2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Twisting-controller gain adaptation

Arie Levant, *Member IEEE*, Mohammed Taleb, and Franck Plestan, *Member IEEE*

***Abstract*— Adaptation of twisting controller is performed in order to diminish the discontinuous control magnitude. The second-order real sliding mode is stably kept due to the instant multiplication of the gain by a chosen constant factor at the moments when the divergence is detected. Then the gain is gradually decreased. The resulting gain maxima are proved not to exceed the unknown equivalent control magnitude multiplied by a certain factor. Computer simulation confirms the theoretical results.**

I. Introduction

CONTROL under heavy uncertainty conditions is one of the main subjects of the modern control theory, and the sliding-mode control is one of the most popular approaches to the problem. The idea is to keep some properly chosen function (sliding variable) at zero by means of high-frequency control switching. Sliding mode (SM) is accurate and insensitive to disturbances [4, 5, 22, 23]. While standard SMs are applicable to nullify sliding variables of the relative degree 1, higher order sliding modes (HOSMs) [2, 3, 6, 11, 12, 16, 17, 18, 21] are used to keep constraints of higher relative degree. One of the main reasons for their application is the possibility [2, 3, 15] to effectively attenuate the so-called chattering effect [1, 7, 8, 9, 22] caused by the high control-switching frequency.

The main idea of the SM application is to suppress the proprietary uncertain dynamics of the sliding variable σ by sufficiently energetic discontinuous control effort. The resulting control magnitude is usually determined by a constant gain, which is to be taken "sufficiently large". In particular, with the relative degree 1 the controller is just a relay of the form $u = -K \operatorname{sign} \sigma$ of the corresponding amplitude K. High order sliding modes are applied with higher relative degrees and sometimes have a complicated structure, but there is always present some gain K determining the discontinuity magnitude.

Since the size of uncertain terms is mostly unknown, the gain K is inevitably taken redundantly large, which leads to excessive system chattering and energy losses. On the other hand, if the uncertainty terms are smoothly changing, one can try to adjust the gain K in real time, so as to diminish the chattering. The idea is to get SM with sufficiently large gain, and then to gradually adjust it, so that the SM is not lost. The approach has been already realized for the first-order sliding modes [20]. Second-order SM (2-SM) are considered in this paper.

A. Levant is with the School of Mathematical Sciences, Tel-Aviv University, Ramat-Aviv, 69978 Tel-Aviv, Israel (levant@post.tau.ac.il).

F. Plestan (Franck.Plestan@irccyn.ec-nantes.fr) and M. Taleb (Mohammed.Taleb@irccyn.ec-nantes.fr) are with IRCCyN, UMR 6597, Ecole Centrale de Nantes-CNRS, France.

The realization of the approach requires a criterion for the detection of the real (i.e. approximate) SM. One may just require the sliding variable to be less than some threshold. The resulting algorithm will be robust in that case, but one cannot expect for better accuracy than the predefined threshold. On the other hand it is known that the characteristic accuracy of r-SM is of the order τ^r, if τ is the sampling period [11], and the sampling errors are of the order of τ^r as well [13]. Obviously, if one requires such accuracy in r-SM adaptation, then also the noise should be at most of the order of τ^r. This approach is adopted in this paper with $r = 2$. The robust algorithm with a predefined accuracy will be considered in a separate paper.

There are still two other options for the problem statement. The first option is that the uncertain equivalent control (the control value which nullifies the rth derivative of the sliding variable) does not have explicit bounds and can be even unbounded. In that case one needs some estimation of the highest rate of its change, so that the adjusting gain K could change faster in order to cope with the uncertainty. This option is considered in another paper of the authors.

The second option is that the uncertainty is bounded, so that one knows in advance the value of K which is sufficient to establish and keep the sliding mode. The problem then is to change the gain with respect to the unknown actual size of the uncertainty. The rate of the uncertainty changing is still supposed to be bounded, *but the bound is unknown*. We show in this paper that in this case one only needs to know the above-mentioned sufficient maximal value of K to effectively adjust its value. Since the method realization considerably depends on the type of the HOSM controller, this paper deals with the well-known twisting controller and the relative degree $r = 2$.

II. Problem Statement and Control Design

A. Problem statement

Consider a dynamic system of the form

$$\dot{x} = a(t,x) + b(t,x)u, \quad \sigma = \sigma(t,x). \qquad (1)$$

Here $x \in \mathbf{R}^n$, a, b and σ: $\mathbf{R}^{n+1} \to \mathbf{R}$ are unknown smooth functions, $u \in \mathbf{R}$, n can be also uncertain (its value is not used), $t \geq 0$. Control can be discontinuous, and solutions are understood in the Filippov sense [5]. Trajectories of (1) are assumed infinitely extendible in time for any uniformly bounded control. The output σ is available in real time. The task is to provide in finite time for accurately keeping $\sigma = 0$ by means of output-feedback control.

The relative degree of the system is assumed to be equal to 2. In other words [10], for the first time the control explicitly appears in the 2nd total time derivative of σ and

$$\ddot{\sigma} = \psi(t,x) + \varphi(t,x)u, \tag{2}$$

where $\psi(t,x)$ and $\varphi(t,x)$ are some unknown smooth functions. It is supposed that for some positive constants ψ_d, φ_m, φ_M, φ_d, $u_{eqM} > 0$ the following inequalities hold:

$$|\dot{\psi}| \le \psi_d, |\dot{\varphi}| \le \varphi_d; \quad \varphi_m \le \varphi \le \varphi_M, \ |\psi/\varphi| \le u_{eqM}. \tag{3}$$

Only u_{eqM} is assumed known. Note that the function $u_{eq} = -\psi/\varphi$ is the so-called equivalent control [22], i.e. the control value which nullifies $\ddot{\sigma}$. Since no bound of $\dot{\psi}$ is available, the problem cannot be solved by standard 3-sliding control methods after the artificial increase of the relative degree to 3. Moreover, such a control would require calculation of $\ddot{\sigma}$ or of the divided second finite differences, which might be troublesome in the presence of noises.

B. Control design and results

Let $\tau > 0$ be the sampling period. Choose the control in the form

$$u = -K(\operatorname{sign}\sigma + \beta \operatorname{sign}\dot{\sigma}), \quad 0.5 < \beta < 1 \tag{4}$$

where β is a constant control parameter. With a constant K the standard twisting controller [11, 14] is obtained. Here $\dot{\sigma}$ is supposed available, the control value remains constant between the measurements. Alternatively $\dot{\sigma}$ can be calculated in real time by standard exact robust SM differentiator [12], since with bounded control under the considered conditions $\ddot{\sigma}$ has an available upper bound. Another option is to use sign $\Delta\sigma$ instead of sign $\dot{\sigma}$ [14].

Since the bound of ψ/φ is available, any constant value $K = K_M$ solves the stated problem of keeping $\sigma \equiv 0$, provided

$$K_M > \tfrac{1}{1-\beta}\, u_{eqM}. \tag{5}$$

Fix a constant K_M satisfying (5). We will provide for $K \le K_M$.

Since $\dot{u}_{eq}$ is bounded according to (3), the variable value $K = \frac{\gamma}{1-\beta}\, u_{eq}$, $\gamma = \text{const} > 1$, would solve the problem after the 2-SM is obtained. Indeed, with small σ and $\dot{\sigma}$ the twisting dynamics is so fast that $u_{eq} = -\psi/\varphi$ could be considered constant. Unfortunately u_{eq} is not available.

Assume that the sliding variable σ is measured at the sampling instants t_i with the sampling period $\tau > 0$. Introduce a criterion for the detection of the real 2-sliding mode with respect to σ. Take a natural number N_t and some $\mu > 0$. Let $t \in [t_i, t_{i+1})$, and define

$$\alpha(t) = \begin{cases} 1 & \text{if } \forall t_j \in [t - N_t\tau, t]: \ |\sigma(t_j)| \le \mu K(t_j)\tau^2, \\ -1 & \text{if } \exists t_j \in [t - N_t\tau, t]: \ |\sigma(t_j)| > \mu K(t_j)\tau^2, \end{cases} \tag{6}$$

where t_j are the sampling instants. *The 2-sliding mode criterion is considered satisfied if* $\alpha = 1$.

Introduce some constants λ, K_m, q satisfying

$$\lambda > 0, \ 0 < K_m \le K_M > 0, \ q > \tfrac{1+\beta}{1-\beta}. \tag{7}$$

Note that there are no other restrictions on λ, and K_m. Let the gradual adaptation law be

$$\dot{K} = \begin{cases} -\lambda K \text{ if } K \ge K_M, \\ -\alpha\lambda \text{ if } K_m < K < K_M, \\ \lambda \text{ if } K \le K_m, \end{cases} \quad K_m \le K(0) \le K_M. \tag{8}$$

Thus, $K_m \le K(t) \le K_M$ is kept, while K_m can be taken arbitrarily small.

The parameter K is not able to track u_{eq}, if λ is not large enough. Instead of it an instant increment is implemented at each sampling instant t_i, if the 2-sliding criterion is violated, i.e. passes from “true” to “false”:

$$K(t_i) = \begin{cases} qK(t_i - 0) \text{ if } \alpha(t_{i-1}) = 1 \,\&\, \alpha(t_i) = -1, \\ K(t_i - 0) \text{ if } \alpha(t_{i-1}) \ne 1 \text{ or } \alpha(t_i) \ne -1. \end{cases} \tag{9}$$

Here $K(t_i - 0)$ is the limit of $K(t)$ as $t \to t_i$ from the left. We will show that actually it is the leaping procedure (9) which keeps the 2-SM.

Theorem 1. *For any sufficiently large μ and sufficiently large $N_t \ge 4$ (chosen after μ) with sufficiently small τ, starting from some moment, the parameter $K(t)$ satisfies $K(t) \in [q_*, q^*]\, |\psi(t,x(t))/\varphi(t,x(t))| \cap \{K \,|K \ge K_m\}$ for some $q^* > q/(1 - \beta)$, $q_* < 1/(1 - \beta)$. Respectively the accuracy $|\sigma| \le \eta_1\tau^2 K(t)$, $|\dot{\sigma}| \le \eta_2\tau K(t)$ is established in finite time. Parameters μ and N_t can be chosen in advance independently of the actual system. The constants η_1, η_2 only depend on the parameters of the algorithm and parameters of the assumptions.*

Obviously, as follows from [11], with μ too small and $K \equiv K_M$ the accuracy $|\sigma| \le \mu\tau^2 K_M$ is unsustainable, which imposes the lower restriction on μ. The larger the number N_t is the closer are q^* to $q/(1 - \beta)$ and q_* to $1/(1 - \beta)$, resulting in K oscillating in a more narrow vicinity of $|u_{eq}|$.

It can be proved that with sufficiently large λ the gain $K(t)$ directly approximates $\max\{|u_{eq}|, K_m\}$, but the choice of λ would require the knowledge of ψ_d, φ_d.

Theorem 2. *The statement of Theorem* 1 *remains true, if the sampling noise magnitude does not exceed $\xi\tau^2$, $\xi > 0$, and the derivative $\dot{\sigma}$ is estimated by the differentiator* [12]*, or it is replaced in* (4) *by the increment of σ between the measurements. Parameters ξ, μ and N_t can be chosen in advance. The constants η_1, η_2 only depend on ξ and on the parameters of the algorithm and the assumptions.*

Obviously $\mu > \xi$ is necessary. The proof of Theorem 2 is obtained by simply taking measurement errors into account [13] in the following proof.

Proof of Theorem 1. Let the control have the form (4), i.e. the exact derivative $\dot{\sigma}$ be used. Introduce a new variable $\Sigma = \sigma/K$. Using relations

$$\dot{\Sigma} = \frac{\dot{\sigma}}{K} - \Sigma\frac{\dot{K}}{K}, \quad \ddot{\Sigma} = \frac{\ddot{\sigma}}{K} - 2\frac{\dot{\sigma}}{K}\frac{\dot{K}}{K} + \Sigma\left(2\frac{\dot{K}^2}{K^2} - \frac{\ddot{K}}{K}\right)$$

obtain that

$$\ddot{\Sigma} = -\varphi\left(\operatorname{sign}\Sigma + \beta\operatorname{sign}\left(\dot{\Sigma} + \Sigma\frac{\dot{K}}{K}\right)\right) + \frac{\psi}{K} - 2\dot{\Sigma}\frac{\dot{K}}{K} - \Sigma\frac{\ddot{K}}{K}. \quad (10)$$

The condition $|\sigma| \le \mu K\tau^2$ from (6) takes the form $|\Sigma| \le \mu\tau^2$. Thus the problem can be reformulated in the terms of Σ.

Following is the plan of the proof. First we show that if $|\Sigma| \le \mu\tau^2$ holds at four successive sampling instants, then on this time interval $\Sigma = O(\tau^2)$ and $\dot{\Sigma} = O(\tau)$ (Lemma 1). Further we show that if $K > \frac{1}{1-\beta-\varepsilon}|\psi|/\varphi$, with some small $\varepsilon > 0$, then (10) is locally uniformly finite-time stable independently of the K variation (8), (9) (Lemma 2). Moreover, with discrete measurements the accuracy $\Sigma = O(\tau^2)$ and $\dot{\Sigma} = O(\tau)$ is established (Lemma 3).

Furthermore, eventually the 2-sliding criterion $\alpha = 1$ is satisfied, for otherwise K increases until its maximal value is attained and then 2-SM is inevitably established. Therefore, in finite time we get $\alpha = 1$ and $\Sigma = O(\tau^2)$, $\dot{\Sigma} = O(\tau)$. From that moment K starts to decrease, which means that (10) might cease to be stable at some moment. We show that if at some moment $\alpha = 1$ is got, then $K > \frac{1}{1-\beta+\varepsilon}|\psi|/\varphi$ for some small ε. Otherwise the inequality $|\Sigma| \le \mu\tau^2$ could not be kept during N_t measurements (Lemma 4). Thus, at the moment when the criterion $\alpha = 1$ is violated the inequality $K > \frac{1}{1-\beta}|\psi|/\varphi$ still holds. Therefore the instant increase (9) of K and its gradual increase (8) immediately reestablish the local finite-time stable dynamics of (10). As a result $\Sigma = O(\tau^2)$ and $\dot{\Sigma} = O(\tau)$ are kept all the time, while K varies in the range $\left[\frac{1}{1-\beta+\varepsilon}, \frac{q}{1-\beta-\varepsilon}\right]|\psi/\varphi| \cap [K_{mm}, \infty)$ (Lemma 5), which corresponds to the statement of the Theorem.

The following Lemmas realize the above plan.

Lemma 1. *Under the conditions of the Theorem there exist such ϖ_1 and ϖ_2 that for any sufficiently small τ for some value of μ keeping $|\Sigma| \le \mu\tau^2$ at 4 successive sampling instants implies that during these three sampling intervals $|\Sigma| \le \varpi_1\tau^2$, $|\dot{\Sigma}| \le \varpi_2\tau$. Here ϖ_1, ϖ_2 depend only on μ, the parameters of the problem and the algorithm.*

Proof. The criterion means that at the ends of the three last sampling periods the condition $|\Sigma| \le \mu\tau^2$ holds. Thus, due to the Lagrange Theorem at some moments t_*, t_{**} during the first and the third period the inequality $|\dot{\Sigma}| \le \mu\tau$ holds. Since $\ddot{\Sigma}$ is uniformly bounded, integrating obtain that $|\dot{\Sigma}|$ remains of the order of τ during the three last sampling periods. Therefore also $|\Sigma|$ remains of the order of τ^2. ■

Lemma 2. *Let $K > \frac{1}{1-\beta-\varepsilon}|\psi|/\varphi$, where $\varepsilon > 0$ and $\beta + \varepsilon < 1$, and let $|\Sigma| < \delta_1$, $|\dot{\Sigma}| < \delta_2$, where δ_1, δ_2 are sufficiently small. Then solutions Σ and $\dot{\Sigma}$ of (10) uniformly converge to zero in finite time.*

Proof. Due to Lemma 1 after a finite-time transient the inequalities $|\Sigma| < \delta_1$, $|\dot{\Sigma}| < \delta_2$ are kept for any sufficiently small δ_1, $\delta_2 > 0$. Rewrite the condition $K > \frac{1}{1-\beta-\varepsilon}|\psi|/\varphi$ as $|\psi|/K < (1 - \beta - \varepsilon)\varphi$. Choose δ_1, δ_2 so that

$$\left(2\lambda|\dot{\Sigma}| + \lambda^2|\Sigma|\right) + \frac{|\psi|}{K} \le (1 - \beta - \tfrac{2}{3}\varepsilon)\varphi, \quad (11)$$

which is possible, since $\varphi \ge \varphi_m$.

Show now that φ can be practically considered constant. Obviously there is a moment t_* when $\Sigma = 0$, since $\ddot{\Sigma}\operatorname{sign}\Sigma < -\frac{2}{3}\varepsilon\varphi < -\frac{2}{3}\varepsilon\varphi_m$. Let $\dot{\Sigma} = \dot{\Sigma}_*$ at that moment, and $|\dot{\Sigma}_*| < \delta_2$. Therefore the time needed to get to $\dot{\Sigma} = 0$ does not exceed $\delta_2/(\frac{2}{3}\varepsilon\varphi_m)$. It is easy to see that the same time is needed to get once more to $\Sigma = 0$, $\dot{\Sigma} = \dot{\Sigma}_{**}$ at the time t_{**}. During the time $\Delta t_* = t_{**} - t_*$ the function φ cannot increase or decrease by a factor larger than $e^{\varphi_d\Delta t_*/\varphi_m}$. Obviously taking δ_2 small enough, one can make $e^{\varphi_d\Delta t_*/\varphi_m}$ as close to 1 as needed.

The convergence condition is $|\dot{\Sigma}_{**}/\dot{\Sigma}_*| \le \text{const} < 1$. Check it. Let for simplicity $\dot{\Sigma}_* > 0$. Taking the initial conditions $\Sigma = 0$, $\dot{\Sigma} = \dot{\Sigma}_*$ construct the majorant curve [11], such that all real trajectories for sure lie between the axis $\Sigma = 0$ and the majorant. Let $\gamma > 0$ be any small number. With sufficiently small $\delta_1 > 0$ obtain that $\lambda|\Sigma| \le \gamma|\Sigma|^{1/2}$ whenever $|\Sigma| < \delta_1$. The majorant is obtained, when remaining negative, $\ddot{\Sigma}$ takes on the minimal possible absolute value with $\dot{\Sigma} > 0$, and the least possible absolute value with $\dot{\Sigma} < 0$. Taking into account (11) obtain that

$$-\left(2 - \tfrac{2}{3}\varepsilon\right)\varphi \le -\left(1+\beta\right)\varphi + \frac{\psi}{K} - 2\lambda\dot{\Sigma} - \lambda^2\Sigma \le -\left(2\beta + \tfrac{2}{3}\varepsilon\right)\varphi,$$

$$-\left(2 - 2\beta - \tfrac{2}{3}\varepsilon\right) \le -\left(1-\beta\right)\varphi + \frac{\psi}{K} - 2\lambda\dot{\Sigma} - \lambda^2\Sigma \le -\tfrac{2}{3}\varepsilon\varphi.$$

Thus, with $\Sigma > 0$ define the majorant by the equations

$$\ddot{\Sigma} = \begin{cases} -(2\beta + \tfrac{2}{3}\varepsilon)\varphi(t_*, x(t_*))e^{-\varphi_d\Delta t_*} & \text{if } \dot{\Sigma} \ge 0, \\ -(2 - \tfrac{2}{3}\varepsilon)\varphi(t_*, x(t_*))e^{\varphi_d\Delta t_*} & \text{if } 0 > \dot{\Sigma} \ge -\gamma\Sigma^{1/2}, \\ -\left(2 - 2\beta - \tfrac{2}{3}\varepsilon\right)\varphi(t_*, x(t_*))e^{\varphi_d\Delta t_*} & \text{if } \dot{\Sigma} < -\gamma\Sigma^{1/2}. \end{cases}$$

Obviously, the fraction $|\dot{\Sigma}_{M**}/\dot{\Sigma}_*|$ of the real trajectory does not exceed the corresponding value calculated for the majorant curve. The calculation shows that with $\gamma = 0$

$$\left|\frac{\dot{\Sigma}_{M**}}{\dot{\Sigma}_*}\right| = \left(\frac{(2-2\beta-\frac{2}{3}\varepsilon)e^{\varphi_d \Delta t_*}}{(2\beta+\frac{2}{3}\varepsilon)e^{-\varphi_d \Delta t_*}}\right)^{1/2} \le \left(\frac{2-2\beta-\frac{2}{3}\varepsilon}{2\beta+\frac{2}{3}\varepsilon}\right)^{1/2} e^{\varphi_d \Delta t_*}.$$

Obviously $\frac{2-2\beta-\frac{2}{3}\varepsilon}{2\beta+\frac{2}{3}\varepsilon} < 1$ since $\beta > 0.5$. Hence, $|\dot{\Sigma}_{M**}/\dot{\Sigma}_*| \le \text{const} < 1$, if Δt_* and γ are small enough. The further proof follows [11].■

Lemma 3. *Under the conditions of Lemma 2 solutions* Σ *and* $\dot{\Sigma}$ *of* (10) *uniformly and in finite time converge to the region of the form* $|\Sigma| \le \varpi_1 \tau^2$, $|\dot{\Sigma}| \le \varpi_2 \tau$, *where* ϖ_1, ϖ_2 *depend only on the parameters of the problem and the algorithm*.

Proof. In a small vicinity of the origin the trajectories satisfy the finite-time-stable homogeneous differential inclusion

$$\ddot{\Sigma} \in -[\varphi_m, \varphi_M]\left(\operatorname{sign}\Sigma + \beta \operatorname{sign}\dot{\Sigma}\right) + \left[-1+\beta-\tfrac{1}{2}\varepsilon, 1-\beta+\tfrac{1}{2}\varepsilon\right] \tag{12}$$

with the weights $\deg \Sigma = 2$, $\deg \dot{\Sigma} = 1$ and the homogeneity degree -1. The Lemma follows now from the general features of finite-time stable homogeneous inclusions [13].■

Lemma 4. *Let* $\varepsilon > 0$ *be any small number. Let the criterion* $|\Sigma| \le \mu\tau^2$ *be satisfied for some value of* μ *at* 4 *successive sampling instants for some values of* μ, *and let* $K < \frac{1}{1+\beta+\varepsilon}|\psi|/\varphi$ *hold at the last instant. Then there exists such natural N that the 2-sliding criterion is violated in N sampling intervals.*

Proof. Due to Lemma 1 at the sampling moment $|\Sigma| \le \varpi_1\tau^2$, $|\dot{\Sigma}| \le \varpi_2\tau$ is kept. In a few sampling intervals at some moment t_* the trajectory enters the quarter $\Sigma\dot{\Sigma} > 0$ and already cannot leave it. During that time $|\psi|/(K\varphi)$ practically does not change. From that moment on the approximate formula $|\Sigma| \ge \frac{1}{2}\varepsilon(t-t_*)^2$ holds. The number N is now easily evaluated.■

The following Lemma finishes the proof of the Theorem.

Lemma 5. *There exists* $\theta > 0$, *such that under conditions of the Theorem from some moment on the local maxima of K do not exceed* $(\frac{q}{1-\beta}+\theta)|\psi|/\varphi$ *whenever* $K > K_{mm}$. *Also the inequalities* $|\Sigma| \le \tilde{\varpi}_1\tau^2$, $|\dot{\Sigma}| \le \tilde{\varpi}_2\tau$ *hold for some* $\tilde{\varpi}_1, \tilde{\varpi}_2$ *which only depend on the parameters of the algorithm and the Assumptions*.

Proof. At some moment the 2-sliding criterion is inevitably satisfied. Indeed, suppose it is not right, then eventually K stabilizes at K_M, and the dynamics (2), (4) turns out to be finite-time stable and homogeneous with the homogeneity degree -1, and weights 2 and 1 of σ and $\dot{\sigma}$ respectively. Thus Σ, $\dot{\Sigma}$ become small and according to Lemma 3 the 2-sliding criterion gets satisfied with some μ_1, μ_2.

Once 2-sliding criterion is established the inequality $\frac{1}{1+\beta+\varepsilon}|\psi|/\varphi < K$ is kept, otherwise it is violated in infinitesimal time (Lemma 4).

Choose any small $\varepsilon > 0$. Once 2-sliding criterion is established K starts to decrease until the criterion is indeed violated (or K stabilizes at K_m). It means that the condition $|\Sigma| \le \mu\tau^2$ is violated only at the last sampling and that N sampling periods earlier $K > \frac{1}{1+\beta+\varepsilon}|\psi|/\varphi$ was held, otherwise the criterion were violated one step earlier. On the other hand at the last but one step $K < \frac{1}{1-\beta-\varepsilon}|\psi|/\varphi$, for otherwise the condition $|\Sigma| \le \mu\tau^2$ would not be violated at the last measurement.

Calculation shows that

$$\frac{d}{dt}\frac{|\psi|}{K\varphi} = \frac{K\varphi\dot{\psi}\operatorname{sign}\psi - |\psi|\left(\dot{K}\varphi + K\dot{\varphi}\right)}{(K\varphi)^2} = \frac{|\psi|}{K\varphi}\left(\frac{\dot{\psi}\operatorname{sign}\psi}{|\psi|} - \frac{\dot{K}}{K} - \frac{\dot{\varphi}}{\varphi}\right)$$

Thus during N sampling steps $|\psi|/(K\varphi)$ practically does not change, and after the instant increment (9) get $\frac{q}{1-\beta-2\varepsilon}|\psi|/\varphi > K > \frac{q}{1+\beta+2\varepsilon}|\psi|/\varphi > \frac{1}{1-\beta-\varepsilon}|\psi|/\varphi$, if ε is sufficiently small.

After the instant increment (9) the conditions $|\Sigma| \le \tilde{\varpi}_1\tau^2$, $|\dot{\Sigma}| \le \tilde{\varpi}_2\tau$ are still valid with somewhat increased coefficients $\tilde{\varpi}_1, \tilde{\varpi}_2$. As a result the conditions of Lemma 3 are satisfied. The convergence time to the invariant set $|\Sigma| \le \varpi_1\tau^2$, $|\dot{\Sigma}| \le \varpi_2\tau$ takes a number of sampling steps only, since, as follows from [13], the convergence time is proportional to the homogeneous norm $|\Sigma|^{1/2} + |\dot{\Sigma}|$. The 2-sliding criterion is satisfied, and K once more starts to decrease until $|\psi|/(K\varphi)$ approaches $\frac{1}{1+\beta+\varepsilon}$. If K decreases until the value K_m, it stops to change until the 2-sliding criterion is violated, which happens when $\frac{1}{1+\beta+\varepsilon}|\psi|/\varphi < K < \frac{1}{1-\beta-\varepsilon}|\psi|/\varphi$.

Since K is a priory bounded by the maximal possible value of $(\frac{q}{1-\beta}+\theta)|\psi|/\varphi \le (\frac{q}{1-\beta}+\theta)|\psi_M|/\varphi_m$, get that $|\Sigma|$, $|\dot{\Sigma}|$ are of the order of τ^2 and τ respectively. ■ ■

III. Simulation

The presented academic example has already appeared in the literature [14, 19]. Performance of various second order sliding-mode controllers combined with sliding-mode differentiators is analyzed in [14]. A second-order sliding mode output-feedback control has been proposed in [19], which only requires the measurement of the position, and does not use the velocity. However, all the previous results required the knowledge of the bounds for the uncertainties and perturbations.

The considered system [14] (Fig. 1) is a variable-length pendulum evolving in a vertical plane. A load of a known mass m moves without friction along the pendulum rod. Its distance from O equals $R(t)$ and is not measured. An engine transmits a torque u, which is considered as control. The task is to track some function x_c given in real time by the angular coordinate x of the rod.

The system is described by the equation

$$\ddot{x} = -2\,\frac{\dot{R}}{R}\,\dot{x} - g\frac{1}{R}\sin x + \frac{1}{mR^2}\,u, \qquad (13)$$

where $m = 1$ and $g = 9.81$ is the gravitational constant. Let

$$0 < R_m \le R \le R_M,$$

$\dot{R}$, $\ddot{R}$, $\dot{x}_c$ and $\ddot{x}_c$ be bounded, $\sigma = x - x_c$ be available. Following are the "unknown" functions R and x_c considered in the simulation:

$$R = 0.8 + 0.1\sin 8t + 0.3\cos 4t,$$
$$x_c = 0.5\sin 0.5t + 0.5\cos t\,.$$

Let $\sigma = x - x_c$. The relative degree of the system equals 2. Due to the unboundedness of $\dot{x}$ assumptions (3) are fulfilled here only locally, and the controller to be applied is effective only for some bounded set of initial conditions. Choosing the maximal acceptable value K_M of the adjusted controller parameter K, the convergence region can be made arbitrarily large.

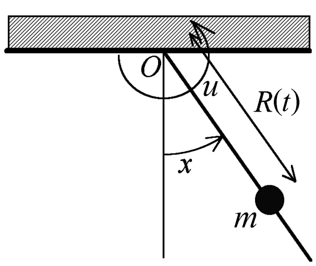

Fig. 1. Variable length pendulum.

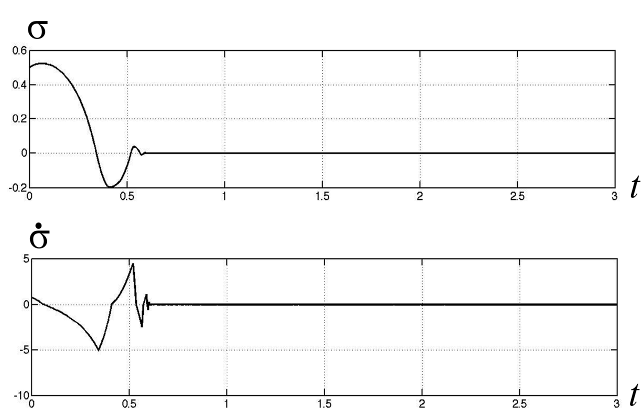

Fig. 2. Top: sliding variable σ versus time (sec). Bottom: time derivative of sliding variable $\dot{\sigma}$ versus time (sec).

The sampling period is $\tau = 0.0001$s. The control law corresponds to (4) with $\beta = 2/3$, and is

$$u = -K\,(\operatorname{sign}\sigma + \tfrac{2}{3}\,\operatorname{sign}\dot{\sigma}).$$

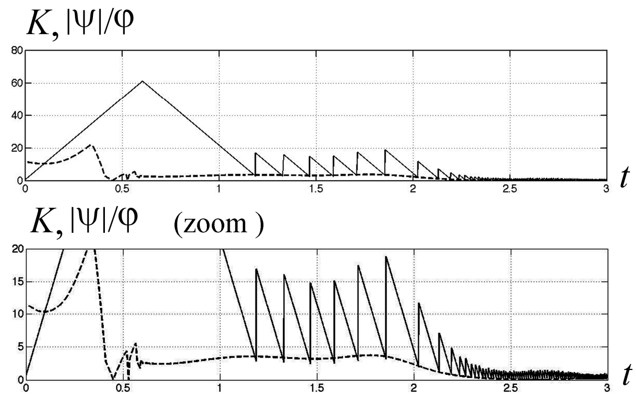

Fig. 3. Top: adjustment of the gain K. Bottom: zoom of the graph, maxima of K are proportional to the equivalent-control absolute value $|u_{eq}| = |\psi|/\varphi$ presented by the dotted line.

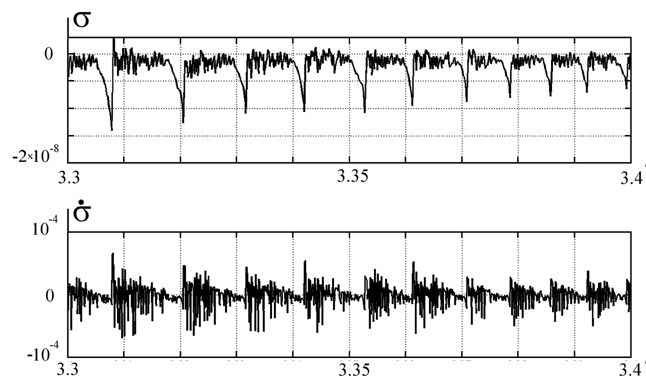

Fig. 4. Top: steady state dynamics of σ with $t \in [3.3, 3.4]$. Bottom: steady state dynamics of $\dot{\sigma}$. The accuracy is proportional to the variable gain K.

According to (7) the inequality $q > 5$ is to hold. Respectively the controller parameters are chosen as

$$\mu = 5,\; q = 6,\; K_M = 100,\; K_m = 0.1,\; \lambda = 100. \qquad (14)$$

The initial values $K(0) = 0.1$, $x(0) = \dot{x}(0) = 0$ are taken.

It appears that system trajectories converge in finite time to the desired trajectories, while the sliding variable σ and its derivative $\dot{\sigma}$ converge to 0 with the accuracy proportional to τ^2 and τ respectively (Fig. 2).

Figure 3 displays the time-varying gain $K(t)$. The second order real sliding mode is detected at $t = 0.6$, when K starts to decrease. It is clearly seen that $K(t)$ is dynamically adapted with respect to the ratio $|\psi|/\varphi$, i.e. with respect to the equivalent-control magnitude. Starting from $t = 1.2$ its maxima do not exceed six absolute values of the equivalent control. As a result, while at $t = 0.6$ the accuracy was $|\sigma| \le 3{\cdot}10^{-6}$, the higher accuracy $|\sigma| \le 5{\cdot}10^{-8}$ is kept near the end of the simulation interval at $t = 3$. Without adaptation the accuracy would remain at about $3{\cdot}10^{-6}$. One can see in Fig. 4 that the fluctuation of σ and $\dot{\sigma}$ follows the dynamics of K with $t \in [3.3, 3.4]$.

IV. Conclusions

Adjustment of the sliding-mode control magnitude, when no a priory upper bounds are known neither for the uncertainties nor for their derivatives is a challenging problem of adaptation, which is currently solved only for the first-order sliding modes. An adaptation sliding-mode strategy is proposed in this paper for the relative degree 2 and the twisting controller.

The adaptation idea is very simple: the gain is to be increased until the sliding mode is attained, then it is decreased until the sliding mode is lost. The sliding mode is detected basing on the sliding mode accuracy checked at a fixed number of sampling instants. The accuracy should be proportional to the squared sampling period and the adaptation gain. Also the noise should be of the order of squared sampling period, in order to apply the algorithm. In practice it means that in the presence of noise one needs to increase the sampling period in order to provide for the better robustness of the system.

From the moment when the second order sliding mode is detected, it is actually kept due to the instant increments of the gain by a specially chosen factor at the very moments when it is about to be lost. As a result the maximal values of the gain remain proportional to the current magnitude of the equivalent control.

Unfortunately the chosen strategy (especially the above multiplication factor) and the corresponding proofs significantly use the concrete controller form (the twisting controller). Therefore the extension of the results to higher orders and other sliding-mode controllers is not a simple task.

References

[1] G. Bartolini, “Chattering phenomena in discontinuous control systems,” *Internat. J. Systems Sci.*, vol. 20, pp. 2471–2481, 1989.

[2] G. Bartolini, A. Ferrara, and E. Usai, “Chattering avoidance by second-order sliding mode control”, IEEE Trans. Automat. Control, Vol. 43, No. 2, pp. 241-246, 1998.

[3] G. Bartolini, A. Pisano, E. Punta, and E. Usai, “A survey of applications of second-order sliding mode control to mechanical systems”, *International Journal of Control*, vol. 76, no. 9/10, pp. 875-892, 2003.

[4] C. Edwards, Spurgeon, S. K. “Sliding Mode Control: Theory and Applications”, Taylor & Francis, 1998.

[5] A.F. Filippov, *Differential Equations with Discontinuous Right-Hand Side*, Kluwer, Dordrecht, the Netherlands, 1988.

[6] T. Floquet, J.-P. Barbot and W. Perruquetti, Higher-order sliding mode stabilization for a class of nonholonomic perturbed systems, *Automatica*, Vol. 39, No. 6, pp. 1077-1083, 2003.

[7] L. Fridman, “Singularly perturbed analysis of chattering in relay control systems”, *IEEE Trans. on Automatic Control*, vol. 47, no.12, pp. 2079-2084, 2002.

[8] L. Fridman, ”Chattering analysis in sliding mode systems with inertial sensors,” *International J. Control,* Vol. 76, no. 9/10, pp. 906-912, 2003.

[9] K. Furuta, Y. Pan, “Variable structure control with sliding sector”, *Automatica,* vol. 36, pp. 211-228, 2000.

[10] A. Isidori. *Nonlinear Control Systems*, second edition, Springer, New York, 1989.

[11] A. Levant (L.V. Levantovsky), “Sliding order and sliding accuracy in sliding mode control”, *International Journal of Control*, vol. 58, no. 6, pp.1247-1263, 1993.

[12] A. Levant, "Higher-order sliding modes, differentiation and output-feedback control", *International Journal of Control*, Vol. 76, No. 9/10, pp. 924-941, 2003.

[13] A. Levant, “Homogeneity approach to high-order sliding mode design”, *Automatica*, vol. 41, no. 5, pp. 823-830, 2005.

[14] A. Levant, “Principles of 2-sliding mode design”, *Automatica*, Vol.43, No.4, pp.576–586, 2007.

[15] A. Levant, “Chattering analysis”, *IEEE Trans. Automat. Control*, vol. 55, no. 6, pp. 1380 – 1389, 2010.

[16] Z. Man, A.P. Paplinski and H.R. Wu, “A robust MIMO terminal sliding mode control for rigid robotic manipulators”, *IEEE Trans. Automat. Control*, Vol. 39, No. 12, pp. 2464-2468, 1994.

[17] T. Massey and Y. Shtessel, “Continuous traditional and high order sliding modes for satellite formation control”, *AIAA J. Guidance, Control, and Dynamics*, Vol. 28, No. 4, 826-831, 2005.

[18] Y. Orlov, Y. Aoustin and C. Chevallereau,"Finite Time Stabilization of a Perturbed Double Integrator—Part I: Continuous Sliding Mode-Based Output Feedback Synthesis", *IEEE Trans. Automat. Control*, Vol. 56, No. 3, pp. 614-618, 2011.

[19] F. Plestan, E. Moulay, A. Glumineau, T. Cheviron, Robust output feedback sampling control based on second-order sliding mode", *Automatica*, vol. 46, no. 6, pp. 1096-1100, 2010.

[20] F. Plestan, Y. Shtessel, V. Brégeault, A. Poznyak, New methodologies for adaptive sliding mode control, *International Journal of Control*, Vol.83, No.9, pp. 1907-1919, 2010

[21] Y.B. Shtessel, I.A. Shkolnikov, and M.D.J. Brown, “An Asymptotic Second-Order Smooth Sliding Mode Control”, *Asian J. of Control*, vol. 5, no. 4, pp. 498-504, 2004

[22] V.I. Utkin, *Sliding Modes in Optimization and Control Problems*, Springer, New York, 1992.

[23] Zinober, A.S.I., (Ed.) *Variable Structure and Lyapunov Control*, Springer-Verlag, Berlin, 1994.