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Lyapunov-Based Cooperative Avoidance Control for Multiple Lagrangian Systems with Bounded Sensing Uncertainties

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***Abstract*— We present a decentralized, real-time, cooperative avoidance control law for a group of nonlinear Lagrangian systems with bounded control inputs, limited sensing ranges, and bounded sensing errors. The control formulation builds on the concept of avoidance control and uses Lyapunov-based analysis to guarantee collision-free trajectories for a group of N vehicles with sensing uncertainties. Advantages of the cooperative avoidance strategy include its easy synthesis with other stable control laws and its null effect on the agent's main task when other vehicles and obstacles are sufficiently away. Two numerical examples are finally presented that illustrate the performance of the proposed control framework.**

I. Introduction

One of the most critical challenges in multi-vehicle systems is to guarantee collision avoidance between neighboring agents and obstacles at all times independently of sensing errors. Unmanned vehicles and mobile robots typically rely on navigation and localization sensors to estimate the distance to nearby agents and obstacles or on wireless communication networks for the broadcast of position coordinates among agents. These sensing mechanisms, in which we include communication networks, may inaccurately estimate the position of obstacles and agents as a result of process delays, interferences, noise, and quantization. For instance, obstacle's position measurements sampled by vision-based sensing mechanisms on many mobile robotic systems are easily affected by weather conditions and light variations [1]. Similarly, underwater localization equipment on board of unmanned vehicles, such as sonar radars and inertial measurements units, may also experience substantial delays, slow sampling rates, and dead reckoning errors [2]. If these estimation errors are not carefully considered when controlling the motion of the vehicle, the system may become vulnerable to collisions. Therefore, avoidance strategies for autonomous navigation must provide robustness to sensing uncertainties.

Collision avoidance strategies coping with sensing uncertainties have been predominantly studied within the field of path planning, where a complete obstacle-free path from the agent's current location to the next target is developed based on estimates of the initial position of obstacles. Examples include the certainty [3] and occupancy grid [4], where the robot's environment is divided into an array of cells with each cell containing a probability of having an obstacle. Then, a safe path, which the robot is meant to follow, is traced according to this probability map. Although these control strategies have been shown to be robust to common sensor uncertainties, they require other agents and obstacles to be static or to move at low speeds such that the agent's initial sensing observation remains true throughout the entire trajectory. An alternate approach with a similar drawback is proposed in [5], where a noncooperative collision avoidance strategy based on the concept of reachable sets [6] is described for zero-velocity obstacles.

In contrast to path planning algorithms, real-time collision avoidance strategies compute the avoidance control inputs online as obstacles are detected, therefore, facilitating (in most cases) the treatment of fast-moving obstacles. Real-time collision avoidance algorithms considering sensing uncertainties have been introduced in [7] and [8] based on a variation of the occupancy grid [4] that incorporates estimates of the obstacles' velocities. Yet, these previous control approaches do not fully investigate the case of time-varying speed obstacles and assume the worst case scenario in which other agents do not try to avoid a collision (i.e., a noncooperative strategy). In [9], a decentralized real-time avoidance strategy for the case of two agents with double integrator dynamics and bounded control inputs is presented using Lyapunov-based analysis. However, the theoretical results are not extended to the general case of multiple nonlinear agents.

In this paper, we now introduce a decentralized, real-time, cooperative avoidance control strategy for a group of heterogeneous nonlinear Lagrangian systems with bounded control inputs and limited sensing. The collision avoidance control formulation is based on the concept of avoidance control [10], [11], yet the avoidance functions and control inputs proposed herein are bounded. The overall control framework is able to cope with bounded sensing errors (including those caused by delays, noise, and quantization) by treating the effect of uncertainties as a disturbance in the control input, similar to [12]. However, the control formulation in [12] does not guarantee robustness with respect to sensing uncertainties and assumes unbounded control inputs. Advantages of the proposed controller also include the activation of the avoidance control only when the vehicle is close to another agent and the relative easy synthesis with other stable control laws. By using Lyapunov-based analysis we are able to present sufficient conditions that guarantee collision-free transit for a group of N nonlinear agents.

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A. Notation

As standard notation, we denote the p-norm of a vector $\mathbf{x} = [x_1, \cdots, x_n]^T \in \Re^n$ as $\|\mathbf{x}\|_p := (|x_i|^p + \cdots + |x_n|^p)^{1/p}$ for $1 \leq p < \infty$ and $\|\mathbf{x}\|_p = \max_i |x_i|$ for $p = \infty$, where $|x_i|$ is the absolute value of a real scalar x_i. We define the induce p-norm of a matrix $A \in \Re^{m \times n}$ as $\|A\|_p := \sup_{\mathbf{x} \neq \mathbf{0}} \frac{\|A\mathbf{x}\|_p}{\|\mathbf{x}\|_p}$. For the 2-norm of a vector or matrix we use the simpler notation $\|\cdot\|$. As a shorthand for a matrix $A \in \Re^{m \times n}$ we use $[a_{kl}]_{m \times n}$, where a_{kl} is the klth entry of A.

II. Multi-Lagrangian System with Bounded Control Inputs and Sensing Uncertainties

Herein, we address the task of controlling a group of N n-degree-of-freedom (DOF) vehicles with Lagrangian dynamics given by

$$M_i(\mathbf{q}_i(t))\ddot{\mathbf{q}}_i(t) + C_i(\mathbf{q}_i(t), \dot{\mathbf{q}}_i(t))\dot{\mathbf{q}}_i(t) = \mathbf{u}_i(t) \qquad (1)$$

where $\mathbf{q}_i \in \Re^n$ are the generalized coordinates, $M_i \in \Re^{n \times n}$ are the positive definite inertia matrices, $C_i \in \Re^{n \times n}$ are the centrifugal and Coriolis matrices, and $\mathbf{u}_i \in \mathcal{U}_i \subset \Re^n$ are the control inputs for $i \in \{1, \cdots, N\}$. We assume that gravitational forces are negligible or compensated via active control and that the magnitudes of the control inputs are radially bounded, i.e., $\exists \mu_i > 0$ such that $\|\mathbf{u}_i(t)\| \leq \mu_i \; \forall i \in \{1, \cdots, N\}, \; t \geq 0$. Moreover, we assume that each agent can locate other near agents with a known bounded error. That is, we suppose that the ith agent is able to sense the jth agent as being located at $\hat{\mathbf{q}}_j^i(t) = \mathbf{q}_j(t) + \mathbf{d}_{ij}(t)$ whenever the jth agent is sufficiently close to the ith agent. The vector $\mathbf{d}_{ij} \in \Re^n$ represents the uncertainty (e.g., due to delays, noise, and quantization) incurred during the localization of the jth vehicle by the ith agent and is considered to be upper bounded by some positive constant Δ_i, i.e., $\|\mathbf{d}_{ij}(t)\| \leq \Delta_i, \; \forall t \geq 0$ and $j \neq i$.

Finally, we make the assumption that the agents under consideration satisfy the following properties, which is true for a wide class of nonlinear systems [13].

Property 2.1: $\dot{M}_i(\mathbf{q}_i) = C_i(\mathbf{q}_i, \dot{\mathbf{q}}_i) + C_i^T(\mathbf{q}_i, \dot{\mathbf{q}}_i)$.

Property 2.2: $\exists$ positive constants $\underline{\lambda}_i$ and $\overline{\lambda}_i$ such that $\underline{\lambda}_i I \leq M_i(\mathbf{q}_i) \leq \overline{\lambda}_i I$, where $I \in \Re^{n \times n}$ is the identity matrix.

III. Control Objective and Definitions

Our control goal is to design decentralized collision avoidance strategies that guarantee the safe navigation of a group of vehicles with sensing uncertainties. Specifically, we would like to guarantee a minimum safe distance between any two vehicles at any time regardless of measurement errors, delays, and noise incurred in the detection process. Additionally, we would like to design the avoidance control strategy to be active only when another vehicle or obstacle is within a short distance. With this in mind, we introduce the following definitions.

First, we define the group $\mathcal{N}_i$ as the set of agents in the vicinity of the ith vehicle. We assume that any jth agent in $\mathcal{N}_i$ can be located by the ith vehicle if the former lies within the bounded *Detection Region*, $\mathcal{D}_{ij}$, of the latter given as

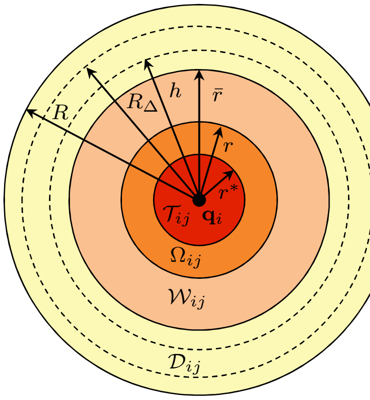

Fig. 1. Antitarget ($\mathcal{T}_{ij}$), Avoidance (Ω_{ij}), Conflict ($\mathcal{W}_{ij}$), and Detection ($\mathcal{D}_{ij}$) Regions for the ith agent.

$$\mathcal{D}_{ij} = \left\{\mathbf{q} : \mathbf{q} \in \Re^{2n}, \|\mathbf{q}_i - \mathbf{q}_j\| \leq R_i\right\}$$

where $R_i > 0$ is the ith vehicle's detection radius and $\mathbf{q}(t) = [\mathbf{q}_i^T(t), \mathbf{q}_j^T(t)]^T$. In addition, we define an *Antitarget Region*, $\mathcal{T}_{ij}$, as the collision zone for the ith agent, i.e.,

$$\mathcal{T}_{ij} = \left\{\mathbf{q} : \mathbf{q} \in \Re^{2n}, \|\mathbf{q}_i - \mathbf{q}_j\| \leq r_{ij}^*\right\}$$

where $r_{ij}^* \in (0, R_i)$ is the minimum safe distance between the ith vehicle and any jth agent in $\mathcal{N}_i$. Similarly, we define an *Avoidance Region*, $\Omega_{ij} \supseteq \mathcal{T}_{ij}$, as a restricted zone for which any agent in $\mathcal{N}_i$ is forbidden. That is,

$$\Omega_{ij} = \left\{\mathbf{q} : \mathbf{q} \in \Re^{2n}, \|\mathbf{q}_i - \mathbf{q}_j\| \leq r_{ij}\right\}$$

where $r_{ij} \in [r_{ij}^*, R_i)$ is the desired minimum distance between the ith and jth agents. Therefore, any collision avoidance strategy designed to avoid Ω_{ij}, will also avoid $\mathcal{T}_{ij}$.

Finally, since the control input and acceleration for the ith vehicle are bounded, any collision avoidance control law must be effected with enough anticipation, such that the ith vehicle has sufficient time to decelerate and prevent a collision. Hence, we define a *Conflict Region*, $\mathcal{W}_{ij}$, as

$$\mathcal{W}_{ij} = \left\{\mathbf{q} : \mathbf{q} \in \Re^{2n}, r_{ij} < \|\mathbf{q}_i - \mathbf{q}_j\| \leq \bar{r}_{ij}\right\}$$

where $\bar{r}_{ij} \in (r_{ij}, R_i)$ is a lower bound on the distance that the ith agent can come from the jth vehicle and still be able to decelerate and avoid Ω_{ij}. Therefore, any collision avoidance strategy for the ith agent must take effect as soon as $\mathbf{q}_i$ and $\mathbf{q}_j$ enter $\mathcal{W}_{ij}$.

Having defined the Antitarget, Avoidance, Conflict, and Detection regions, we can state the control objective as follows. Given $\{\Delta_1, \cdots, \Delta_N\}$, $\mathcal{T} = \bigcup_{i \in N, j \in \mathcal{N}_i} \mathcal{T}_{ij}$, and $\mathcal{D} = \bigcup_{i \in N, j \in \mathcal{N}_i} \mathcal{D}_{ij}$, design control inputs $\{\mathbf{u}_i(t), \cdots, \mathbf{u}_N(t)\}$ such that $[\mathbf{q}_1^T(t), \cdots, \mathbf{q}_N^T(t)]^T \notin \Omega = \bigcup_{i \in N, j \in \mathcal{N}_i} \Omega_{ij}$ for all $t \geq 0$, where $\Omega \supseteq \mathcal{T}$.

For simplicity, we define $R = \min_i\{R_i\}$, $r^* = \max_i\{r_{ij}^*\}$, and $\Delta = \max_i\{\Delta_i\}$, and let $\bar{r}_{ij} = \bar{r} \; \forall i \in \{1, \cdots, N\}$, $j \in \mathcal{N}_i$. An illustration is presented in Fig. 1.

IV. Control Framework

In order to achieve our control objectives, we consider the following control input

$$\mathbf{u}_i = \mathbf{u}_i^o + \mathbf{u}_i^a, \qquad \|\mathbf{u}_i\| \le \mu_i \tag{2}$$

where $\mathbf{u}_i^o$ and $\mathbf{u}_i^a$ are the objective and collision avoidance control laws, respectively. The objective input is taken to be a stable control law designed to achieve a particular task such as trajectory tracking or set-point regulation. The collision avoidance input is a control policy aimed to guarantee collision-free transit among agents independently of bounded sensing uncertainties. Ideally, $\mathbf{u}_i^a$ must be designed such that it does not interfere with the objective control $\mathbf{u}_i^o$ when no potential collision is present.

According to this formulation, we propose the objective control law to be computed as

$$\mathbf{u}_i^o = -\bar{\mu}_i \frac{\partial V_i^{o^T}(\mathbf{q}_i)}{\partial \mathbf{q}_i} \tag{3}$$

where V_i^o is an objective function satisfying the following two properties.

Property 4.1: $0 \le V_i^o(\mathbf{q}_i) \le \alpha_i$, for some $\alpha_i > 0$.

Property 4.2: $\|\partial V_i^o(\mathbf{q}_i)/\partial \mathbf{q}_i\| \le \beta_i$, for some $\beta_i > 0$.

On the other hand, we propose the collision avoidance control to be given as

$$\mathbf{u}_i^a = -\bar{\mu}_i \sum_{j\in\mathcal{N}_i} \frac{\partial V_{ij}^{a^T}(\mathbf{q}_i, \hat{\mathbf{q}}_j^i)}{\partial \mathbf{q}_i} - \sum_{j\in\mathcal{N}_i} \gamma_i \boldsymbol{\theta}_{ij}(\dot{\mathbf{q}}_i, \mathbf{q}_i, \hat{\mathbf{q}}_j^i) \tag{4}$$

where $\bar{\mu}_i = \frac{\mu_i}{N-1+\varepsilon+\beta_i}$, $\gamma_i = \varepsilon\bar{\mu}_i$, and $\boldsymbol{\theta}_{ij} = \boldsymbol{\theta}_{ij}(\dot{\mathbf{q}}_i, \mathbf{q}_i, \hat{\mathbf{q}}_j^i)$ is given by

$$\boldsymbol{\theta}_{ij} = \begin{cases} \dfrac{\dot{\mathbf{q}}_i}{(N-1)\|\dot{\mathbf{q}}_i\|}, & \text{if } \|\dot{\mathbf{q}}_i\| > 0 \text{ and } \left\|\mathbf{q}_i - \hat{\mathbf{q}}_j^i\right\| \le R \\ \mathbf{0}, & \text{otherwise} \end{cases}$$

for some $\varepsilon \in [0, 1)$. The avoidance function V_{ij}^a, illustrated in Figure 2, is defined as

$$V_{ij}^a(\mathbf{q}_i, \mathbf{q}_j) = \begin{cases} 0, & \text{if } \|\mathbf{q}_i - \mathbf{q}_j\| \ge R_\Delta \\ \frac{R_\Delta + h}{2} - \|\mathbf{q}_i - \mathbf{q}_j\|, & \text{if } \|\mathbf{q}_i - \mathbf{q}_j\| < h \\ \frac{(\|\mathbf{q}_i - \mathbf{q}_j\| - R_\Delta)^2}{2(R_\Delta - h)}, & \text{otherwise} \end{cases} \tag{5}$$

where $R_\Delta = R - \Delta$ and $h = \bar{r} + \Delta < R_\Delta$. The reader can easily verify that V_{ij}^a is positive semi-definite, almost everywhere continuously differentiable, and that its partial derivative is given by

$$\frac{\partial V_{ij}^{a^T}(\mathbf{q}_i, \mathbf{q}_j)}{\partial \mathbf{q}_i} = \begin{cases} \mathbf{0}, & \text{if } \|\mathbf{q}_i - \mathbf{q}_j\| \ge R_\Delta \\ \text{not defined}, & \text{if } \|\mathbf{q}_i - \mathbf{q}_j\| = 0 \\ -\frac{\mathbf{q}_i - \mathbf{q}_j}{\|\mathbf{q}_i - \mathbf{q}_j\|}, & \text{if } 0 < \|\mathbf{q}_i - \mathbf{q}_j\| < h \\ \left(1 - \frac{R_\Delta}{\|\mathbf{q}_i - \mathbf{q}_j\|}\right) \frac{\mathbf{q}_i - \mathbf{q}_j}{R_\Delta - h}, & \text{otherwise} \end{cases} \tag{6}$$

Note that in contrast to the avoidance function and control inputs in [11], [14], both V_{ij}^a and $\mathbf{u}_i^a$ (proposed herein and depicted in Fig. 2) are bounded. Moreover, as the next lemma

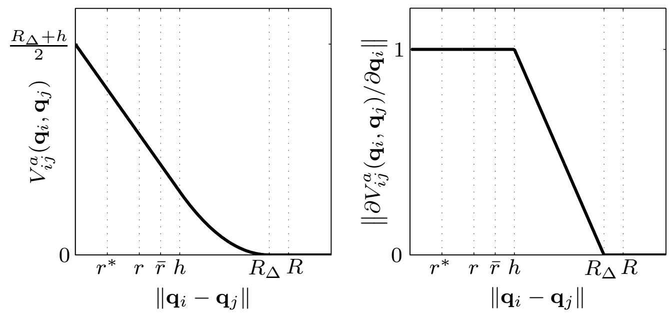

Fig. 2. Bounded avoidance function and bounded avoidance control.

will show, the gradient of the avoidance function (6) is locally Lipschitz.

Lemma 4.1: $\partial V_{ij}^a(\mathbf{q}_i, \mathbf{q}_j)/\partial \mathbf{q}_i$ is Lipschitz continuous in $\mathbf{q}_j$ on the domain $\mathcal{Y} = \{\mathbf{q} : \mathbf{q} \in \Re^{2n}, \|\mathbf{q}_i - \mathbf{q}_j\| \ge r - \Delta\}$ with Lipschitz constant given by

$$L = \max\{L_1, \min\{L_{2a}, L_{2b}\}\} \tag{7}$$

where

$$L_1 = \frac{3 + 2\sqrt{n-1}}{4(r-\Delta)} \tag{8}$$

$$L_{2a} = \frac{1}{h} + \frac{R_\Delta\sqrt{n}}{h(R_\Delta - h)} \tag{9}$$

$$L_{2b} = \frac{1}{R_\Delta - h} + \frac{(3 + 2\sqrt{n-1})R_\Delta}{4h(R_\Delta - h)}. \tag{10}$$

Proof: Define $[a_{kl}]_{n\times n} = \partial^2 V_{ij}^a(\mathbf{q}_i, \mathbf{q}_j)/\partial \mathbf{q}_i \partial \mathbf{q}_j$ and let a_{kl} denote the klth entry of $\partial^2 V_{ij}^a(\mathbf{q}_i, \mathbf{q}_j)/\partial \mathbf{q}_i \partial \mathbf{q}_j$. Then, we have that (6) is locally Lipschitz continuous on $\mathcal{Y}$ if

$$\left\| \frac{\partial^2 V_{ij}^a(\mathbf{q}_i, \mathbf{q}_j)}{\partial \mathbf{q}_i \partial \mathbf{q}_j} \right\| = \left\| [a_{kl}]_{n\times n} \right\| \le L$$

for some non-negative constant L, except possibly on a set of Lebesgue measure zero. In addition, since $\partial^2 V_{ij}^a(\mathbf{q}_i, \mathbf{q}_j)/\partial \mathbf{q}_i \partial \mathbf{q}_j$ is symmetric, we have that

$$\left\| \frac{\partial^2 V_{ij}^a(\mathbf{q}_i, \mathbf{q}_j)}{\partial \mathbf{q}_i \partial \mathbf{q}_j} \right\|_\infty = \left\| \frac{\partial^2 V_{ij}^a(\mathbf{q}_i, \mathbf{q}_j)}{\partial \mathbf{q}_i \partial \mathbf{q}_j} \right\|_1$$

and, therefore,

$$\begin{aligned} \left\| \frac{\partial^2 V_{ij}^a(\mathbf{q}_i, \mathbf{q}_j)}{\partial \mathbf{q}_i \partial \mathbf{q}_j} \right\| &\le \sqrt{\left\| \frac{\partial^2 V_{ij}^a(\mathbf{q}_i, \mathbf{q}_j)}{\partial \mathbf{q}_i \partial \mathbf{q}_j} \right\|_1 \left\| \frac{\partial^2 V_{ij}^a(\mathbf{q}_i, \mathbf{q}_j)}{\partial \mathbf{q}_i \partial \mathbf{q}_j} \right\|_\infty} \\ &= \left\| \frac{\partial^2 V_{ij}^a(\mathbf{q}_i, \mathbf{q}_j)}{\partial \mathbf{q}_i \partial \mathbf{q}_j} \right\|_1. \end{aligned} \tag{11}$$

Hence, the choice of Lipschitz constant L is invariant under the use of $\|\cdot\|_1$, $\|\cdot\|$, or $\|\cdot\|_\infty$.

Now, let us divide the problem in three domains: $\mathcal{Y}_1 = \{\mathbf{q} : \mathbf{q} \in \Re^{2n}, r - \Delta \le \|\mathbf{q}_i - \mathbf{q}_j\| < h\}$, $\mathcal{Y}_2 = \{\mathbf{q} : \mathbf{q} \in \Re^{2n}, h < \|\mathbf{q}_i - \mathbf{q}_j\| < R_\Delta\}$, and $\mathcal{Y}_3 = \{\mathbf{q} : \mathbf{q} \in \Re^{2n}, \|\mathbf{q}_i - \mathbf{q}_j\| > R_\Delta\}$.

For $\mathbf{q}_i = [x_1, \cdots, x_n]^T \in \Re^n$, $\mathbf{q}_j = [y_1, \cdots, y_n]^T \in \Re^n$, and $\mathbf{q} \in \mathcal{Y}_1$ we have that

$$\left\| \frac{\partial^2 V_{ij}^a(\mathbf{q}_i, \mathbf{q}_j)}{\partial \mathbf{q}_i \partial \mathbf{q}_j} \right\|_1 = \max_l \sum_{k=1}^n |a_{kl}|$$

where

$$a_{kl} = \begin{cases} \dfrac{\|\mathbf{q}_i - \mathbf{q}_j\|^2 - (x_k - y_k)^2}{\|\mathbf{q}_i - \mathbf{q}_j\|^3}, & \text{if } k = l \\ \dfrac{-(x_k - y_k)(x_l - y_l)}{\|\mathbf{q}_i - \mathbf{q}_j\|^3}, & \text{if } k \neq l \end{cases}.$$

Hence,

$$\begin{aligned} \left\| \frac{\partial^2 V_{ij}^a(\mathbf{q}_i, \mathbf{q}_j)}{\partial \mathbf{q}_i \partial \mathbf{q}_j} \right\|_1 &\leq \frac{\left| \|\mathbf{q}_i - \mathbf{q}_j\|^2 - |x_l - y_l|^2 \right|}{\|\mathbf{q}_i - \mathbf{q}_j\|^3} \\ &\quad + \frac{|x_l - y_l| \sum_{k=1, k\neq l}^n |x_k - y_k|}{\|\mathbf{q}_i - \mathbf{q}_j\|^3} \\ = \frac{\|\mathbf{q}_i - \mathbf{q}_j\|^2 - |x_l - y_l|^2 + \sqrt{n-1}\, |x_l - y_l|\, \|\mathbf{q}_i - \mathbf{q}_j\|}{\|\mathbf{q}_i - \mathbf{q}_j\|^3}. \end{aligned}$$

By noting that the numerator is maximized for $|x_l - y_l| = \frac{\|\mathbf{q}_i - \mathbf{q}_j\|}{2}$ and recalling (11), we then obtain that

$$\left\| \frac{\partial^2 V_{ij}^a(\mathbf{q}_i, \mathbf{q}_j)}{\partial \mathbf{q}_i \partial \mathbf{q}_j} \right\| \leq \frac{3 + 2\sqrt{n-1}}{4\|\mathbf{q}_i - \mathbf{q}_j\|} \leq \frac{3 + 2\sqrt{n-1}}{4(r - \Delta)} = L_1$$

for all $\mathbf{q} = [\mathbf{q}_i^T, \mathbf{q}_j^T]^T \in \mathcal{Y}_1$.

Now, for $\mathbf{q} \in \mathcal{Y}_2$ we have that

$$a_{kl} = \begin{cases} \dfrac{-\|\mathbf{q}_i - \mathbf{q}_j\|^3 + R_\Delta \|\mathbf{q}_i - \mathbf{q}_j\|^2 - R_\Delta (x_k - y_k)^2}{(R_\Delta - h)\|\mathbf{q}_i - \mathbf{q}_j\|^3}, & \text{if } k = l \\ \dfrac{-R_\Delta (x_k - y_k)(x_l - y_l)}{(R_\Delta - h)\|\mathbf{q}_i - \mathbf{q}_j\|^3}, & \text{if } k \neq l \end{cases}.$$

Therefore,

$$\begin{aligned} \left\| \frac{\partial^2 V_{ij}^a(\mathbf{q}_i, \mathbf{q}_j)}{\partial \mathbf{q}_i \partial \mathbf{q}_j} \right\|_1 &\leq \frac{\|\mathbf{q}_i - \mathbf{q}_j\|^2 (R_\Delta - \|\mathbf{q}_i - \mathbf{q}_j\|)}{(R_\Delta - h)\|\mathbf{q}_i - \mathbf{q}_j\|^3} \\ &\quad + \frac{R_\Delta |x_l - y_l| \sum_{k=1}^n |x_k - y_k|}{(R_\Delta - h)\|\mathbf{q}_i - \mathbf{q}_j\|^3} \\ \leq \frac{\|\mathbf{q}_i - \mathbf{q}_j\|^2 (R_\Delta - \|\mathbf{q}_i - \mathbf{q}_j\|) + R_\Delta \|\mathbf{q}_i - \mathbf{q}_j\| \|\mathbf{q}_i - \mathbf{q}_j\|_1}{(R_\Delta - h)\|\mathbf{q}_i - \mathbf{q}_j\|^3} \\ \leq \frac{R_\Delta - \|\mathbf{q}_i - \mathbf{q}_j\| + R_\Delta \sqrt{n}}{(R_\Delta - h)\|\mathbf{q}_i - \mathbf{q}_j\|} \end{aligned}$$

which can be reduced to

$$\left\| \frac{\partial^2 V_{ij}^a(\mathbf{q}_i, \mathbf{q}_j)}{\partial \mathbf{q}_i \partial \mathbf{q}_j} \right\| \leq \frac{1}{h} + \frac{R_\Delta \sqrt{n}}{h(R_\Delta - h)} = L_{2a}, \quad \forall \mathbf{q} \in \mathcal{Y}_2. \tag{12}$$

Alternatively, we can compute a different upper bound for (12). After some calculations we can also show that

$$\left\| \frac{\partial^2 V_{ij}^a(\mathbf{q}_i, \mathbf{q}_j)}{\partial \mathbf{q}_i \partial \mathbf{q}_j} \right\| \leq \frac{4h + (3 + 2\sqrt{n-1})R_\Delta}{4h(R_\Delta - h)} = L_{2b} \tag{13}$$

which becomes a less conservative upper bound on the Lipschitz property if

$$h > \frac{(1 + 4\sqrt{n} - 2\sqrt{n-1})R_\Delta}{8}.$$

Therefore, by combining (12) and (13) we obtain that

$$\left\| \frac{\partial^2 V_{ij}^a(\mathbf{q}_i, \mathbf{q}_j)}{\partial \mathbf{q}_i \partial \mathbf{q}_j} \right\| \leq \min\{L_{2a}, L_{2b}\}, \qquad \text{for } \mathbf{q} \in \mathcal{Y}_2.$$

Finally, since $\partial V_{ij}^a(\mathbf{q}_i, \mathbf{q}_j)/\partial \mathbf{q}_i \equiv \mathbf{0}$ for $\mathbf{q} \in \mathcal{Y}_3$, we have that $\left\| \partial^2 V_{ij}^a(\mathbf{q}_i, \mathbf{q}_j)/\partial \mathbf{q}_i \partial \mathbf{q}_j \right\| = 0$. Consequently,

$$\left\| \frac{\partial^2 V_{ij}^a(\mathbf{q}_i, \mathbf{q}_j)}{\partial \mathbf{q}_i \partial \mathbf{q}_j} \right\| \leq \max\{L_1, \min\{L_{2a}, L_{2b}\}\} = L$$

for all $\mathbf{q} \in \mathcal{Y}$, except on the set $\mathcal{Y}_0 = \{\mathbf{q} : \mathbf{q} \in \Re^{2n}, \|\mathbf{q}_i - \mathbf{q}_j\| \in \{h, R_\Delta\}\}$ of Lebesgue measure zero. Thereby, we can conclude that (6) is locally Lipschitz continuous with Lipschitz constant given by (7). ■

V. Collision Avoidance Analysis

We now show that the proposed collision avoidance control law, along with the control objective input, guarantees collision-free trajectories for a group of N vehicles with bounded control inputs, limited sensing range, and detection uncertainties.

Theorem 5.1: (Collision Avoidance for Multiple Agents with Sensing Uncertainties): Consider the multi-Lagrangian system in (1) with control inputs given by (2) to (6). Suppose that $\|\dot{\mathbf{q}}_i(0)\| \leq \eta_i$, $\|\mathbf{q}_i(0) - \mathbf{q}_j(0)\| \geq R$, and $\|\mathbf{d}_{ij}(t)\| \leq \Delta$ for some known $\eta_i \geq 0$ and $\forall i, j,\ i \neq j$. Let $\varepsilon \in [0, 1)$, $\bar{L} \in \left(0, \frac{\varepsilon}{(N-1)\Delta}\right]$, and choose r, h, and α_i such that

(*i*) $r^* \leq \frac{(3+2\sqrt{n-1})}{4\bar{L}} + \Delta \leq r < R_\Delta$,

(*ii*) $r < \underline{h} \leq h \leq \bar{h} < R_\Delta$, where

$$\underline{h} = \min\left\{ \frac{\bar{L}R_\Delta + 1 - \sqrt{1 + \bar{L}^2 R_\Delta^2 - 2\bar{L}R_\Delta\left(1 + 2\sqrt{n}\right)}}{2\bar{L}}, \frac{\bar{L}R_\Delta - 1 - \sqrt{1 + \bar{L}^2 R_\Delta^2 - \bar{L}R_\Delta\left(5 + 2\sqrt{-1+n}\right)}}{2\bar{L}} \right\}$$

$$\bar{h} = \max\left\{ \frac{\bar{L}R_\Delta + 1 + \sqrt{1 + \bar{L}^2 R_\Delta^2 - 2\bar{L}R_\Delta\left(1 + 2\sqrt{n}\right)}}{2\bar{L}}, \frac{\bar{L}R_\Delta - 1 + \sqrt{1 + \bar{L}^2 R_\Delta^2 - \bar{L}R_\Delta\left(5 + 2\sqrt{-1+n}\right)}}{2\bar{L}} \right\}$$

(*iii*) and $\sum_{i=1}^N \alpha_i < \frac{R_\Delta + h}{2} - r - \sum_{i=1}^N \frac{\bar{\lambda}_i \eta_i^2}{2\bar{\mu}_i}$, where $\overline{\lambda}_i$ is the larger eigenvalue of M_i.

Then, $[\mathbf{q}_i, \cdots, \mathbf{q}_N] \notin \Omega\ \forall t \geq 0$.

Proof: Consider the following Lyapunov function

$$\begin{aligned} V = &\frac{1}{2} \sum_{i=1}^N \sum_{j \in \mathcal{N}_i} V_{ij}^a(\mathbf{q}_i, \mathbf{q}_j) + \sum_{i=1}^N V_i^o(\mathbf{q}_i) \\ &+ \frac{1}{2} \sum_{i=1}^N \frac{1}{\bar{\mu}_i} \dot{\mathbf{q}}_i^T M_i(\mathbf{q}_i) \dot{\mathbf{q}}_i. \end{aligned} \tag{14}$$

Taking its time-derivative and invoking Property 2.1 yields

$$\begin{aligned}\dot{V} =& \frac{1}{2}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\left(\frac{\partial V_{ij}^a(\mathbf{q}_i,\mathbf{q}_j)}{\partial \mathbf{q}_i}\dot{\mathbf{q}}_i + \frac{\partial V_{ij}^a(\mathbf{q}_i,\mathbf{q}_j)}{\partial \mathbf{q}_j}\dot{\mathbf{q}}_j\right)\\ &+\sum_{i=1}^{N}\frac{\partial V_i^o(\mathbf{q}_i)}{\partial \mathbf{q}_i}\dot{\mathbf{q}}_i\\ &+\sum_{i=1}^{N}\frac{1}{\bar{\mu}_i}\left(\dot{\mathbf{q}}_i^T M_i(\mathbf{q}_i)\ddot{\mathbf{q}}_i + \frac{1}{2}\dot{\mathbf{q}}_i^T \dot{M}(\mathbf{q}_i)\dot{\mathbf{q}}_i\right)\\ =& \underbrace{\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\frac{\partial V_{ij}^a(\mathbf{q}_i,\mathbf{q}_j)}{\partial \mathbf{q}_i}\dot{\mathbf{q}}_i}_{\text{Due to symmetry}} + \sum_{i=1}^{N}\frac{\partial V_i^o(\mathbf{q}_i)}{\partial \mathbf{q}_i}\dot{\mathbf{q}}_i\\ &+\sum_{i=1}^{N}\frac{1}{\bar{\mu}_i}\left(-\bar{\mu}_i\sum_{j\in\mathcal{N}_i}\frac{\partial V_{ij}^a(\mathbf{q}_i,\hat{\mathbf{q}}_j^i)}{\partial \mathbf{q}_i}\right.\\ &\left.-\gamma_i\sum_{j\in\mathcal{N}_i}\boldsymbol{\theta}_{ij}^T(\dot{\mathbf{q}}_i,\mathbf{q}_i,\hat{\mathbf{q}}_j^i) - \bar{\mu}_i\frac{\partial V_i^o(\mathbf{q}_i)}{\partial \mathbf{q}_i}\right)\dot{\mathbf{q}}_i\\ =& \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\left(\frac{\partial V_{ij}^a(\mathbf{q}_i,\mathbf{q}_j)}{\partial \mathbf{q}_i} - \frac{\partial V_{ij}^a(\mathbf{q}_i,\hat{\mathbf{q}}_j^i)}{\partial \mathbf{q}_i}\right)\dot{\mathbf{q}}_i\\ &-\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\frac{\gamma_i}{\bar{\mu}_i}\boldsymbol{\theta}_{ij}^T(\dot{\mathbf{q}}_i,\mathbf{q}_i,\hat{\mathbf{q}}_j^i)\dot{\mathbf{q}}_i.\end{aligned}$$

Now, by applying Lemma 4.1 we obtain

$$\begin{aligned}\dot{V} \leq& \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} L\Delta\left\|\dot{\mathbf{q}}_i\right\| - \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\frac{\gamma_i}{\bar{\mu}_i(N-1)}\left\|\dot{\mathbf{q}}_i\right\|\\ =& \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\left(L\Delta - \frac{\varepsilon}{(N-1)}\right)\left\|\dot{\mathbf{q}}_i\right\|\end{aligned}$$

where L is the Lipschitz constant for $\partial V_{ij}^a(\mathbf{q}_i,\mathbf{q}_j)/\partial \mathbf{q}_i$. It is easy to show that if we choose r and h according to conditions (i) and (ii), then $L \leq \bar{L} \leq \frac{\varepsilon}{(N-1)\Delta}$ and $\dot{V} \leq 0$. The semi-negative definite property on $\dot{V}(t)$ implies that $V(t) \leq V(0) \leq \sum_{i=1}^{N}\left(\frac{\bar{\lambda}_i\eta_i^2}{2\bar{\mu}_i} + \alpha_i\right)$, $\forall t \geq 0$, where we have used Property 2.2.

Now assume that (iii) holds and suppose that for at least some pair i,j, $\|\mathbf{q}_i(t) - \mathbf{q}_j(t)\| \to r$. As a consequence, $V(t) \geq V_{ij}^a(t) \to \frac{R_\Delta + h}{2} - r > \sum_{i=1}^{N}\left(\frac{\bar{\lambda}_i\eta_i^2}{2\bar{\mu}_i} + \alpha_i\right)$. Since we reached a contradiction, we conclude that $\|\mathbf{q}_i(t) - \mathbf{q}_j(t)\| \notin \Omega_{ij}$ for all $i,j,\ i \neq j$ and $t \geq 0$. ∎

Theorem 5.1 provides sufficient conditions for the safe navigation of a group of N Lagrangian vehicles. It, however, does not provide information about the fulfillment of the objective control. We can only deduce that whenever the agents are outside of the Detection Regions, i.e., $[\mathbf{q}_i, \cdots, \mathbf{q}_N] \notin \mathcal{D}$, the collision avoidance control inputs do not affect the objective control laws.

In the following we posit sufficient conditions for collision avoidance of nonlinear vehicles with zero uncertainties but limited sensing range and bounded control inputs. The results along this line are of relevance given that vehicles with bounded control inputs and accelerations cannot react (e.g., evade or escape) instantaneously to a collision threat.

Corollary 5.1: (Collision Avoidance for Multiple Agents without Sensing Uncertainties): Consider the multi-Lagrangian system in (1) with control inputs given by (2) to (6) for $\varepsilon = 0$. Suppose that $\|\mathbf{d}_{ij}(t)\| = \Delta = 0$ and $\|\dot{\mathbf{q}}_i(0)\| \leq \eta_i$, $\|\mathbf{q}_i(0) - \mathbf{q}_j(0)\| \geq R\ \forall i,j,\ i \neq j$. Furthermore, assume $\exists\ r \geq r^*$, $h < R$, and $\alpha_i > 0$ such that

$$\sum_{i=1}^{N}\alpha_i < \frac{R+h}{2} - r - \sum_{i=1}^{N}\frac{\bar{\lambda}_i\eta_i^2}{2\bar{\mu}_i} \tag{15}$$

Then, $[\mathbf{q}_i, \cdots, \mathbf{q}_N] \notin \Omega\ \forall t \geq 0$.

Proof: Consider the Lyapunov candidate function given in (14). Its time-derivative along the trajectories of the system are computed as

$$\begin{aligned}\dot{V} =& \sum_{i=1}^{N}\frac{\partial V_i^o(\mathbf{q}_i)}{\partial \mathbf{q}_i}\dot{\mathbf{q}}_i + \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\frac{\partial V_{ij}^a(\mathbf{q}_i,\mathbf{q}_j)}{\partial \mathbf{q}_i}\dot{\mathbf{q}}_i\\ &+\sum_{i=1}^{N}\left(-\sum_{j\in\mathcal{N}_i}\frac{\partial V_{ij}^a(\mathbf{q}_i,\mathbf{q}_j)}{\partial \mathbf{q}_i} - \frac{\partial V_i^o(\mathbf{q}_i)}{\partial \mathbf{q}_i}\right)\dot{\mathbf{q}}_i \leq 0.\end{aligned}$$

Therefore, $V(t) \leq V(0) \leq \sum_{i=1}^{N}\left(\frac{\bar{\lambda}_i\eta_i^2}{2\bar{\mu}_i} + \alpha_i\right)$. Now suppose (15) is satisfied and that for at least some pair $i,j,i \neq j$ and some $t \geq 0$, $[\mathbf{q}_i(t), \mathbf{q}_j(t)] \to \Omega_{ij}(t)$. This would implies that $V(t) \geq V_{ij}^a(t) \to \frac{R+h}{2} - r > \sum_{i=1}^{N}\left(\frac{\bar{\lambda}_i\eta_i^2}{2\bar{\mu}_i} + \alpha_i\right)$, which is a contradiction. Consequently, we can conclude that $[\mathbf{q}_i(t), \cdots, \mathbf{q}_N(t)] \notin \Omega(t)\ \forall t \geq 0$. ∎

The need to satisfy the inequality constraints in Theorem 5.1 and Corollary 5.1 in order to guarantee collision-free transit for a group of agents may seems very restrictive at first glance. However, the sufficient conditions established in the previous statements can be easily satisfied if the detection radii for the agents are large enough. For instance, if the agents have unlimited detection radius (i.e., the vehicles can detect any other agent in their environment), then we can always find a set of parameters r, h, ε, and α_i for which collision avoidance can be guaranteed. The next section will illustrate two examples for which the conditions in Theorem 5.1 and Corollary 5.1 are satisfied.

VI. EXAMPLES

In order to validate the proposed avoidance strategy, we now present two numerical examples. The first example illustrates the performance of the cooperative collision avoidance strategy under sensing uncertainties, whereas the second evaluates its performance under perfect sensing information.

A. Collision Avoidance with Bounded Sensing Error

We simulate the behavior of four 2-DOF vehicles with dynamics governed by

$$m_i\ddot{\mathbf{q}}_i = \mathbf{u}_i - \rho_i\dot{\mathbf{q}}_i \tag{16}$$

where $m_i = 1.5\text{kg}$, $\rho_i = \frac{1}{15}\text{kg/s}$, and $\|\mathbf{u}_i\| \leq 100\text{kgm/s}^2$ for $i = \{1, 2, 3, 4\}$. The minimum safety distance and detection radius for all vehicles are assumed to be $r^* = 2\text{m}$ and $R = 20\text{m}$, respectively. In addition, we assume that the sensing uncertainty for all agents can be characterized as $\mathbf{d}_{ij}(t) = \boldsymbol{\zeta}_{ij}(t)$, where $\boldsymbol{\zeta}_{ij}$ is a random noise with uniform distribution on the set $\mathcal{Z}_{ij} = \{\boldsymbol{\zeta} : \boldsymbol{\zeta} \in \Re^2, \|\boldsymbol{\zeta}\| \leq 0.15\text{m}\}$. Therefore, $\Delta = 0.15\text{m}$.

The control objective is to safely drive the vehicles from an initial configuration to a desired final position. To reach this goal, we propose the use of the following objective function

$$V_i^o(\mathbf{q}_i) = \alpha_i \left(1 - \operatorname{sech}\left(\frac{\|\mathbf{q}_i - \mathbf{q}_i^d\|}{\sigma_i}\right)\right)^{\delta_i} \tag{17}$$

where $\alpha_1 = \alpha_2 = 3$, $\alpha_3 = \alpha_4 = 2.5$, $\sigma_i = 5$, and $\delta_i = 7$ for $i \in \{1, 2, 3, 4\}$. The desired final configurations are chosen as $\mathbf{q}_1^d = -\mathbf{q}_3^d = [-15\text{m}, 15\text{m}]^T$ and $\mathbf{q}_2^d = -\mathbf{q}_4^d = [15\text{m}, 15\text{m}]^T$. The collision avoidance control input for the four agents are computed according to Theorem 5.1 as $r = 3.67\text{m}$, $h = 10.19\text{m}$, $\beta_1 = \beta_2 = 0.234$, $\beta_3 = \beta_4 = 0.196$, $\bar{\mu}_1 = \bar{\mu}_2 = 29.46\text{N}$, $\bar{\mu}_3 = \bar{\mu}_4 = 29.80\text{N}$, $\gamma_1 = \gamma_2 = 4.71\text{N}$, and $\gamma_3 = \gamma_4 = 4.78\text{N}$, where we have chosen $\varepsilon = 0.16$ and assumed that initial velocities for all agents are bounded by $\eta_1 = \eta_2 = 1\text{m/s}$, $\eta_3 = 2\text{m/s}$, and $\eta_4 = 3\text{m/s}$.

The response of the four agents to the objective and collision avoidance control inputs is illustrated in Fig. 3. The agents start from positions $\mathbf{q}_1(0\text{s}) = -\mathbf{q}_3(0\text{s}) = [35\text{m}, 0\text{m}]^T$ and $\mathbf{q}_2(0\text{s}) = -\mathbf{q}_4(0\text{s}) = [0\text{m}, -35\text{m}]^T$ moving in a counter-clock wise direction (see Fig. 3(b)) with initial velocities given by $\dot{\mathbf{q}}_i(0\text{s}) = \eta_i \frac{\mathbf{q}_i^d}{\|\mathbf{q}_i^d\|}$. Notice that at $t \approx 19\text{s}$, the second and fourth agent come into close proximity to the third agent, entering its Detection Region. They, however, managed to keep a safe distance among each other while continuing toward their final destination. Similarly, observe that at $t \approx 44\text{s}$ (corresponding to Fig. 3(e)), the first and second agent entered each other's Detection Region. This event repeats twice, after which all agents are able to converge to their desired locations as illustrated in Fig. 3(f).

Fig. 4 depicts the distances among the four agents. Note that the pairs of agents $\{2, 3\}$, $\{3, 4\}$, and $\{1, 2\}$ entered the Detection Regions at different instances of time. Yet, no collision took place.

B. Collision Avoidance with Zero Sensing Uncertainty

We now evaluate the performance of the control strategy considering zero sensing uncertainty. We simulate the response of three 2-DOF vehicles with dynamics governed by (16), where $\rho_1 = \rho_2/2 = \rho_3 = 0.2\text{kg/s}$, $m_i = 2\text{kg}$, and $\|\mathbf{u}_i\| \leq 100\text{kgm/s}^2$ for $i \in \{1, 2, 3\}$. The minimum safety distance and detection radius for all vehicles are assumed to be $r^* = 2\text{m}$ and $R = 10\text{m}$. The control objective is chosen such that the first agent remains at the origin (i.e., $\mathbf{q}_1^d = [0\text{m}, \text{m}]^T$), while the second and third vehicle are driven toward opposite corners of their workspace (i.e., $\mathbf{q}_2^d = -\mathbf{q}_3^d = [10\text{m}, 9\text{m}]^T$). Accordingly, the objective function is constructed as in (17) with $\alpha_1 = 1$, $\alpha_2 = \alpha_3 = 2.12$, $\sigma_1 = 3$,

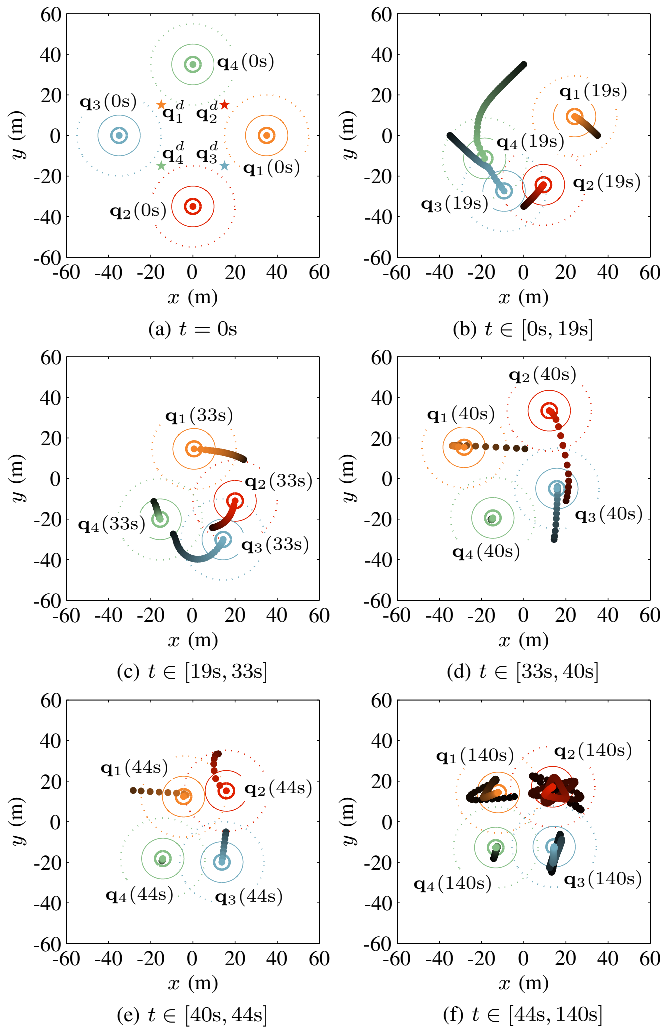

Fig. 3. Collision avoidance with bounded sensing error. The initial positions of the four agents at the start of each simulation interval are indicated by the small circles of darker color. Subsequence positions are traced by circular markers of lighter color and time-spaced by 0.5s. The desired final positions are indicated by the star-shaped markers in plot (a). The Avoidance, Conflict, and Detection Region for all agents at the end of each simulation interval are delimited by the bold, thin, and dashed lines, respectively.

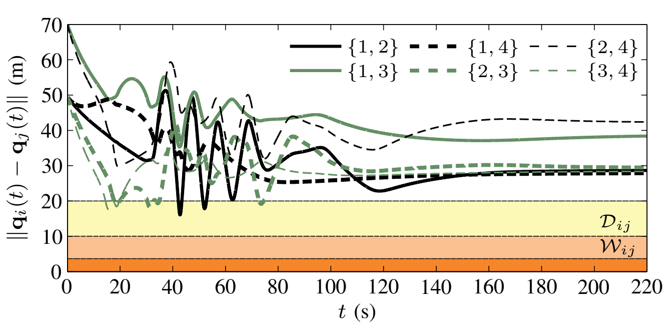

Fig. 4. Distances among agents.

$\sigma_2 = \sigma_3 = 7$, and $\delta_1 = \delta_2 = \delta_3 = 2$. The avoidance control input parameters are chosen satisfying Corollary 5.1 as $r = 2\text{m}$, $h = 5\text{m}$, $\beta_i = 0.15$, and $\bar{\mu}_i = 46.51\text{N}$, where we have assumed that $\eta_1 = 0\text{m/s}$ and $\eta_2 = \eta_3 = 3\text{m/s}$.

Fig. 5 illustrates the response of the three agents to the objective and avoidance control inputs. The agents start from positions $\mathbf{q}_1(0\mathrm{s}) = \mathbf{q}_1^d$ and $\mathbf{q}_2(0\mathrm{s}) = -\mathbf{q}_3(0\mathrm{s}) = [-9\mathrm{m}, -10\mathrm{m}]^T$, and with initial velocities $\dot{\mathbf{q}}_i(0\mathrm{s}) = \eta_i \frac{\mathbf{q}_i^d}{\|\mathbf{q}_i^d\|}$. Note that shortly after the initial time, the second and third agent enter the first vehicle's Detection Region as they try to move toward their desired configurations (see Fig. 5(b)). The second and third agent immediately react by retreating from the potential collision as seen in Fig. 5(c). Gradually, the agents are able to resolve the conflict by departing slightly from their objective paths (refer to Fig. 5(d)) ultimately reaching their final destinations (see Fig. 5(f)).

The distances among the three agents are reported in Fig. 6. Observe that the second and third agent entered repeatedly the first agent's Detection Region. Despite these conflicts, they successfully evaded the Avoidance Region.

VII. Conclusion

In this paper, we presented a real-time, cooperative collision avoidance control strategy for a group of nonlinear Lagrangian systems with bounded control inputs and bounded sensing uncertainties. By applying Lyapunov-based analysis and constructing avoidance control inputs based on avoidance functions, we were able to derive sufficient conditions for the collision-free navigation of a group of N vehicles independently of obstacle localization errors. It is shown that if the detection and avoidance radii satisfy a set of design inequalities, then we can formulate control laws that keep the agents safely apart.

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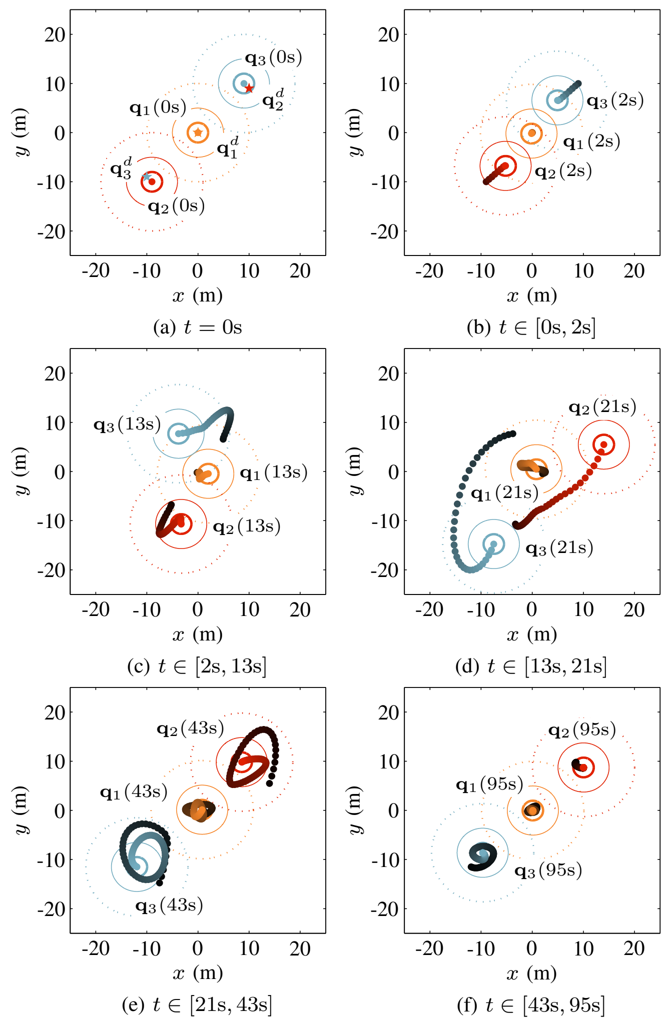

Fig. 5. Collision avoidance with zero uncertainty. The agents' initial positions at the start of each simulation interval are indicated by the small circles of darker color. Subsequence positions are traced by circular markers of lighter color and time-spaced by 0.25s. The desired final positions are indicated by the star-shaped markers in plot (a). The Avoidance, Conflict, and Detection Region for all agents at the end of each simulation interval are delimited by the bold, thin, and dashed lines, respectively.

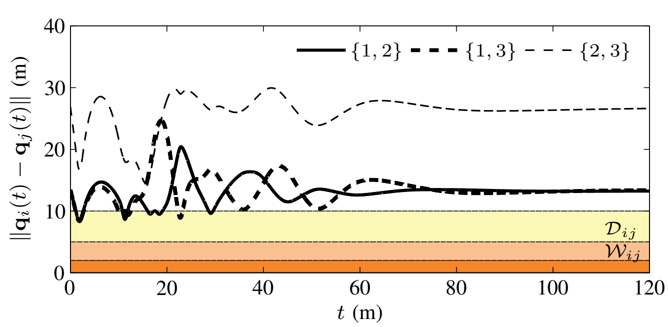

Fig. 6. Distances among agents.