCs allows to take into account uncertainties 9 times larger. These two examples prove that our observer gives the same performance as methods previously presented in this framework, but also allows to take into account cases non tractable with these methods.

VII. CONCLUSION

In this paper, we present a new structure: the Augmented-Gain Observer, which allows to observe uncertain systems such as systems with LTV/LTI uncertainties, or some unstable uncertain systems. This structure is very general, and encompasses most of the previous structures. We have presented the solution of the L_2 -gain sub-optimal Augmented-Gain Observer based on finite dimensional optimization, and a separation of graphs theorem. It has also been proved that the resolution presented could be used for other structures, such as static-gain observers. This solution is still conservative, but the conservatism could be reduced by increasing the order of the dynamic IQCs $\Pi(j\omega)$ at the cost of the number of variables. This resolution paves the way to a general approach of robust estimation for uncertain LFT systems. One of the closest perspectives is to compute an H_2 sub-optimal augmented-gain observer.

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