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# Wind Energy Aggregation: A Coalitional Game Approach

E. Baeyens, E.Y. Bitar, P.P. Khargonekar, K. Poolla

*Abstract*— **In this paper we explore the extent to which a group of $N$ wind power producers can exploit the statistical benefits of *aggregation* and *quantity risk sharing* by *forming a willing coalition* to pool their variable power to jointly offer their aggregate power output as single entity into a forward energy market. We prove that wind power generators will always improve their expected profit when they aggregate their generated power and use tools from coalitional game theory to design *fair* sharing mechanisms to allocate the payoff among the coalition participants. We show that the corresponding coalitional game is super-additive and has a nonempty core. Hence, there always exists a mechanism for profit-sharing that makes the coalition stable. However, the game is not convex and the celebrated Shapley value may not belong to the core of the game. An allocation mechanism that minimizes the worst-case dissatisfaction is proposed.**

## I. INTRODUCTION

Motivated by concerns over global warming, there are worldwide efforts to increase the penetration of renewable energy resources serving electrical loads. Wind and solar electric energy resources posses tremendous potential to reduce the use of carbon emitting fuel sources such as coal, oil, and natural gas [3]. However, wind and solar power generation differ from these traditional sources of electric power, because they are inherently *variable*. Due to natural variations in wind speed, wind power output from a wind turbine exhibits major fluctuations (over various time scales). Additionally, wind resources have limited dispatchability and are extremely difficult to forecast. Because of the need to maintain instantaneous balance between load and generation, this inherent variability presents a central challenge to large-scale integration of renewable energy into the electric grid. The interested reader is referred to [12], [5], [7], [8] for a thorough review of the challenges facing the integration of variable renewable generation into the electric grid.

It is generally believed [5], [12] that the aggregation of geographically diverse wind energy resources has significant potential to mitigate wind power variability. Indeed, this approach has been successfully monetized by aggregators such as Iberdrola Renewables [9]. Also, the EWITS report [5] states, "Both variability and uncertainty of aggregate wind decrease percentage wise with more wind and larger geographic areas." This attenuation of output variability of wind resources aggregated over large spatial regions is derived from the the tendency of wind speed at different geographic locations to decorrelate with increasing spatial separation. In this paper, we analyze and quantify the financial benefit of wind power aggregation through *coalitional bidding* in a competitive two-settlement market setting. The central idea is that a set of independent wind power producers (WPP) can exploit the statistical benefits of aggregation by forming a willing coalition to pool their variable power to jointly offer their aggregate output as single entity into a forward energy market. As deviations from offered contracts are penalized, this amounts to an act of *quantity risk sharing* among the members of the coalition. Assuming that coalitional bidding results in profit increase beyond that achievable through individual market participation, a central question arises in this setting. *What are fair sharing mechanisms to allocate the additional profit among the coalition to ensure its stability?*

We formalize this question in the setting of cooperative games using tools from coalitional game theory [14]. We define the *value of a coalition* of WPPs as the maximum expected profit achievable through joint bidding of the aggregate wind power in a two-settlement market. Using this *value function*, it can be shown that, except for degenerate cases, coalition formation always results in a net increase in expected profit and that there always exist stabilizing rules for sharing the profit. Moreover, via a counterexample, we show that this game is *not convex* and that the famous *Shapley mechanism* is not stabilizing. We propose the use of the *imputation*, which minimizes the worst-case dissatisfaction (excess), as a profit sharing mechanism and show that it is stabilizing for every coalition member in that it satisfies certain *fairness axioms*.

As the value function, associated with our coalitional game for wind energy aggregation, is defined in the metric of optimal *expected* profit, an imputation belonging to the corresponding *core*, represents the payment that each wind power producer should receive *in expectation*. In practice, however, the realized profit for will vary day to day, as the profit is inherently a random variable given its explicit dependence on the stochastic wind power production and imbalance prices. To account for this issue, in Section IV-C we propose a *daily* payoff allocation mechanism to distribute the realized profit among the coalition members, such that the payment that each member receives – averaged over an increasing number of days – approaches an imputation in the core, *almost surely*.

Although different in application and formulation, our problem has significant connections with the classical

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E. Baeyens is with the Instituto de las Tecnologías Avanzadas de la Producción, Universidad de Valladolid, Spain enrbae@eis.uva.es

E. Bitar is with the Department of Mechanical Engineering, U.C. Berkeley, USA ebitar@berkeley.edu

P. P. Khargonekar is with the Department of Electrical and Computer Engineering, University of Florida, USA ppk@ece.ufl.edu

K. Poolla is with the Department of Electrical Engineering and Computer Science, U.C. Berkeley, USA poolla@berkeley.edu

newsvendor problem [21] in operations research. In both cases, the optimal contract offering is given in terms of a probabilistic quantile. Moreover, coalitional game theory has also been applied in the newsvendor setting [6] where it has been shown that the *core is nonempty* [10].

The paper is organized as follows. In Section II, we begin with a formulation of the WPP coalitional bidding problem in a two-settlement market setting. We follow this, in Section III with a brief review of certain key results from coalitional game theory. Finally, in Section IV, we state our main results and provide illustrations with some numerical examples. Due to space constraints, we omit statement of proofs.

## II. Problem Formulation

### A. Aggregate Wind Power Model

Consider a group of $N$ independent wind power producers (WPP) indexed by $i \in \mathcal{N} := \{1, 2, \ldots, N\}$. The power $w_i(t) \in [0, W_i]$ produced at wind farm $i$ is modeled as a scalar valued random process. Denote the collection of wind power production as a vector-valued random process, $w(t) = [\ w_1(t), \cdots, w_N(t)\ ]^T$, whose cumulative distribution function (CDF) at each time $t$ is given by

$$\Phi(w;t) = \mathbb{P}\{w(t) \leq w\}. \tag{1}$$

The distribution $\Phi(w;t)$ has support on a subset of $\mathbb{R}_+^N$ where $W_i$ denotes the nameplate capacity of wind power plant $i$. The corresponding probability density function is denoted by $\phi(w;t)$. We assume that

**A1** the group $\mathcal{N}$ of WPPs are connected to a common bus in the power network.

Consequently, the group $\mathcal{N}$ of WPPs face common market prices and can directly aggregate their power without regard to transmission capacity constraints. Accordingly, it is natural to consider scenarios in which individual wind power producers form willing coalitions $\mathcal{S} \subseteq \mathcal{N}$ to aggregate their wind power production and jointly bid into electricity markets for energy. The aggregate output corresponding to a coalition $\mathcal{S} \subseteq \mathcal{N}$ is denoted by

$$w_{\mathcal{S}}(t) = \sum_{i \in \mathcal{S}} w_i(t). \tag{2}$$

The stochastic process corresponding to the aggregate power output of a coalition $\mathcal{S} \subseteq \mathcal{N}$ is denoted by

$$\mathbf{w}_{\mathcal{S}} = \{w_{\mathcal{S}}(t) \mid t \in [t_0, t_f]\ \}.$$

Similarly, the CDF corresponding to the aggregate wind power $w_{\mathcal{S}}(t)$ at time $t$ is defined as

$$\Phi_{\mathcal{S}}(w;t) = \mathbb{P}\left\{w_{\mathcal{S}}(t) \leq w\right\}. \tag{3}$$

with support $[0, \sum_{i \in \mathcal{S}} W_i]$. Throughout the paper, we will work with wind power processes defined on the interval $[t_0, t_f]$ of width $T = t_f - t_0$. Of importance is the *time-averaged* CDF corresponding to the coalition $\mathcal{S} \subseteq \mathcal{N}$.

$$F_{\mathcal{S}}(w) \quad = \quad \frac{1}{T}\int_{t_0}^{t_f} \Phi_{\mathcal{S}}(w;t)dt \tag{4}$$

Also, define $F_{\mathcal{S}}^{-1} : [0,1] \to [0, \sum_{i \in \mathcal{S}} W_i]$ as the *quantile function* corresponding to the coalitional CDF $F_{\mathcal{S}}$. More precisely, for $\beta \in [0,1]$, the $\beta$-quantile of $F_{\mathcal{S}}$ is given by

$$F_{\mathcal{S}}^{-1}(\beta) = \inf\left\{x \in [0,1]\ :\ \beta \leq F_{\mathcal{S}}(x)\right\}. \tag{5}$$

### B. Market Model and Metrics

We assume that the coalition $\mathcal{S} \subseteq \mathcal{N}$ of wind power producers (WPP) is participating in a *competitive two-settlement market system* operated as a power exchange. See [19] for a detailed description of such markets. Generally, the two-settlement system consists of two *ex-ante* markets (a day-ahead (DA) forward market and a real-time (RT) spot market) and an *ex-post* imbalance settlement mechanism to penalize uninstructed deviations from contracts scheduled ex-ante. The pricing scheme for penalizing contract deviations reflects the energy imbalance of the control area as a whole and the spot price of balancing energy in the RT market. Hence, the imbalance prices are assumed unknown during the DA forward market and are not revealed until the RT spot market, on which they are based, is cleared.

In order to identify conditions under which *coalitions form* and *fair profit sharing mechanisms*, we first analyze the problem of optimizing the offering of a coalition constant power contract $C$ in a *single* ex-ante *DA forward market*, scheduled to be delivered continuously over a *single* time interval $[t_0, t_f]$ (typically of length one hour). The clearing price in the DA forward market is denoted by $p \in \mathbb{R}_+$ (\$/MWh). As the WPP has no energy storage capabilities for possible price arbitrage, the decision of how much constant power to offer over any individual hour-long time interval is independent of the decision for every other time interval. Hence, the problems decouple with respect to contract intervals. We assume that deviations from said contract $C$ are penalized ex-post at a price $q \in \mathbb{R}$ (\$/MWh) for negative deviations and a price $\lambda \in \mathbb{R}$ (\$/MWh) for positive deviations.

We make the following *assumptions* regarding prices and production costs.

**A2** The WPPs are assumed to be *price takers* in the forward market, as the individual WPP capacity is assumed small relative to the whole market. As such, the forward settlement price $p$ is assumed *fixed* and *known*.

**A3** The WPPs are assumed to have a *zero marginal cost of production.*

**A4** As imbalance prices $(q, \lambda) \in \mathbb{R}^2$ tend to exhibit volatility and are difficult to forecast, they are modeled as *random* variables, with expectations denoted by

$$\mu_q = \mathbb{E}[\ q\ ], \quad \mu_\lambda = \mathbb{E}[\ \lambda\ ]$$

The imbalance prices $(q, \lambda)$ are assumed to be *statistically independent* of the wind $w(t)$.

**A5** The imbalance prices are assumed to be *non-negative*, i.e., $(q, \lambda) \in \mathbb{R}_+^2$. Hence, it is never profitable to deviate from offered contracts.

*Profit Metric:* In accordance with the preceding market rules, the profit acquired by a coalition $\mathcal{S} \subseteq \mathcal{N}$ for an offered contract $C$ on the time interval $[t_0, t_f]$ is defined as

$$\Pi(C, \mathbf{w}_{\mathcal{S}}, q, \lambda) = \int_{t_0}^{t_f} pC \; - \; q \; [C - w_{\mathcal{S}}(t)]^+ \; - \; \lambda \; [w_{\mathcal{S}}(t) - C]^+ \; dt \tag{6}$$

where $x^+ := \max\{x, 0\}$ for all $x \in \mathbb{R}$. Define the *expected* profit

$$J_{\mathcal{S}}(C) \; = \; \mathbb{E} \; \Pi(C, \mathbf{w}_{\mathcal{S}}, q, \lambda). \tag{7}$$

### C. Initial Results

The *profit maximizing contract* $C^*_{\mathcal{S}}$ corresponding to a coalition $\mathcal{S} \subseteq \mathcal{N}$ can be obtained by solving the following optimization problem

$$C^*_{\mathcal{S}} = \arg\max_{C \geq 0} \; J_{\mathcal{S}}(C). \tag{8}$$

The solution to this problem is explored in depth in [1]. For completeness, the main result is restated below for the important case of $\mu_q \geq p$.

*Theorem 2.1 ([1]):* Define the time-averaged distribution $F_{\mathcal{S}}(w)$ as in (4). An *optimal contract* $C^*_{\mathcal{S}}$ is given by

$$C^*_{\mathcal{S}} \; = \; F^{-1}_{\mathcal{S}}(\gamma), \qquad \text{where} \quad \gamma = \frac{p + \mu_\lambda}{\mu_q + \mu_\lambda}. \tag{9}$$

The *optimal expected profit* is given by

$$\frac{J_{\mathcal{S}}(C^*)}{T} \; = \; \mu_q \int_0^\gamma F^{-1}_{\mathcal{S}}(x) \; dx \quad - \quad \mu_\lambda \int_\gamma^1 F^{-1}_{\mathcal{S}}(x) \; dx. \tag{10}$$

In this paper, one of our objectives is to quantify the financial benefit of coalitional bidding in two-settlement markets. As a motivating result, it is straightforward to show that the act of *risk sharing* through coalitional bidding leads to an increase in collective profit *almost surely*.

*Theorem 2.2:* Let $\{C_1, \cdots, C_N\}$ be a set of $N$ individual contracts. For $C_{\mathcal{N}} = \sum_{i=1}^N C_i$ we have *almost surely* that

$$\Pi(C_{\mathcal{N}}, \; \mathbf{w}_{\mathcal{N}}, \; q, \; \lambda) \quad \geq \quad \sum_{i=1}^N \Pi \; (C_i, \; \mathbf{w}_i, \; q, \; \lambda). \tag{11}$$

It follows from Theorem 2.2 that coalitional bidding will always result in a net profit increase that can be shared between the coalition participants. Unfortunately, the expression for optimal expected profit (10) does not provide any clue as to how the added income should be shared among the coalition participants. Naïve sharing mechanisms, such as equal distribution of the profit among the participants, are not satisfactory, because certain members of the coalition may obtain a greater profit if they were to break up the coalition and form a smaller one. Thus, our primary objective is to identify *stabilizing* payoff allocation mechanisms for wind farm coalitions. This is the subject of the remainder of the paper.

The problem of sharing collective profits has been extensively studied in cooperative game theory [13]. We will show that our problem can be modeled as a coalitional game and we will study its properties and identify sharing mechanisms that are fair from an axiomatic perspective. In the next section we review some basic concepts and results of the coalitional game theory. The interested reader may see [13], [11], [14] for a more detailed exposition on the topic.

Finally, we close this section by introducing a functional $\Psi$ that will be vital in analyzing the properties of the coalitional game associated with wind power aggregation. Let $\mathbf{x} = \{x(t) \mid t \in \mathbb{R}\}$ be a scalar stochastic process that takes nonnegative values on the interval $[t_0, t_f]$ and define the functional $\Psi[\mathbf{x}]$ as a mapping from the space of square integrable stochastic processes to the positive reals. The functional $\Psi[\mathbf{x}]$ represents the maximal expected profit achievable under the random process $\mathbf{x}$. Specifically,

$$\Psi[\mathbf{x}] \quad := \quad \max_{C \geq 0} \quad \mathbb{E} \; \Pi(C, \mathbf{x}, q, \lambda) \tag{12}$$

where $\Pi$ is defined in equation (6). The following lemma establishes certain properties of the functional $\Psi$ that will be used to characterize the coalitional game in the sequel.

*Lemma 2.3:* The functional $\Psi$ as defined in (12) is *positively homogeneous (of degree one)* and *superadditive* in the underlying random process. For any pair of random processes $\mathbf{x} = \{x(t) \mid t \in \mathbb{R}\}$ and $\mathbf{y} = \{y(t) \mid t \in \mathbb{R}\}$, we have

(i) (positive homogeneity) $\Psi[\alpha\mathbf{x}] = \alpha\Psi[\mathbf{x}] \;\; \forall \;\; \alpha \geq 0$
(ii) (superadditivity) $\Psi[\mathbf{x}] + \Psi[\mathbf{y}] \leq \Psi[\mathbf{x} + \mathbf{y}]$

where $\alpha\mathbf{x} = \{\alpha x(t) \mid t \in \mathbb{R}\}$ and $\mathbf{x}+\mathbf{y} = \{x(t)+y(t) \mid t \in \mathbb{R}\}$.

## III. Background: Results from Coalitional Game Theory

Game theory deals with rational behavior of economic agents in a mutually interactive setting. In a game, several interacting agents aim to maximize certain expected utility by making particular decisions. The final payoff of each agent depends on the decisions taken by all the agents. The *game* is specified by the set of participants, the possible decisions taken by each agent and the set of all possible payoffs. The agents in the game are called the players. A game is called *cooperative* if the players are allowed to form alliances or teams. Cooperative games [13] are also known as coalitional games and have been used extensively in diverse disciplines such as social science, economics, philosophy, psychology [11] and more recently in engineering and communication networks [15].

Let $\mathcal{N} := \{1, 2, \ldots, N\}$ denote a finite collection of players.

*Definition 3.1 (Coalition):* A *coalition* is any subset $\mathcal{S} \subseteq \mathcal{N}$. The cardinality of the coalition $\mathcal{S}$ is its number of players and is denoted by $|\mathcal{S}|$. The *set of all possible coalitions* is defined as the power set $2^{\mathcal{N}}$ of $\mathcal{N}$. The *grand coalition* $\mathcal{N}$ is the coalition that comprises every player in the game.

*Definition 3.2 (Coalitional game and value):* A coalitional game is defined by a pair $(\mathcal{N}, v)$ where $v : 2^{\mathcal{N}} \to \mathbb{R}$ is the *value function* that assigns a real value to each possible coalition $\mathcal{S} \subseteq \mathcal{N}$. The *value of the coalition* $\mathcal{S}$ is defined as $v(\mathcal{S})$.

*Definition 3.3 (Superadditive game):* A coalitional game $(\mathcal{N}, v)$ is *superadditive* if its value function is superadditive, *i.e.*, for any pair of disjoint coalitions $\mathcal{S}, \mathcal{T} \subset \mathcal{N}$ with $\mathcal{S} \cap \mathcal{T} = \emptyset$,

$$v(\mathcal{S}) + v(\mathcal{T}) \ \leq \ v(\mathcal{S} \cup \mathcal{T}) \tag{13}$$

*Remark 3.4:* Superadditivity implies that the value of a coalition cannot be improved by splitting it up into two smaller coalitions. □

A central problem in coalitional game theory is the identification of payoff allocation mechanisms that *fairly* share the coalition value $v(\mathcal{S})$ among all of the members of the coalition $\mathcal{S}$. The use of payoff allocation mechanisms that *do not fairly share* the coalition value among the members may result in certain members exiting the coalition to form more profitable sub-coalitions. We make this more precise by presenting an *axiomatic formulation of fairness* in definition 3.8. Additionally, we are interested in the class of coalitional games with *transferable payoff*.

*Definition 3.5 (Transferable payoff):* A coaltional game with *transferable payoff* is characterized by the property that there is no restriction on the sharing of coalition value between members of the coalition.

*Definition 3.6 (Payoff allocation):* A *payoff allocation* for the coalition $\mathcal{S} \subseteq \mathcal{N}$ is a vector $x \in \mathbb{R}^{|\mathcal{S}|}$ whose entries represent payoffs to each member of the coalition.

1) *(Efficiency)* An allocation $x$ is said to be *efficient* if the payoffs add up to the value of the coalition,

$$x^T \mathbf{1} = \sum_{i \in \mathcal{S}} x_i = v(\mathcal{S}).$$

2) *(Individually rational)* An allocation is said to be *individually rational* if each player gets a payoff that is at least as good as that obtained by playing alone,

$$x_i \geq v(\{i\}), \quad \forall \ \ i \in \mathcal{S}.$$

*Definition 3.7 (Imputation):* A payoff allocation $x$ for the grand coalition $\mathcal{N}$ is said to be an *imputation* if it is simultaneously efficient and individually rational. The set of all imputations $\mathcal{I}$ for the game $(\mathcal{N}, v)$ is defined as follows

$$\mathcal{I} := \left\{ x \in \mathbb{R}^N \left| \sum_{i \in \mathcal{N}} x_i = v(\mathcal{N}), \quad x_i \geq v(\{i\}), \ \forall \ i \in \mathcal{N} \right. \right\}$$

We next define a fundamental solution concept for coalitional games known as the *core*. It can be interpreted as being analogous to Nash equilibria for non-cooperative games [13].

*Definition 3.8 (The Core):* Consider a coalitional game $(\mathcal{N}, v)$ with *transferable payoff*. The *core* is defined to be the set of imputations such that no coalition can obtain a payoff which is better than the sum of the members current payoffs. Consequently, for an imputation in the core, no subgroup of players has an incentive to leave the grand coalition to form another coalition $\mathcal{S} \subset \mathcal{N}$. A mathematical expression for the core is given by:

$$\mathcal{C} := \left\{ x \in \mathbb{R}^N \left| \sum_{i \in \mathcal{N}} x_i = v(\mathcal{N}), \quad \sum_{i \in \mathcal{S}} x_i \geq v(\mathcal{S}), \forall \mathcal{S} \subseteq \mathcal{N} \right. \right\} \tag{14}$$

*Definition 3.9:* A payoff allocation $x \in \mathbb{R}^N$ is said to be *stabilizing* if if belongs to the core $\mathcal{C}$.

## A. Existence of a Nonempty Core

Certain coalitional games have a empty cores. Two important classes of games with a nonempty core are *convex games* and *balanced games*.

*Definition 3.10 (Convex game):* A coalitional game $(\mathcal{N}, v)$ is *convex* if its value function is *supermodular*, *i.e.*

$$v(\mathcal{S}) + v(\mathcal{T}) \ \leq \ v(\mathcal{S} \cup \mathcal{T}) + v(\mathcal{S} \cap \mathcal{T}), \quad \text{for all} \ \ \mathcal{S}, \mathcal{T} \subset \mathcal{N} \tag{15}$$

*Lemma 3.11 (Supermodularity):* A value function $v$ is supermodular $\iff$ for all $i \in \mathcal{N}$ and every set of coalitions $\mathcal{S} \subset \mathcal{T} \subset \mathcal{N}$ such that $\mathcal{S} \cap \{i\} = \mathcal{T} \cap \{i\} = \emptyset$, the following inequality holds:

$$v(\mathcal{S} \cup \{i\}) - v(\mathcal{S}) \ \leq \ v(\mathcal{T} \cup \{i\}) - v(\mathcal{T}) \tag{16}$$

Generally speaking, a game is convex if an individual's marginal contribution to a coalition increases if he joins a larger coalition.

*Theorem 3.12 ([17]):* A *convex* coalitional game has a *nonempty core*.

Convexity of a coalitional game is a strong condition and many real-world games are not convex. A weaker condition is balancedness of a coalitional game. In order to define a balanced coalitional game, we need to introduce the concept of a balanced map.

*Definition 3.13 (Balanced map):* A map $\alpha : 2^{\mathcal{N}} \to [0, 1]$ is said to be *balanced* if for any $i \in \mathcal{N}$,

$$\sum_{\mathcal{S} \in 2^{\mathcal{N}}} \alpha(\mathcal{S}) \mathbf{1}\{i \in \mathcal{S}\} = 1 \tag{17}$$

where $\mathbf{1}\{\cdot\}$ denotes the indicator function.

Thus, a balanced map provides a weight for each coalition in the game such that for each player $i \in \mathcal{N}$, the sum of the weights corresponding to all coalitions that contain the player $i$ equals one.

*Definition 3.14 (Balanced game):* A game $(\mathcal{N}, v)$ is *balanced* if for any balanced map $\alpha$,

$$\sum_{\mathcal{S} \in 2^{\mathcal{N}}} \alpha(\mathcal{S}) v(\mathcal{S}) \ \leq \ v(\mathcal{N}). \tag{18}$$

A balanced coalitional game always has a nonempty core. In fact, [2] and [18] independently proved, using duality in linear programming, the following result.

*Theorem 3.15: (Bondareva-Shapley Theorem)* A coalitional game has a nonempty core $\iff$ it is balanced.

However, not every coalitional game is balanced. For such games, alternative solution concepts have been introduced. The most important among these are the *Shapley value* and the *nucleolus*.

## B. Shapley Value and Nucleolus

*1) The Shapley Value:* The Shapley value takes an axiomatic approach to value allocation in a coalitional game. For a coalitional game $(\mathcal{N}, v)$, the *Shapley value* $\chi_i(v)$ denotes the payoff to each player $i \in \mathcal{N}$. The Shapley value must satisfy five basic axioms.

1) *(Individual rationality)* $\chi_i(v) \geq v(\{i\})$ for all $i \in \mathcal{N}$.
2) *(Efficiency)* $\sum_{i \in \mathcal{N}} \chi_i(v) = v(\mathcal{N})$.

3) *(Symmetry)* Let $\mathcal{S} \cap \{i, j\} = \emptyset$, if $v(\mathcal{S} \cup \{i\}) = v(\mathcal{S} \cup \{j\})$ then $\chi_i(v) = \chi_j(v)$.
4) *(Dummy action)* Let $\mathcal{S} \cap \{i\} = \emptyset$, if $v(\mathcal{S} \cup \{i\}) = v(\mathcal{S})$ then $\chi_i(v) = 0$.
5) *(Additivity)* If $v_1$ and $v_2$ are two value functions then $\chi_i(v_1 + v_2) = \chi_i(v_1) + \chi_i(v_2)$.

*Theorem 3.16:* Consider a coalitional game $(\mathcal{N}, v)$. An analytical expression for the corresponding *Shapley value* is given by

$$\chi_i(v) = \sum_{\mathcal{S} \subset \mathcal{N} \setminus \{i\}} \frac{|\mathcal{S}|!(N - |\mathcal{S}| - 1)!}{N!} \left[ v(\mathcal{S} \cup \{i\}) - v(\mathcal{S}) \right]. \tag{19}$$

The Shapley value $\chi_i(v)$ can be interpreted as the expected marginal contribution of player $i$ to the grand coalition $\mathcal{N}$ when it joins at a uniformly at random order. The weight is the probability that player $i$ enters right after every player in the coalition $\mathcal{S}$.

*Remark 3.17: (Relation to the core)* The Shapley value always exists but is *not necessarily in the core.* If a coalitional game has a nonempty core and if in addition the imputation defined by the Shapley value lies in the core, then this imputation shares the stability properties of the core and the fairness established by the axioms of the Shapley value. As a matter of fact, for a convex game, the imputation corresponding to the Shapley value is always in the core [17]. However, this is not true, in general, for a balanced game. □

*2) The Nucleolus:* The *nucleolus* of a coalitional game $(\mathcal{N}, v)$ is an imputation that minimizes the *dissatisfaction* of the players. Let $x \in \mathbb{R}^N$ be an imputation associated with the coalitional game $(\mathcal{N}, v)$. The *dissatisfaction* of a coalition $\mathcal{S}$ with respect to the imputation $x$ is measured by the *excess* defined as follows:

$$e(x, \mathcal{S}) = v(\mathcal{S}) - \sum_{i \in \mathcal{S}} x_i. \tag{20}$$

For a given imputation $x$, define the associated *excess vector*, $\theta(x) \in \mathbb{R}^{2^N - 2}$, as a vector whose entries are the excesses for all coalitions (excluding the grand coalition) arranged in nonincreasing order, *i.e.*

$$\theta_i(x) \leq \theta_j(x) \quad \text{for all} \quad i, j \in \mathbb{N} \quad \text{such that} \quad i \geq j.$$

Let $\Theta$ denote the set of excess vectors associated with each imputation $x \in \mathcal{I}$ for a coalitional game $(\mathcal{N}, v)$.

$$\Theta = \{\theta(x) : x \in \mathcal{I}\} \tag{21}$$

*Definition 3.18 (Lexicographic order):* Define a lexicographic order on the elements of $\Theta$ as follows: $\theta(x) \leq_{lex} \theta(y)$ if there exists an index $k \in \mathbb{N}$ such that for all $i < k$, $\theta_i(x) = \theta_i(y)$ and $\theta_k(x) \leq \theta_k(y)$.

*Definition 3.19 (Nucleolus):* The nucleolus of the game $(\mathcal{N}, v)$ is the *lexicographically minimal imputation* based on this ordering.

*Remark 3.20: (Relation to the core)* The core can be easily related to the nucleolus solution concept [4]. The nucleolus always exists and is unique. Moreover, the nucleolus belongs to the core, if the core is non-empty, as the the core is the set of all imputations with negative or zero excesses. □

## IV. A Coalitional Game for Wind Energy Aggregation

Let $\mathcal{N} = \{1, \cdots, N\}$ denote the set of $N$ wind power producers (WPP) connected to a common bus in the network. Using tools from coalitional game theory, we aim to (i) prove that a collection of wind power producers (WPP) can improve their expected optimal profit, in aggregate, by forming a coalition to jointly offer their aggregate power as a single entity and (ii) identify stabilizing mechanisms to allocate the additional profit among the members of the coalition.

We model the formation of a willing coalition among wind power producers to jointly offer a contract for energy in a two-settlement market as a *coalitional game* $(\mathcal{N}, v)$, where the *value function* $v(\mathcal{S})$ is defined as the expected profit corresponding to an optimal coalitional offer (Theorem 2.1) of the aggregate wind power $\mathbf{w}_{\mathcal{S}}$ associated with the coalition $\mathcal{S} \subseteq \mathcal{N}$.

$$v(\mathcal{S}) = \Psi[\mathbf{w}_{\mathcal{S}}] \;=\; \max_{C \geq 0} \;\; \mathbb{E}\, \Pi(C, \mathbf{w}_{\mathcal{S}}, q, \lambda) \tag{22}$$

In section IV-A, we prove that the corresponding coalitional game is *superadditive*, from which it follows that the the formation of a grand coalition $\mathcal{N}$ is optimal from the perspective of maximizing the WPPs collective expected profit. We also prove that the coalitional game is *balanced* and hence has a *nonempty core* (i.e., $\mathcal{C} \neq \emptyset$). This guarantees the existence of a *fair* payoff allocation in the core.

The challenge is to find an *imputation* $x^* \in \mathbb{R}^N$ *in the core* $\mathcal{C}$. Through counterexample, we show in section IV-B that the coalitional game for wind energy aggregation is *not convex* and that the *Shapley value* does not necessarily belong to the core. Although the nucleolus belongs to the core for a balanced game, its calculation can be computationally demanding, as it requires the solution of a sequence of $o(2^N)$ linear programs [16]. As an alternative, we propose the use of a candidate imputation that minimizes the *worst-case excess* for every coalition.

Finally, as the coalitional value function $v$ (22) is defined in the metric of optimal *expected* profit, an imputation $x^* \in \mathbb{R}^N$ belonging to the corresponding *core* $\mathcal{C}$, represents the payment that each WPP (coalition member) should receive in expectation. In practice, the realized profit will vary day by day, as the profit (6) is a random variable. Hence, given any realization of the profit, we propose, in section IV-C, a payoff allocation mechanism to distribute the realized profit among the coalition members, such that the payment that each member receives – averaged over an increasing number of days – approaches the imputation $x^* \in \mathcal{C}$.

### A. Properties of the Coalitional Game

*Theorem 4.1:* The coalitional game $(\mathcal{N}, v)$ for wind energy aggregation is superadditive.

*Remark 4.2:* (*Positively Correlated Wind Processes*) Superadditivity of the game $(\mathcal{N}, v)$ guarantees that coalition formation will never detract from the members' expected profit in aggregate. However, in the worst case of *perfectly positively correlated* wind power process, the coalition optimal expected profit equals the sum of the individuals' optimal expected profits if they were to participate in the market independently – i.e., $v(\mathcal{N}) = \sum_{i=1}^{N} v(\{i\})$. $\square$

Theorem 4.1 guarantees that wind power producers can improve their expected profit by forming coalitions with other producers to jointly offer a contract for their aggregate power. Moreover, the larger the coalition the greater the improvement in the aggregate expected profit – indicating that the most profitable coalition is the *grand coalition*. Superadditivity, however, does not guarantee the existence of a *stabilizing* payoff allocation – i.e., the existence of a *non-empty core*. In Theorem 4.3, we prove nonemptiness of the core corresponding to the game $(\mathcal{N}, v)$. Homogeneity and superadditivity of the functional $\Psi$, as in Lemma 2.3, are instrumental in the proof of this theorem.

*Theorem 4.3:* The coalitional game $(\mathcal{N}, v)$ for wind energy aggregation is *balanced* and thus has a *nonempty core*.

### *B. Sharing of Expected Coalition Profit*

As the coalitional game for wind energy aggregation has a nonempty core, there exists an imputation in the core that guarantees that no wind power producer can improve its expected profit by defecting from the grand coalition.

*1) The Shapley Value Is Not in the Core:* For convex games, the Shapley value provides a closed-form expression for an imputation that belongs to the core. It can be shown through a counterexample, however, that our coalitional games is not convex and that the Shapley value does not necessarily specify an imputation belonging to the core.

*Example 4.4 (Counterexample):* Consider a coalitional game involving three independent wind power producers, $\mathcal{N} = \{1, 2, 3\}$, offering contracts on the time interval $[t_0, t_f]$ of length one hour. Each wind power process $\mathbf{w}_i$ $(i = 1, 2, 3)$ is assumed to be *stationary in the strict sense* with discrete marginal distributions. The wind power processes $\mathbf{w}_1$ and $\mathbf{w}_2$ are assumed to be *independent* and have *identical marginal distributions* defined by

$$w_i(t) = \begin{cases} 1, & w.p. \quad 0.5 \\ 2, & w.p. \quad 0.5 \end{cases} \qquad i = 1, 2 \quad \text{for all} \quad t.$$

The wind power process $\mathbf{w}_3$ is assumed to be *perfectly positively correlated* to $\mathbf{w}_2$, i.e., $w_3(t) = w_2(t)$ for all $t$. The forward market clearing price and expected imbalance prices are set at $p = 0.5$, $\mu_q = 1$, and $\mu_\lambda = 0$.

Consider the coalitional game $(\mathcal{N}, v)$. The time-averaged cumulative distribution function $F_{\mathcal{S}}(w)$ and value $v(\mathcal{S})$ associated with each coalition $\mathcal{S} \subseteq \mathcal{N}$ are depicted in Figure 1. The shaded blue area depicts the value $v(\mathcal{S})$ for each coalition. The numerical values are given by the following:

$$\begin{aligned} v(\{i\}) &= \Psi[w_i] = 0.5, \;\; i \in \{1, 2, 3\} \\ v(\{1, i\}) &= \Psi[w_1 + w_i] = 1.25, \;\; i \in \{2, 3\} \\ v(\{2, 3\}) &= \Psi[w_2 + w_3] = 2v(\{2\}) = 1 \\ v(\{1, 2, 3\}) &= \Psi[w_1 + w_2 + w_3] = 1.75 \end{aligned}$$

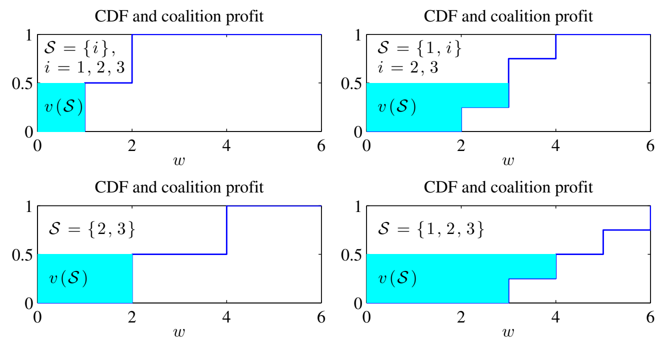


Fig. 1. This figure depicts the time-averaged cumulative distribution function $\Phi_{\mathcal{S}}(w)$ and value $v(\mathcal{S})$ associated with each coalition $\mathcal{S}$ in the power set of $\mathcal{N}$. The shaded blue area depicts the value $v(\mathcal{S})$ for each coalition.

As indicated in Theorem 4.3, this coalitional game is balanced and, consequently, has a nonempty core. However, this game is *not convex*, as the *value function is not supermodular*. Take for example,

$$\begin{aligned} v(\{1, 2, 3\}) - v(\{1, 2\}) &= 0.50 \\ &< v(\{1, 3\}) - v(\{1\}) = 0.75, \end{aligned}$$

which contradicts the supermodularity property defined in equation (16).

We now show that the imputation given by the Shapley value is *not in the core*. An imputation $x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$ is in the core if it satisfies the following conditions, as defined by equation (14).

$$\begin{aligned} x_i &\geq v(\{i\}) = 0.5, \;\; i \in \{1, 2, 3\} && (23) \\ x_1 + x_i &\geq v(\{1, i\}) = 1.25, \;\; i \in \{2, 3\} && (24) \\ x_2 + x_3 &\geq v(\{2, 3\}) = 1.0 && (25) \\ x_1 + x_2 + x_3 &= v(\{1, 2, 3\}) = 1.75 && (26) \end{aligned}$$

The imputation given by the Shapley value can be easily computed using the closed form expression in equation (19):

$$\chi_1(v) = \frac{2}{3}, \quad \chi_2(v) = \frac{1.625}{3}, \quad \chi_3(v) = \frac{1.625}{3}$$

It is straightforward to see that the Shapley value violates condition (24):

$$\chi_1(v) + \chi_2(v) = \frac{3.625}{3} = 1.2083 < 1.25 = v(\{1, 2\}).$$

Hence, *the imputation given by the Shapley value is not in the core* for this particular game. $\blacksquare$

*2) The Nucleolus and Minimizing Worst-Case Excess:* With respect to the coalitional game for wind energy aggregation, the previous counterexample 4.4 proves that the game *not convex* and, consequently, the imputation given by the Shapley value is not guaranteed to belong to the core. The strength in application of the Shapley value resides in its closed form characterization – providing computational efficiency. However, as the Shapley value for a non-convex game is not guaranteed to belong to the core, one must seek alternative solution concepts to obtain imputations in the core.

As noted in Section III, the *nucleolus* is guaranteed to belong to the core for a balanced game. However, as the nucleolus is defined as the imputation with the lexicographically minimal excess vector, its computation requires the solution of a sequence of $o(2^N)$ linear programs [16]. This can be computationally demanding.

To surmount this difficulty we propose the use of a candidate imputation that *minimizes the worst-case excess* for every coalition. This imputation is defined as follows:

$$e^* = \min_{x\in\mathbb{R}^N} \max_{\mathcal{S}\in 2^{\mathcal{N}}} e(x,\mathcal{S}) \text{ s.t } \begin{cases} e(x,\mathcal{N}) \ = \ 0 \\ v(\{i\}) - x_i \ \leq \ 0 \ \ \forall\, i\in\mathcal{N} \end{cases}$$

In contrast to the nucleolus solution concept, computation of the imputation that minimizes the worst-case excess can be recast as a *single linear program*:

$$e^* = \min_{x\in\mathbb{R}^N,\ e\in\mathbb{R}} e \text{ s.t. } \begin{cases} v(\mathcal{S}) - \sum_{i\in\mathcal{S}} x_i - e \ \leq \ 0, \forall \mathcal{S}\subset\mathcal{N} \\ v(\mathcal{N}) - \sum_{i\in\mathcal{N}} x_i = 0 \\ v(\{i\}) - x_i \ \leq \ 0, \ \ \forall\, i\in\mathcal{N} \end{cases} \tag{27}$$

Although the imputation that minimizes the worst-case excess in problem (27) is not guaranteed to belong to the core, it is a simple matter to check feasibility with respect to the core.

*Lemma 4.5:* A feasible imputation $x^*$ achieving the minimal cost $e^*$ in problem (27) belongs to the core if $e^* \leq 0$.

The following example depicts an instance where this imputation that minimizes worst-case excess belongs to the core and the Shapley value does not.

*Example 4.6:* Consider again the coalitional game corresponding to the Example 4.4 in Section IV. Recall that the Shapley value of this game does not belong to the core. Since the coalitional game is balanced, it has a nonempty core. Using problem formulation (27), we can solve a linear program (LP) to compute an imputation that minimizes the worst-case excess for any possible coalition in the game. Such an imputation is computed by solving the following LP corresponding to our game.

$$\begin{array}{ll} \text{Minimize} & e \\ \text{subject to} & e + x_i - 0.5 \geq 0,\ i\in\{1,2,3\} \\ & e + x_1 + x_i - 1.25 \geq 0,\ i\in\{2,3\} \\ & e + x_2 + x_3 - 1.0 \geq 0 \\ & x_1 + x_2 + x_3 = 1.75 \\ & x_i - 0.5 \geq 0,\ i\in\{1,2,3\} \end{array}$$

The minimal cost $e^*$ and corresponding imputation $x^*$ are given by

$$e^* = 0, \qquad x_1^* = 0.75, \ \ x_2^* = 0.5, \ \ x_3^* = 0.5.$$

Moreover, in contrast to the Shapley value for this game, the imputation $x^*$ *belongs to the core* as $e^* = 0$. ■

### C. Sharing of Realized Coalition Profit

We have thus far focused our attention on the computation of payoff allocations that fairly distribute the *optimal expected profit* among coalition members. This approach stems from our formulation of the coalitional game $(\mathcal{N}, v)$ as having a value function $v$ defined in the metric of optimal *expected* profit,

$$v(\mathcal{S}) \ = \ \max_{C\geq 0} \ \ \mathbb{E}\,\Pi(C,\mathbf{w}_{\mathcal{S}},q,\lambda) \quad \text{for all} \ \ \mathcal{S}\subseteq\mathcal{N}.$$

Consequently, an imputation $x^*\in\mathbb{R}^N$ belonging to the corresponding *core* $\mathcal{C}$, represents the payment that each WPP (coalition member) should receive in expectation. In practice, however, the realized profit for the grand coalition will vary day to day, as the profit (6) is inherently a random variable given its dependence on the random wind power process $\mathbf{w}_{\mathcal{S}}$ and imbalance prices $(q,\lambda)$. A natural question thus arises. Does there exist a profit allocation mechanism to distribute the *realized profit* among the coalition members, such that the payment that each member receives – averaged over an increasing number of days – approaches the imputation $x^*\in\mathcal{C}$? Under certain assumptions, the answer is yes.

**A6** We assume that the wind power process $\mathbf{w}_{\mathcal{S}}^k$ (for all $\mathcal{S}\subseteq\mathcal{N}$) and imbalance prices $(q^k,\lambda^k)$ are *iid* across days indexed by $k$.

$$w_{\mathcal{S}}^k(t) \perp\!\!\!\perp w_{\mathcal{S}}^j(t), \quad q^k \perp\!\!\!\perp q^j, \quad \lambda^k \perp\!\!\!\perp \lambda^j$$

for all times $t\in[t_0,t_f]$ and days $k\neq j$.

It follows that the optimal profit (28), corresponding to any coalition $\mathcal{S}\subseteq\mathcal{N}$, is likewise an *iid* sequence $\{\Pi_{\mathcal{S}}^k\}$ across days.

$$\Pi_{\mathcal{S}}^k := \Pi(C_{\mathcal{S}}^*,\mathbf{w}_{\mathcal{S}}^k,q^k,\lambda^k), \quad \text{where} \quad C_{\mathcal{S}}^* = F_{\mathcal{S}}^{-1}(\gamma^k) \tag{28}$$

*Remark 4.7:* (*Cyclostationarity*) The assumption of distribution stationarity, across days, is motivated by the empirical observation of strong diurnal periodicity in the underlying wind speed and price processes [20]. □

*Remark 4.8:* (*Negative Profit Realization*) Whereas the expected optimal profit is guaranteed to be nonnegative, it is important to note that realized optimal profit can take on negative values. Consequently, there may occur a day such that certain members of the coalition have to *pay* for their contribution to the cost of contract imbalance. □

*1) A Consistent Approach to Daily Profit Allocation:* Denote the allocation of the profit realized on day $k$ by

$$\varrho^k = \begin{bmatrix}\varrho_1^k & \cdots & \varrho_N^k\end{bmatrix}^T \in\mathbb{R}^N,$$

where coalition member $i$ receives $\varrho_i^k$ of the realized profit on day $k$.

*Definition 4.9 (Budget Balanced):* A profit allocation $\varrho_k\in\mathbb{R}^N$ is *budget balanced* with respect to the profit realized on day $k$ if

$$\sum_{i=1}^N \varrho_i^k \ = \ \Pi_{\mathcal{N}}^k.$$

*Definition 4.10 (Consistency):* A mechanism for daily profit allocation $\varrho_k$ is *strongly consistent* with respect to a fixed allocation $x\in\mathbb{R}^N$ if

$$\frac{1}{K}\sum_{k=1}^K \varrho_i^k \ \xrightarrow{a.s.} \ x_i.$$

Consider the following naïve mechanism for daily profit allocation. Let $x^* \in \mathbb{R}^N$ be an imputation in the core $\mathcal{C}$ for the coalitional game defined by the value function (22). Given a realization of profit $\Pi_{\mathcal{N}}^k$ on day $k$ for the grand coalition $\mathcal{N}$, *distribute the profit among the coalition members according to the following rule:*

$$\varrho_i^k \;=\; \beta_i \; \Pi_{\mathcal{N}}^k, \quad \text{where} \quad \beta_i = \frac{x_i^*}{\sum_{j=1}^N x_j^*} \tag{29}$$

*Theorem 4.11:* The naïve profit allocation mechanism (29) is both *budget balanced* and *strongly consistent* with respect to the corresponding imputation $x^* \in \mathcal{C}$ on which it is based.

*Remark 4.12:* (*Defection in the Short Run*) The sharing of the realized coalition profit in accordance with (29) – although fair in the long run – may lead to the defection of certain coalition members in the short run if said members consistently receive payments that are below that which would have been attainable through independent participation in the market, i.e., if the event

$$\varrho_i^k < \Pi(C_i^*, \mathbf{w}_i^k, q^k, \lambda^k) \tag{30}$$

occurs with a sufficiently high frequency.

We are currently exploring alternative formulations of the coalitional game to discourage defection of coalition members in the short run. For example, consider a formulation where the value function is defined as the realized optimal profit (31), rather than the expected optimal profit (22):

$$v(\mathcal{S}) \;=\; \Pi(C_{\mathcal{S}}^*, \mathbf{w}_{\mathcal{S}}, q, \lambda), \quad \text{for all} \quad \mathcal{S} \subseteq \mathcal{N}. \tag{31}$$

Working with such a stochastic formulation of the coalition game $(\mathcal{N}, v)$, one can directly compute *stabilizing* profit allocations explicitly as a function of the realized wind power production and imbalance prices. Moreover, assuming the existence of a nonempty core for such a game, a daily payoff allocation given by $\varrho^k = x^{*,k}$, where $x^{*,k}$ is an imputation in the core associated with day $k$, would guarantee that event (30) never occurs – among other beneficial properties. □

## V. Conclusion

Using coalitional game theory as a vehicle for our analysis, we have analyzed the benefits of aggregation attainable through the formation of a willing coalition among wind power producers (WPP) to pool their variable power to jointly offer the aggregate output as single entity into a forward energy market. Having assumed transferable payoff and a value function defined as the maximum expected profit attainable through competitive bidding, we have shown that the associated coalitional game is *superadditive* and *balanced*. Consequently, the *core* of such a game is necessarily *nonempty* – or more simply, there exists a *stabilizing* profit sharing rule that is satisfactory from the perspective of every coalition participant. To this end, we propose an sharing rule – that minimizes worst-case excess for each coalition in the game – to fairly allocate the expected profit among coalition members.

Our results demonstrate that wind power aggregation and coalitional bidding can serve as an effective means for improving wind power profitability in the face of future production uncertainty. However, our results are limited to the setting in which all WPPs are connected to a common single bus in the network. As the transmission network can severely constrain a coalition's ability to directly aggregate wind power generated at different buses, we are presently working on extensions of these results to the multi-bus network setting to account for transmission effects.

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