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Almost sure stability of Markov jump linear systems with dwell-time constrained switching dynamics

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Abstract— **The stability of a class of Markov jump linear systems (MJLS) characterized by constant transition rates and piecewise-constant system dynamics is investigated. For these Switching Dynamics Markov jump linear systems (SD-MJLS), almost sure exponential stability (ASE-stability) is analyzed by applying the ergodic law of large numbers under the assumption that suitable average contractivity conditions are satisfied. The main result is a sufficient condition that guarantees ASE-stability under constraints on the dwell-time between switching instants.**

I. Introduction

The study of hybrid systems, characterized by the interconnection of logical and continuous dynamics, is motivated by a variety of applications in diverse fields [7], [26], [24], [15], [19], [9], [1]. The commutations between logical variables can be given either a probabilistic or deterministic description.

In the former case, an important class of hybrid systems is that of Markov jump systems whose logical states are subject to stochastic jumps governed by a Markov chain. The stability analysis of these systems is a particular case of stability analysis of random systems, see [21], [22], [23]. It is important to note that there exist several different notions of stochastic stability including mean square, δ-moment, and almost sure stability. Even for the restricted class of Markov jump linear systems (MJLS), numerous works were devoted to the derivation of sufficient and/or necessary conditions for different types of stability [17], [16], [14], [8], [12], [3], [6], [4], [29], [28]. Recent stability results for neural Markov jump network systems are reported in [30], [31]. It is worth observing that mean square stability is somehow easier to analyze since Lyapunov-type equations can be employed. A less conservative and more useful notion is almost sure stability as it guarantees (with probability one) the convergence to zero of sample paths of the state trajectory. However, almost sure stability results are generally more difficult to establish, as the determination of the sign of the top Lyapunov exponent is usually a rather difficult task [2], [13]. A possible way to prove almost sure stability relies on the ergodic law of large numbers under the assumption that suitable average contractivity properties hold. In particular, in [14] almost sure exponential (ASE) stability was established under an average instantaneous contractivity condition. More recently, sufficient conditions, based on average contractivity on a finite interval or over a single jump, were established [3], [4], [28].

When it is not possible, or desirable, to assume a probabilistic model, the evolution of the logical state can be described as a deterministic switching signal. This might be the case when commutations are not governed by nature but orchestrated by a supervisor attempting to improve or optimize system performance. In this case, the analysis aims to guarantee stability for all switching signals belonging to a suitable class. In particular, it is rather natural to impose dwell-time constraints to the switching signal, meaning that there exists a minimum dwell-time between two consecutive switching instants. Some results on the stability under this kind of constraint are reported in [25], [27], [20], [18].

In the present paper, we consider hybrid systems subject to both stochastic jumps and deterministic switches. Given that Markov jumps are an appropriate model for random faults and unexpected events, these stochastic-deterministic hybrid systems are well suited to describe the dynamics of fault-prone systems managed by a supervisor whose actions are represented by deterministic switches. Stochastic stability of switching MJLS was first studied in [5], where sufficient conditions for mean square stability were proven. As already mentioned, almost sure stability is less restrictive than mean square stability, but is also more difficult to prove. The main purpose of the present paper is to derive an easy-to-check sufficient condition ensuring ASE stability of a class of switching MJLS under dwell-time constraints on the deterministic switching signal. In particular, we consider deterministic switchings between MJLS that share the underlying Markov chain. These systems are dubbed as Switching Dynamics Markov jump linear systems (SD-MJLS). The extension to MJLS with different underlying Markov chains is nontrivial because the uniqueness of the Markov chain is instrumental to the application of the ergodic law of large numbers, that represents the key point of the stability proof. The obtained ASE stability condition is easily verified on the basis of a few scalars providing bounds on the norm of the transition matrices of the subsystems. As a corollary, it is shown that, under average instantaneous contractivity, ASE stability holds for an arbitrarily short dwell-time.

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Notation Throughout the paper, $\|A\|$ will denote the usual spectral norm of A, i.e. its maximum singular value. The notation $A^{[\gamma(t)]}$ will denote the dependence of matrix A on the piecewise constant switching signal $\gamma(t)$. Conversely, the notation $A_{\sigma(t)}$ will denote the dependence of matrix A on the Markovian form process $\sigma(t)$.

II. Preliminaries and problem formulation

A continuous-time Markov jump linear system (MJLS) is a stochastic system described by the state equation

$$\dot{x}(t) = A_{\sigma(t)}x(t) \quad , \quad t \geq 0 \tag{1}$$

where $x(t) \in I\!R^n$, and the form process $\sigma(t)$ is a finite-state, time homogeneous, Markov stochastic process taking values in a finite set $\mathcal{S} = \{1, 2, \ldots, N\}$, with stationary transition probabilities

$$\Pr\{\sigma(t+h) = j|\sigma(t) = i\} = \lambda_{ij}h + o(h) \quad , \quad i \neq j \tag{2}$$

where $h > 0$, and $\lambda_{ij} \geq 0$ is the transition rate from mode i at time t to mode j at time $t+h$. Letting

$$\lambda_{ii} = -\sum_{j=1, j\neq i}^{N} \lambda_{ij}$$

and defining $\Lambda = [\lambda_{ij}]$, the matrix Λ is called the transition rate matrix of the Markov process.

Let $p_i(t) = \Pr\{\sigma(t) = i\}$ and

$$p(t) = \left[\begin{array}{ccc} p_1(t) & \ldots & p_N(t) \end{array}\right]'$$

Given an initial probability distribution

$$p_0 = \left[\begin{array}{ccc} p_{01} & \ldots & p_{0N} \end{array}\right]'$$

where $p_{0i} := \Pr\{\sigma(0) = i\}$, the probability distribution $p(t)$ obeys the differential equation

$$\dot{p}(t)' = p(t)'\Lambda \tag{3}$$

Hereafter, we assume that the Markov chain is recurrent, irreducible and aperiodic, so that ergodicity is guaranteed, see e.g. [9]. Under this assumption, it is well known that the solution of (3) asymptotically converges to a constant vector

$$\pi = \left[\begin{array}{ccc} \pi_1 & \ldots & \pi_N \end{array}\right]'$$

representing the stationary probability distribution of the logical state.

The class of systems studied in this paper is given by MJLS with switching dynamics (SD-MJLS). More precisely, consider a set of M MJLS sharing the same state order n, the same number of modes N, and the same underlying Markov chain, but having different dynamic matrices. The overall dynamics undergoes deterministic, yet unknown, switching between the M elements of this set. More formally, we consider the system

$$\dot{x}(t) = A_{\sigma(t)}^{[\gamma(t)]}x(t) \quad , \quad t \geq 0 \tag{4}$$

where $x(t) \in I\!R^n$ and the stochastic jumping signal $\sigma(t)$ is a form process, taking values in the finite set $\mathcal{S} = \{1, 2, \ldots, N\}$, with transition rate matrix Λ. The deterministic switching signal $\gamma(t) \in \mathcal{M} = \{1, 2, \ldots, M\}$ is an exogenous piecewise constant function.

We are interested in the stability properties of the SD-MJLS (4). Since the system is stochastic, different notions of stability can be used, see e.g. [16].

Definition 1: For a given switching signal $\gamma(t)$, the SD-MJLS (4) is said to be mean square stable (MS-stable) if

$$\lim_{t\to\infty} E[\|x(t)\|^2] = 0$$

for any initial condition $x(0)$ and any initial probability distribution $p(0)$.

Definition 2: For a given switching signal $\gamma(t)$, the SD-MJLS (4) is said to be almost sure exponentially stable (ASE-stable) if there exists $\rho > 0$ such that, for any $x(0) \in I\!R^n$ and any initial distribution $p(0)$, it results that

$$Pr\{\limsup_{t\to\infty} \frac{1}{t} \ln \|x(t)\| \leq -\rho\} = 1$$

□

It is immediate to see that Definition 2 implies that $Pr\{\lim_{t\to\infty} \|x(t)\| = 0\} = 1$, i.e. almost all realizations of the stochastic process $x(t)$ converge to zero. As such, ASE-stability is related to sample-path stability properties and is usually regarded as the most appropriate stability notion for practical purposes. Conversely, MS-stability is an ensemble property but has the advantage of being more easily assessed. For non-switching MJLS, it is known that MS-stability implies ASE-stability [11]. It can be easily seen that the same holds for SD-MJLS.

Checking ASE-stability of a MJLS involves the determination of the sign of the top Lyapunov exponent, which is usually a rather difficult task [2], [13]. This motivated the development of easy-to-check sufficient conditions for ASE-stability, based on the notion of average contractivity (namely the expectation of the logarithm of the norm of the transition matrix over a suitable interval is negative). A first result in this direction hinges on the definition of AINC (Average Instantaneous Norm-Contractivity), based on the notion of matrix measure $\mu(A)$ of a square matrix (see e.g. [10]). The matrix measure $\mu(A)$ is defined as

$$\mu(A) = \lim_{t\to 0} \frac{\|I + At\| - 1}{t}$$

where I is the identity matrix. Note that $\mu(A)$ is the derivative of the norm of $\exp(At)$ at $t = 0$. It is easy to see that, if $\mu(A) < 0$, then A is "instantaneously norm-contractive", in the sense that $\|\exp(At)\| \leq \exp(\mu(A)t), \forall t \geq 0$. In [14], the following sufficient condition for ASE-stability can be found.

Theorem 1: If $\sum_{i=1}^{N} \pi_i\mu(A_i) < 0$, then the MJLS (1) is ASE-stable. □

Such a condition is tantamount to requiring that

$E[\mu(A_\sigma)] < 0$, namely that the system is "averagely instantaneously norm-contractive" (AINC).

A second less conservative condition for ASE-stability was originally developed in [28], and it is here recalled with slightly different notation.

Theorem 2: Consider the MJLS (1) and let $\alpha_i \geq 0$, and β_i, $i \in \mathcal{S}$ be such that

$$\|e^{A_i t}\| \leq e^{\alpha_i - \beta_i t}, \quad t \geq 0$$

Then, the system is ASE-stable if

$$\sum_{i=1}^{N} (\lambda_{ii}\alpha_i + \beta_i)\, \pi_i > 0$$

□

The above condition is sufficient to ensure average norm-contractivity of the system over a single jump of the underlying Markov chain (Average One-Jump Norm-Contractivity, AOJNC). It is easily seen that the sufficient condition of Theorem 1 implies the fulfillment of the sufficient condition of Theorem 2, with $\alpha_i = 0$, $\beta_i = -\mu_i$, $\forall i \in \mathcal{S}$.

In this paper we restrict our attention to the class of switching signals $\gamma(t)$ with minimum dwell-time $T \geq 0$, i.e.

$$\Gamma_T = \{\gamma(\cdot) | t_{k+1} - t_k \geq T\}$$

where $t_0, t_1, t_2, \ldots$ denote the switching time instants of the deterministic switching signal. Hereafter, without loss of generality, we take $t_0 = 0$.

Our aim is to derive sufficient conditions to guarantee that the SD-MJLS (4) is ASE-stable for all $\gamma(t) \in \Gamma_T$. In this case, we will say that the system is ASE-stable in Γ_T. The obtained conditions will lead to a computable upper bound of the minimum dwell-time ensuring ASE-stability. Since the set Γ_T includes all constant signals $\gamma(t) = i, \forall t, i \in \mathcal{M}$, a necessary condition for ASE-stability in Γ_T is that all the M individual MJLS are ASE-stable.

III. ASE-stability with dwell-time

In order to derive the main result, we first introduce some useful notation. For each $A_i^{[j]}$, $i \in \mathcal{S}$, $j \in \mathcal{M}$, let $\alpha_i^{[j]} \geq 0$, and $\beta_i^{[j]}$ be two real constants such that

$$\|e^{A_i^{[j]} t}\| \leq e^{\alpha_i^{[j]} - \beta_i^{[j]} t}, \quad t \geq 0$$

Moreover define

$$\eta^{[j]} = \sum_{i=1}^{N} \left(\lambda_{ii}\alpha_i^{[j]} + \beta_i^{[j]}\right) \pi_i$$

and

$$\tilde{\eta}^{[j]} = \sum_{i=1}^{N} \left(\lambda_{ii}\bar{\alpha}_i + \beta_i^{[j]}\right) \pi_i \tag{5}$$

where $\bar{\alpha}_i = \max_j \alpha_i^{[j]}$. According to Theorem 2, the condition $\eta^{[j]} > 0$ is sufficient to guarantee ASE-stability of the j-th MJLS. Moreover, by recalling that $\lambda_{ii} < 0$, it is immediate to see that $\tilde{\eta}^{[j]} > 0$ implies that $\eta^{[j]} > 0$, and hence ASE-stability of the j-th MJLS.

We are now in a position to prove the main result of the paper.

Theorem 3: If

$$\tilde{\eta}^{[j]} > 0, \forall j \in \mathcal{M}, \quad T > \frac{\max_i \bar{\alpha}_i}{\min_j \tilde{\eta}^{[j]}}$$

then the SD-MJLS (4) is ASE-stable in Γ_T. □

Proof: For a given switching signal $\gamma(t)$, let $\Psi(t,0)$ be the transition matrix of the stochastic system (4). The thesis will be proved by showing that the associated top Lyapunov exponent is negative, i.e.

$$\limsup_{t\to\infty} \frac{1}{t} \ln \|\Psi(t,0)\| < 0, \quad \text{a.s.} \tag{6}$$

Assume that k switches occur in the interval $(0,t)$, and let $\Phi^{[j]}(t,\tau)$ be the stochastic transition matrix of the j-th MJLS. Note that

$$\begin{aligned}
\ln\|\Psi(t,0)\| &\leq \ln\|\Phi^{[\gamma(t_k)]}(t,t_k)\| \qquad (7)\\
&\quad + \sum_{m=0}^{k-1} \ln\|\Phi^{[\gamma(t_m)]}(t_{m+1},t_m)\| \\
&\leq \sum_{i=1}^{N} \left(\alpha_i^{[\gamma(t_k)]} N_i(t,t_k) - \beta_i^{[\gamma(t_k)]} T_i(t,t_k)\right) \\
&\quad + \sum_{m=0}^{k-1}\sum_{i=1}^{N} \alpha_i^{[\gamma(t_m)]} N_i(t_{m+1},t_m) \\
&\quad - \sum_{m=0}^{k-1}\sum_{i=1}^{N} \beta_i^{[\gamma(t_m)]} T_i(t_{m+1},t_m) \qquad (8)
\end{aligned}$$

where $N_i(t,\tau)$ represents the number of activations of mode i in the interval $[\tau,t)$, and $T_i(t,\tau)$ represents the cumulative residence time of mode i in the interval $[\tau,t)$. Notice that $N_i(t,\tau)$ and $T_i(t,\tau)$ do not depend on $\gamma(t)$ because the underlying Markov process is unique.

It is useful to introduce the following indicator functions:

$$J_i(t) = \begin{cases} 1, & \sigma(t) = i \\ 0, & \sigma(t) \neq i \end{cases}, \quad I^{[j]}(t) = \begin{cases} 1, & \gamma(t) = j \\ 0, & \gamma(t) \neq j \end{cases}$$

Moreover, let

$$r^{[j]}(t) = \frac{1}{t}\int_0^t I^{[j]}(\tau)d\tau$$

denote the fraction of time in the interval $[0,t]$ when the j-th MJLS is active. Finally, let

$$T_i^{[j]}(t) = \int_0^t I^{[j]}(\tau) J_i(\tau) d\tau$$

be the cumulative residence time of mode i when the j-th MJLS is active.

By rearranging the terms in the sums of (7), we obtain

$$\begin{aligned}
\ln\|\Psi(t,0)\| &\leq \sum_{i=1}^{N}\alpha_i^{[\gamma(t_k)]}N_i(t,t_k) \\
&\quad +\sum_{i=1}^{N}\sum_{m=0}^{k-1}\alpha_i^{[\gamma(t_m)]}N_i(t_{m+1},t_m) \\
&\quad -\sum_{i=1}^{N}\beta_i^{[\gamma(t_k)]}T_i(t,t_k) \\
&\quad -\sum_{i=1}^{N}\sum_{m=0}^{k-1}\beta_i^{[\gamma(t_m)]}T_i(t_{m+1},t_m) \\
&\leq \sum_{i=1}^{N}\bar{\alpha}_i\left(N_i(t,t_k)+\sum_{m=0}^{k-1}N_i(t_{m+1},t_m)\right) \\
&\quad -\sum_{i=1}^{N}\sum_{j=1}^{M}\beta_i^{[j]}T_i^{[j]}(t)
\end{aligned}$$

Note that, almost surely,

$$N_i(t,t_k)+\sum_{m=0}^{k-1}N_i(t_{m+1},t_m) > N_i(t,0)$$

due to the fact that, with probability 1, $\sigma(t)$ does not change across the switching instants t_m. Hence, the summation on the left hand side contains extra terms with respect to $N_i(t,0)$. As a matter of fact,

$$N_i(t,t_k)+\sum_{m=0}^{k-1}N_i(t_{m+1},t_m) = N_i(t,0)+k_i, \quad \sum_{i=1}^{N}k_i = k$$

where k_i is the number of switching instants t_m occurring when the mode i is active. Hence

$$\begin{aligned}
\ln\|\Psi(t,0)\| &\leq \sum_{i=1}^{N}\bar{\alpha}_i\left(N_i(t,0)+k_i\right)+ \\
&\quad -\sum_{i=1}^{N}\sum_{j=1}^{M}\beta_i^{[j]}T_i^{[j]}(t) \\
&\leq k\bar{\alpha}_{max}+\sum_{i=1}^{N}\bar{\alpha}_iN_i(t,0)+ \\
&\quad -\sum_{i=1}^{N}\sum_{j=1}^{M}\beta_i^{[j]}T_i^{[j]}(t)
\end{aligned}$$

where $\bar{\alpha}_{max}=\max_i\bar{\alpha}_i$.

As proved in Lemma 1, when the Markov process is at steady state, it holds that

$$\begin{aligned}
E[N_i(t,\tau)] &= \pi_i-\pi_i\lambda_{ii}(t-\tau) \\
E[T_i(t,\tau)] &= \pi_i(t-\tau) \\
E[T_i^{[j]}(t)] &= r^{[j]}(t)\pi_i t
\end{aligned}$$

Observing that $\gamma(t)\in\Gamma_T$ implies that $k\leq t/T$, and exploiting the ergodic law of large numbers, one obtains that, with probability 1,

$$\begin{aligned}
\limsup_{t\to\infty}\frac{1}{t}\ln\|\Psi(t,0)\| &\leq \frac{\bar{\alpha}_{max}}{T}-\sum_{i=1}^{N}\bar{\alpha}_i\pi_i\lambda_{ii}+ \\
&\quad -\sum_{i=1}^{N}\pi_i\sum_{j=1}^{M}\beta_i^{[j]}\bar{r}^{[j]} \qquad (9)
\end{aligned}$$

where

$$\bar{r}^{[j]}=\limsup_{t\to\infty}r^{[j]}(t)\leq 1$$

It is also easily seen that

$$\sum_{j=1}^{M}\bar{r}^{[j]}\geq 1 \qquad (10)$$

It is only left to show that the right hand side of inequality (9) is strictly negative. To this purpose, note that the assumptions of the Theorem, inequality (10) and the definition (5) of $\tilde{\eta}^{[j]}$ entail that

$$\begin{aligned}
\frac{\bar{\alpha}_{max}}{T} &< \min_j\tilde{\eta}^{[j]}\leq\sum_{j=1}^{M}\bar{r}^{[j]}\tilde{\eta}^{[j]}= \\
&= \sum_{j=1}^{M}\bar{r}^{[j]}\sum_{i=1}^{N}\bar{\alpha}_i\pi_i\lambda_{ii}+\sum_{i=1}^{N}\pi_i\sum_{j=1}^{M}\beta_i^{[j]}\bar{r}^{[j]}
\end{aligned}$$

Since $\bar{\alpha}_i\pi_i\lambda_{ii}<0,\forall i$ and recalling (10), then

$$\frac{\bar{\alpha}_{max}}{T}<\sum_{i=1}^{N}\pi_i\left(\bar{\alpha}_i\lambda_{ii}+\sum_{j=1}^{M}\beta_i^{[j]}\bar{r}^{[j]}\right)$$

so that, from (9), it results that

$$\limsup_{t\to\infty}\frac{1}{t}\ln\|\Psi(t,0)\|<0, \quad \text{a.s.,} \quad \forall\gamma(t)\in\Gamma_T$$

and the proof is completed. □

Remark 1: A special case occurs when all MJLS are AINC (averagely instantaneously norm-contractive), see Theorem 1. Then, by applying Theorem 3 with $\alpha_i^{[j]}=0$, $\beta_i^{[j]}=-\mu(A_i^{[j]})$, $\forall i,j$, it turns out that the SD-MJLS is ASE-stable $\forall T>0$, i.e. under arbitrary switching.

Conversely, when all MJLS are AOJNC (averagely one-jump norm-contractive), see Theorem 2, the fulfillment of condition $\tilde{\eta}^{[j]}>0,\forall j\in\mathcal{M}$ of Theorem 3 is not guaranteed. Comparing $\tilde{\eta}^{[j]}$ and $\eta^{[j]}$, it appears that our main Theorem needs a slightly stronger assumption on the properties of the individual MJLS.

IV. Numerical example

Consider the SD-MJLS (4) with $M=2$, $N=2$, $n=2$, and

$$\begin{aligned}
A_1^{[1]} &= \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_2^{[1]}=\begin{bmatrix} -5 & 1 \\ 0 & -2 \end{bmatrix} \\
A_1^{[2]} &= \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \quad A_2^{[2]}=\begin{bmatrix} -5 & 0 \\ 0 & -2 \end{bmatrix}
\end{aligned}$$

Let the transition matrix of the Markov process $\sigma(t)$ be given by

$$\Lambda = \begin{bmatrix} -1 & 1 \\ 0.96 & -0.96 \end{bmatrix}$$

so that the stationary probability distribution of the logical state is $\pi = [0.49 \quad 0.51]'$. The matrix measures of the matrices $A_i^{[j]}$ are

$$\mu(A_1^{[1]}) = 2, \quad \mu(A_2^{[1]}) = -1.92$$
$$\mu(A_1^{[2]}) = 1.12, \quad \mu(A_2^{[2]}) = -2$$

Moreover, it can be verified that suitable values for $\alpha_i^{[j]}$ and $\beta_i^{[j]}$ ar given by

$$\alpha_1^{[1]} = 0, \quad \alpha_2^{[1]} = 0.058, \quad \beta_1^{[1]} = -2, \quad \beta_2^{[1]} = 2$$
$$\alpha_1^{[2]} = 0.010, \quad \alpha_2^{[2]} = 0, \quad \beta_1^{[2]} = -1.12, \quad \beta_2^{[2]} = 2$$

It turns out that the condition of Theorem 1 is satisfied for the second MJLS but not for the first one. Hence, it is not true that both MJLS are AINC. However, by applying Theorem 2, it can be shown that both MJLS are AOJNC, so that the procedure presented in section III can be used to establish the existence of a dwell time for which ASE-stability of the SD-MJLS is guaranteed. To this purpose, compute $\tilde{\eta}^{[j]}$, for $j = 1, 2$, recalling that $\bar{\alpha}_1 = 0.010$, $\bar{\alpha}_2 = 0.058$. Since it turns out that

$$\tilde{\eta}^{[1]} = 0.0067 > 0, \quad \tilde{\eta}^{[2]} = 0.4379 > 0$$

the first condition of Theorem 3 is satisfied. According to this theorem, the SD-MJLS is ASE-stable in Γ_T, with $T > 0.058/0.0067 = 8.66$.

In summary, both MJLS are ASE-stable, but ASE-stability of the overall system can be guaranteed only if the switches of the deterministic signal $\gamma(t)$ are sufficiently far apart in time. Of course, since the conditions of Theorem 3 are only sufficient, the actual minimum dwell-time preserving stability may well be lower than the calculated bound.

V. Concluding remarks

In this paper almost sure exponential stability for Markov jump linear systems subject to deterministic switching dynamics has been investigated. A sufficient condition for stability under a dwell-time constraint has been provided. The proof is based on the ergodic law of large numbers and exploits average contractivity properties of each individual Markov jump subsystem. The extension to switching probabilities is nontrivial due to the impossibility of resorting to ergodic arguments, and, as such, is an open and challenging research topic.

Acknowledgments

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Appendix

Lemma 1: Consider the SD-MJLS (4), where $\sigma(t)$ is a form process, taking values in the finite set $\mathcal{S} = \{1, 2, \ldots, N\}$, where Λ is the transition rate matrix and π_i denote the stationary probability of mode i. The function $\gamma(t) \in \mathcal{M} = \{1, 2, \ldots, M\}$ is a deterministic switching signal. Moreover recall that

$N_i(t, \tau)$ is the number of activations of mode i in the interval $[\tau, t)$;

$T_i(t, \tau)$ is the cumulative residence time of mode i in the interval $[\tau, t)$;

$r^{[j]}(t)$ is the fraction of time in the interval $[0, t]$ when the j-th MJLS is active;

$T_i^{[j]}(t)$ is the cumulative residence time of mode i when the j-th MJLS is active.

Then, when the Markov process is in stationary conditions,

(i) $E[T_i(t, \tau)] = \pi_i(t - \tau)$
(ii) $E[N_i(t, \tau)] = \pi_i - \pi_i \lambda_{ii}(t - \tau)$
(iii) $E[T_i^{[j]}(t)] = r^{[j]}(t)\pi_i t$

Proof: The proof of (i) is immediate in view of the very definition of π_i.

To prove (ii), observe that, in stationary conditions,

$$E[N_i(t, \tau)|T_i(t, \tau)] = -\lambda_{ii} T_i(t, \tau) + \pi_i$$

As a matter of fact, the first term on the right-hand side is the average number of transitions out of mode i within the interval $[\tau, t]$ (recalling that these transitions are Poisson events). The second term takes into account the probability that the i-th mode is active at the final time t of the interval. Now, the proof follows from the total probability theorem:

$$\begin{aligned} E[N_i(t, \tau)] &= \int E[N_i(t, \tau)|T_i(t, \tau)] dp(T_i) \\ &= -\lambda_{ii} E[T_i(t, \tau)] + \pi_i \\ &= \pi_i - \pi_i \lambda_{ii}(t - \tau) \end{aligned}$$

Property (iii) follows immediately from property (i) in view of the very definition of $r^{[j]}(t)$ $\square$

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