

Decentralized control using compositional analysis techniques

F. Kerber and A.J. van der Schaft

Abstract—Decentralized control strategies aim at achieving a global control target by means of distributed local controllers acting on individual subsystems of the overall plant. In this sense, decentralized control is a dual problem to compositional analysis where a global verification task is decomposed into several local tasks involving components of the overall system. In this paper we apply recently developed compositional reasoning techniques to decentralized control problems for linear systems. We assume the global plant and global specification to be both given as series of feedback interconnections. In this setting compositional and assume-guarantee reasoning schemes can be shown to be valid. Provided the local controllers are such that the locally controlled subsystems of the plant satisfy their respective sub-specifications the network of locally controlled plants is then guaranteed to satisfy the global specification.

I. INTRODUCTION

Decentralized control [6], [4], [11] is the attempt to control a global plant by locally controlling its subsystems in such a way that the overall controlled system satisfies a given global specification. Decentralized control has several important advantages. Restrictions due to limited communication and controller action between component systems can be incorporated naturally in the design of decentralized control schemes. Likewise, distributed sensor and actuator locations, such as in structural monitoring [7], process control [10], and distributed robotic networks [2], restrict the communication and information flow between subsystems, thus also requiring decentralized control strategies. Furthermore, compared to the closed loop system using a global controller, the design procedures and consequently the hardware requirements for the network of locally controlled plants are less complex. However, the challenge of decentralized control strategies is to guarantee that the interconnection of locally controlled subsystems of the plant satisfies the desired global control target. In this respect, decentralized control can be seen as a complementary notion of compositional analysis for verification of complex systems. In this paper we want to make use of compositional analysis techniques for linear systems as developed in [5] for guaranteeing the validity of decentralized control schemes.

Originally developed to verify properties of transition systems [8], [3], formal methods based on (bi)simulation relations have recently been adopted to dynamical control systems [1], [9], [13]. The main principle of compositional and assume-guarantee reasoning is to split a global verification goal formulated for the overall system into several

tasks for the components involved. Thus, the complexity of the original problem can be reduced significantly. This makes these techniques well-suited for decentralized control problems. Assuming that the global specification can be decomposed according to the decomposition of the overall plant system, the global control target can be split into several local targets involving subsystems of the plant and their respective sub-specifications. In this paper, we present two different strategies based on compositional and circular assume-guarantee reasoning, respectively. We prove that these reasoning schemes hold true in a decentralized setting for series of feedback interconnections of linear systems. As a result, we obtain conditions under which a network of locally controlled subsystems of the plant satisfies the global specification.

II. PROBLEM SETTING

We consider the global plant system Σ_P to be an interconnection of k component systems $\Sigma_{P_i}, i = 1, \dots, k$,

$$\Sigma_P := \Sigma_{P_1} \parallel \dots \parallel \Sigma_{P_k}, \quad (1)$$

where each plant subsystem Σ_{P_i} is of the form (see Figure 1)

$$\Sigma_{P_i} : \begin{cases} \dot{x}_{P_i} &= A_{P_i}x_{P_i} + B_{P_i}u_{P_i} + G_{P_i}e_{P_i} \\ y_{P_i} &= C_{P_i}x_{P_i} \\ z_{P_i} &= H_{P_i}x_{P_i} \end{cases} \quad (2)$$

where u_{P_i}, y_{P_i} are pairs of variables used for interconnection with local controller systems, and e_{P_i}, z_{P_i} are pairs of external variables used to interconnect the subsystems of the plant with each other. All variables take values in

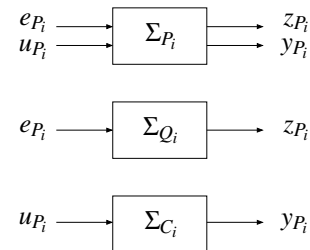


Fig. 1. Plant, specification and controller subsystems.

vector spaces of appropriate dimensions, $x_{P_i} \in \mathcal{X}_{P_i}, u_{P_i} \in \mathcal{U}_{P_i}, e_{P_i} \in \mathcal{E}_{P_i}, y_{P_i} \in \mathcal{Y}_{P_i}, z_{P_i} \in \mathcal{Z}_{P_i}$. The global specification, denoted by Σ_Q , is assumed to be similarly decomposable into local sub-specifications $\Sigma_{Q_i}, i = 1, \dots, k$, corresponding to the respective plant subsystems Σ_{P_i} , i.e.,

$$\Sigma_Q := \Sigma_{Q_1} \parallel \dots \parallel \Sigma_{Q_k} \quad (3)$$

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Each component Σ_{Q_i} of the global specification Σ_Q defines the desired external behavior of the respective component of the plant and is given as

$$\Sigma_{Q_i} : \begin{cases} \dot{x}_{Q_i} &= A_{Q_i}x_{Q_i} + G_{Q_i}e_{Q_i} \\ z_{Q_i} &= H_{Q_i}x_{Q_i} \end{cases} \quad (4)$$

The topology of the global system model is determined by the type of interconnection between the individual components. In the remainder of this paper, we consider series of feedback interconnections with respect to the external variables e_i, z_i to model decompositions of the overall plant system, as well as of the global specification.

Definition 1: Consider k systems $\Sigma_i, i = 1, \dots, k$, of the form (2) or (4) with external variables e_i, z_i , and control interconnection variables u_i, y_i . Then define the *series interconnection* $\Sigma_1 \parallel \dots \parallel \Sigma_k$ with respect to the external variables e, z using feedback interconnections as follows:

$$\begin{aligned} z_i^- &= e_{i-1}^+, & z_i^+ &= e_{i+1}^-, & i &= 2, \dots, k-1 \\ e_i^- &= z_{i-1}^+, & e_i^+ &= z_{i+1}^-, & & \end{aligned}$$

$$\begin{aligned} z_1^- &= z_1, & z_1^+ &= e_2^-, & z_k^- &= e_{k-1}^+, & z_k^+ &= z_k \\ e_1^- &= e_1, & e_1^+ &= z_2^-, & e_k^- &= z_{k-1}^+, & e_k^+ &= e_k \end{aligned}$$

The matrices G_i and H_i corresponding to the external inputs $e_i = (e_i^+, e_i^-)$ and outputs $z_i = (z_i^+, z_i^-)$ are partitioned accordingly into submatrices

$$G_i = \begin{bmatrix} G_i^+ \\ G_i^- \end{bmatrix}, \quad H_i = \begin{bmatrix} H_i^+ \\ H_i^- \end{bmatrix}, \quad i = 1, \dots, k$$

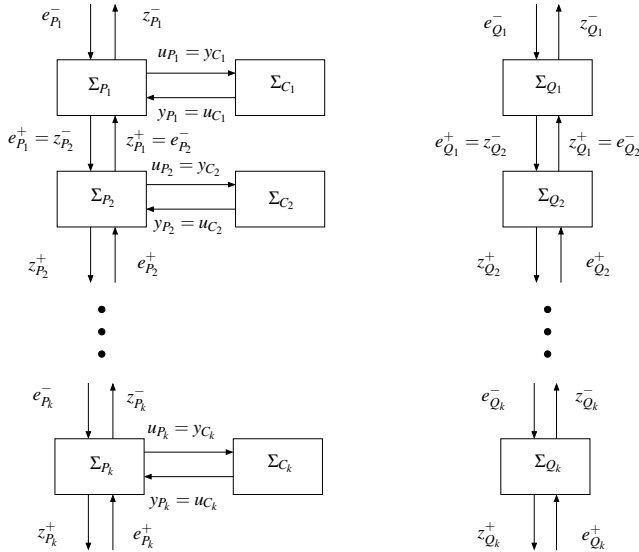


Fig. 2. Decompositions of the global closed loop system $(\Sigma_{P_1} \parallel_{u,y} \Sigma_{C_1}) \parallel \dots \parallel (\Sigma_{P_k} \parallel_{u,y} \Sigma_{C_k})$ and of the global specification $\Sigma_{Q_1} \parallel \dots \parallel \Sigma_{Q_k}$.

The controller systems $\Sigma_{C_i}, i = 1, \dots, k$, are defined as linear systems

$$\Sigma_{C_i} : \begin{cases} \dot{x}_{C_i} &= A_{C_i}x_{C_i} + B_{C_i}u_{C_i} \\ y_{C_i} &= C_{C_i}x_{C_i} \end{cases} \quad (5)$$

Plant-controller interconnections $\Sigma_{P_i} \parallel_{u,y} \Sigma_{C_i}$ are defined by relating the control interconnection variables u_i, y_i of the plant components to inputs and outputs of the controller systems by means of standard output feedback

$$u_i = y_{C_i}, \quad u_{C_i} = y_i, \quad i = 1, \dots, k, \quad (6)$$

leading to the closed loop components

$$\begin{cases} \dot{x}_{P_i} \\ \dot{x}_{C_i} \end{cases} = \begin{bmatrix} A_{P_i} & B_{P_i}C_{C_i} \\ B_{C_i}C_{P_i} & A_{C_i} \end{bmatrix} \begin{bmatrix} x_{P_i} \\ x_{C_i} \end{bmatrix} + \begin{bmatrix} G_{P_i} \\ 0 \end{bmatrix} e_{P_i} \\ z_{P_i} = \begin{bmatrix} H_{P_i} & 0 \end{bmatrix} \begin{bmatrix} x_{P_i} \\ x_{C_i} \end{bmatrix} \quad (7)$$

denoted by $\Sigma_{P_i} \parallel_{u,y} \Sigma_{C_i}, i = 1, \dots, k$.

The resulting system $(\Sigma_{P_1} \parallel_{u,y} \Sigma_{C_1}) \parallel \dots \parallel (\Sigma_{P_k} \parallel_{u,y} \Sigma_{C_k})$ consisting of k locally controlled plant subsystems $\Sigma_{P_i} \parallel_{u,y} \Sigma_{C_i}$ interconnected in series by output feedback is depicted in Figure 2.

Analogous to the decomposition of the global plant, the global specification Σ_Q is assumed to be given as a series of k sub-specifications Σ_{Q_i} interconnected by output feedback. The performance targets are expressed in terms of the external variables $e_i^{+, -}, z_i^{+, -}, i = 1, \dots, k$. Roughly speaking, the closed loop system $(\Sigma_{P_1} \parallel_{u,y} \Sigma_{C_1}) \parallel \dots \parallel (\Sigma_{P_k} \parallel_{u,y} \Sigma_{C_k})$ meets its specification $\Sigma_{Q_1} \parallel \dots \parallel \Sigma_{Q_k}$ if the behavior of the closed loop system with respect to these external variables is contained in the external behavior of the global specification. The formal definition of when the plant behavior is included in the specification behavior (i.e., when the plant satisfies the specification) will be given by means of simulation relations, see Section III.

Remark 2: Although we restrict ourselves in this paper to series of feedback interconnections of linear systems, the results can be shown to hold for other network topologies as well.

Within this setting, we formulate the following decentralized control problems:

Problem 3: Which conditions do the local controllers Σ_{C_i} have to fulfill in order to guarantee that the closed loop system $(\Sigma_{P_1} \parallel_{u,y} \Sigma_{C_1}) \parallel \dots \parallel (\Sigma_{P_k} \parallel_{u,y} \Sigma_{C_k})$ satisfies the global specification $\Sigma_{Q_1} \parallel \dots \parallel \Sigma_{Q_k}$?

Problem 4: What are necessary and sufficient conditions such that there exist local controllers Σ_{C_i} that satisfy the conditions determined by Problem 3?

In this paper, we concentrate on solutions to Problem 3. A follow-up version of this paper will deal with the solution of Problem 4.

III. SIMULATION THEORY FOR LINEAR SYSTEMS

In this section we recall compositional analysis techniques for linear systems based on simulation relations.

Originating from computer science, (bi)simulation relations have first been introduced for dynamical control systems in [1]. In [9], [12] the existence of a (bi)simulation relation between two linear systems was reformulated as a geometric control problem allowing for linear-algebraic characterizations of (bi)simulation relations between linear

systems. Intuitively speaking, bisimulation relations define a concept of external equivalence. Applied to the decentralized setting considered in this paper, equivalence by bisimulation is expressed by requiring the external variables e_i, z_i to remain equal.

Definition 5: A linear subspace $S \subset \mathcal{X}_P \times \mathcal{X}_C \times \mathcal{X}_Q$ is a *simulation relation* of $\Sigma_P \|_{u,y} \Sigma_C$ by Σ_Q with $\Sigma_P \|_{u,y} \Sigma_C$ of the form (7) and Σ_Q of the form (4), if it satisfies the following properties: Take any $(x_P, x_C, x_Q) \in S$ and any joint external input function $e(\cdot) = e_P(\cdot) = e_Q(\cdot)$. Then the resulting state trajectories $x_P(\cdot), x_C(\cdot)$ and $x_Q(\cdot)$, starting at $x_i(0) = x_{i0}, i \in \{P, C, Q\}$, satisfy

- (i): $(x_P(t), x_C(t), x_Q(t)) \in S \quad \forall t \geq 0$
- (ii): $z_P(t) = z_Q(t) \quad \forall t \geq 0$

A simulation relation S is called *full* and denoted by $\Sigma_P \|_{u,y} \Sigma_C \preceq \Sigma_Q$ if the projection on the first state component covers the whole state space, i.e., $\Pi_{\mathcal{X}_P \times \mathcal{X}_C} S = \mathcal{X}_P \times \mathcal{X}_C$.

A *bisimulation relation* R between $\Sigma_P \|_{u,y} \Sigma_C$ and Σ_Q , $\Sigma_i, i \in \{P, C, Q\}$, like before, is a linear subspace $R \subset \mathcal{X}_P \times \mathcal{X}_C \times \mathcal{X}_Q$ with the following property: R defines a simulation relation of $\Sigma_P \|_{u,y} \Sigma_C$ by Σ_Q and $R^{-1} := \{(x_Q, x_P, x_C) \mid (x_P, x_C, x_Q) \in R\}$ defines a simulation relation of Σ_Q by $\Sigma_P \|_{u,y} \Sigma_C$. Moreover, R is full if $\Pi_{\mathcal{X}_i} R = \mathcal{X}_i, i \in \{P, C, Q\}$, which will be denoted by $\Sigma_P \|_{u,y} \Sigma_C \approx \Sigma_Q$.

Hence, the closed loop system satisfies the global specification if

$$(\Sigma_{P_1} \|_{u,y} \Sigma_{C_1}) \parallel \dots \parallel (\Sigma_{P_k} \|_{u,y} \Sigma_{C_k}) \preceq \Sigma_{Q_1} \parallel \dots \parallel \Sigma_{Q_k}. \quad (8)$$

Informally this means that the controller action should be such that any trajectory of the closed loop system with respect to the external variables $e_i^{+, -}, z_i^{+, -}$ is matched ('simulated') by a trajectory of the global specification.

Verifying the simulation (8) is in general a complex task. However, using compositional analysis techniques the verification task (8) can be simplified by splitting it in into several subtasks involving components of the overall systems. In [5] results for compositional and assume-guarantee reasoning of two linear systems were derived. We state without proof the main results of [5] that will be needed in the following. These results are of two types: *compositional reasoning* and *circular assume-guarantee reasoning*.

Compositional reasoning decomposes the global proof obligation into proof obligations for the components.

Theorem 6: For linear systems $\Sigma_i, i \in \{P_1, P_2, Q_1, Q_2\}$ of the form (2) or (4), compositional reasoning

$$\left. \begin{array}{l} \Sigma_{P_1} \preceq \Sigma_{Q_1} \\ \Sigma_{P_2} \preceq \Sigma_{Q_2} \end{array} \right\} \implies \Sigma_{P_1} \| \Sigma_{P_2} \preceq \Sigma_{Q_1} \| \Sigma_{Q_2} \quad (9)$$

holds true.

A more involved scheme is *circular assume-guarantee reasoning*, which deals with interconnections of the individual components with the corresponding *sub-specifications*, see [5] for a detailed description.

Theorem 7: For linear systems $\Sigma_i, i \in \{P_1, P_2, Q_1, Q_2\}$ of the form (2) or (4) circular assume-guarantee reasoning is

always¹ valid, i.e., the implication

$$\left. \begin{array}{l} S_I : \Sigma_{P_1} \| \Sigma_{Q_2} \preceq \Sigma_{Q_1} \| \Sigma_{Q_2} \\ S_{II} : \Sigma_{Q_1} \| \Sigma_{P_2} \preceq \Sigma_{Q_1} \| \Sigma_{Q_2} \end{array} \right\} \implies S : \Sigma_{P_1} \| \Sigma_{P_2} \preceq \Sigma_{Q_1} \| \Sigma_{Q_2} \quad (10)$$

holds true.

IV. DECENTRALIZED CONTROL USING COMPOSITIONAL ANALYSIS TECHNIQUES

As a first result of this paper we extend compositional reasoning, as treated in [5] for the interconnection of two system, to an arbitrary number of systems. The resulting deduction scheme then immediately yields a solution to the decentralized control problem 3.

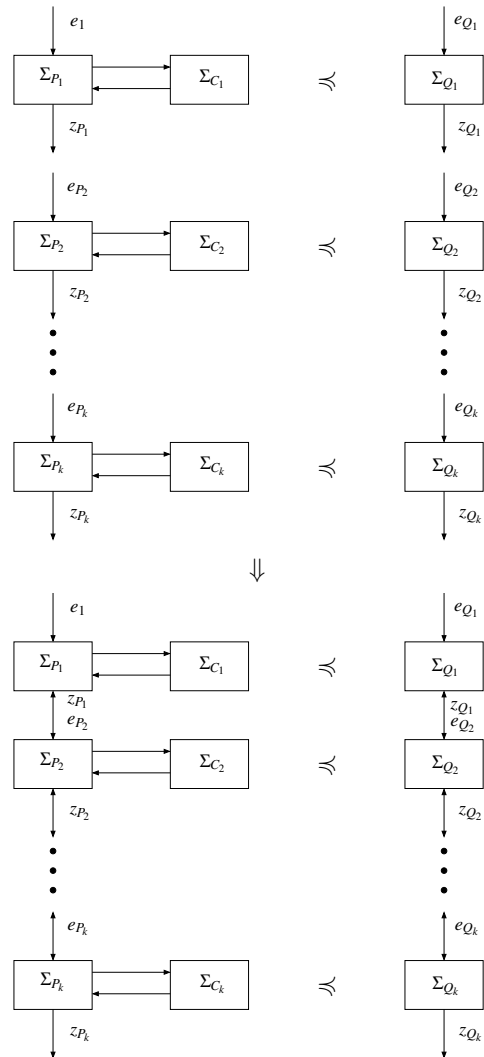


Fig. 3. Decentralized control scheme based on compositional reasoning.

Theorem 8: Consider k plant-controller interconnections $\Sigma_{P_i} \|_{u,y} \Sigma_{C_i}, i = 1, \dots, k$, of the form (7) and k specifications

¹Somewhat surprisingly no additional conditions, as is the case for transition systems, need to be made.

Σ_{Q_i} of the form (4). Then compositional reasoning holds true for series interconnections of k control systems, i.e.,

$$\begin{aligned} \Sigma_{P_i} \parallel_{u,y} \Sigma_{C_i} &\preceq \Sigma_{Q_i}, i = 1, \dots, k \\ \Downarrow \\ (\Sigma_{P_1} \parallel_{u,y} \Sigma_{C_1}) \parallel \dots \parallel (\Sigma_{P_k} \parallel_{u,y} \Sigma_{C_k}) &\preceq \Sigma_{Q_1} \parallel \dots \parallel \Sigma_{Q_k} \end{aligned} \quad (11)$$

Proof: The proof uses induction over k . Theorem 6 contains the proof for the case $k = 2$. Assume now that the series interconnection Σ_P of k plant-controller systems $\Sigma_{P_1} \parallel \dots \parallel \Sigma_{P_k} =: \Sigma_P$ fulfills a composed specification Σ_Q of the form $\Sigma_Q := \Sigma_{Q_1} \parallel \dots \parallel \Sigma_{Q_k}$, i.e.,

$$\Sigma_P \preceq \Sigma_Q \quad (12)$$

Moreover, let there exist a full simulation relation of $\Sigma_{P_{k+1}}$ by $\Sigma_{Q_{k+1}}$, i.e.,

$$\Sigma_{P_{k+1}} \preceq \Sigma_{Q_{k+1}} \quad (13)$$

Taking the product of the full simulation relations for (12) and (13) yields, after reordering the state components, a full simulation relation of

$$\Sigma_P \parallel \Sigma_{P_{k+1}} \preceq \Sigma_Q \parallel \Sigma_{Q_{k+1}} \quad (14)$$

This proves the induction step. \blacksquare

Theorem 8 immediately implies the validity of the following decentralized control scheme.

Corollary 9: Given local controllers $\Sigma_{C_i}, i = 1, 2, \dots, k$, that are such that the locally controlled components $\Sigma_{P_i} \parallel \Sigma_{C_i}$ satisfy the local specifications Σ_{Q_i} , i.e.,

$$\Sigma_{P_i} \parallel_{u,y} \Sigma_{C_i} \preceq \Sigma_{Q_i} \quad (15)$$

then the global system consisting of series interconnections of these locally controlled plant components is guaranteed to fulfill the global specification given by the series interconnection of the local specifications, see Figure 3.

An alternative requirement for the controllers Σ_{C_i} , less strict than (15), can be based on *circular assume-guarantee reasoning*. Rather than formulating conditions on the component level as in Theorem 8, we formulate them for interconnections of locally controlled plants and sub-specifications.

Theorem 10: Consider $k \geq 2$ plant-controller interconnections $\Sigma_{P_i} \parallel_{u,y} \Sigma_{C_i}, i = 1, \dots, k$, of the form (7) and k corresponding specifications Σ_{Q_i} of the form (4). Let k circularly dependent conditions

$$\begin{aligned} S_1 : (\Sigma_{P_1} \parallel_{u,y} \Sigma_{C_1}) \parallel \Sigma_{Q_2} \parallel \dots \parallel \Sigma_{Q_k} &\preceq \Sigma_{Q_1} \parallel \Sigma_{Q_2} \parallel \dots \parallel \Sigma_{Q_k} \\ S_2 : \Sigma_{Q_1} \parallel (\Sigma_{P_2} \parallel_{u,y} \Sigma_{C_2}) \parallel \dots \parallel \Sigma_{Q_k} &\preceq \Sigma_{Q_1} \parallel \Sigma_{Q_2} \parallel \dots \parallel \Sigma_{Q_k} \\ \vdots &\vdots \\ S_k : \Sigma_{Q_1} \parallel \Sigma_{Q_2} \parallel \dots \parallel (\Sigma_{P_k} \parallel_{u,y} \Sigma_{C_k}) &\preceq \Sigma_{Q_1} \parallel \Sigma_{Q_2} \parallel \dots \parallel \Sigma_{Q_k} \end{aligned} \quad (16)$$

be satisfied. Then

$$(\Sigma_{P_1} \parallel_{u,y} \Sigma_{C_1}) \parallel \dots \parallel (\Sigma_{P_k} \parallel_{u,y} \Sigma_{C_k}) \preceq \Sigma_{Q_1} \parallel \Sigma_{Q_2} \parallel \dots \parallel \Sigma_{Q_k} \quad (17)$$

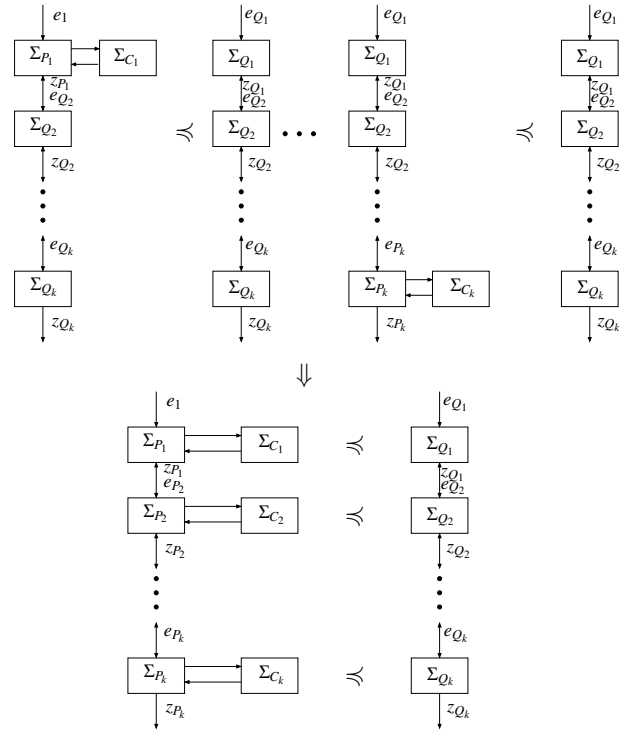


Fig. 4. Decentralized control scheme based on circular assume-guarantee reasoning.

Thus the decentralized controlled plant fulfills the global specification. Moreover, if (16) holds with bisimilarity then (17) also holds with bisimilarity.

Figure 4 depicts the second decentralized control scheme based on circular assume-guarantee reasoning. Each local controller Σ_{C_i} should be such that the global specification Σ_Q is satisfied, *assuming* that the other sub-specifications $\Sigma_{Q_k}, k \neq i$, are already satisfied. Hence, the k conditions S_1, S_2, \dots, S_k are circularly dependent.

This circular dependence complicates the proof of Theorem 8. We first need an auxiliary result that extends the proof rule of Theorem 7 by interconnecting arbitrary systems from the left and right.

Lemma 11: Consider six linear control systems $\Sigma_i, i \in \{P_1, P_2, Q_1, Q_2, L, R\}$ of the form (4) or (7). Then the following reasoning is valid:

$$\begin{aligned} S_I : \Sigma_L \parallel \Sigma_{P_1} \parallel \Sigma_{Q_2} \parallel \Sigma_R &\preceq \Sigma_L \parallel \Sigma_{Q_1} \parallel \Sigma_{Q_2} \parallel \Sigma_R \\ S_{II} : \Sigma_L \parallel \Sigma_{Q_1} \parallel \Sigma_{P_2} \parallel \Sigma_R &\preceq \Sigma_L \parallel \Sigma_{Q_1} \parallel \Sigma_{Q_2} \parallel \Sigma_R \end{aligned} \quad (18)$$

\Downarrow

$$S : \Sigma_L \parallel \Sigma_{P_1} \parallel \Sigma_{P_2} \parallel \Sigma_R \preceq \Sigma_L \parallel \Sigma_{Q_1} \parallel \Sigma_{Q_2} \parallel \Sigma_R$$

Proof: In order to prove this lemma, we extend $S_i, i = I, II$, in two steps. First, consider

$$S'_I = \{(x_L, x_{P_1}, x_{Q_2}, x_R, x_L, x_{Q_1}, \bar{x}_{Q_2}, x_R) \mid \exists x'_L, x'_R : (x_L, x_{P_1}, x_{Q_2}, x_R, x'_L, x_{Q_1}, \bar{x}_{Q_2}, x'_R) \in S_I\}$$

$$S'_{II} = \{(x_L, x_{Q_1}, x_{P_2}, x_R, x_L, \bar{x}_{Q_1}, x_{Q_2}, x_R) \mid \exists x'_L, x'_R : (x_L, x_{Q_1}, x_{P_2}, x_R, x'_L, \bar{x}_{Q_1}, x_{Q_2}, x'_R) \in S_{II}\}$$

The fact that $S'_i, i = I, II$, define full simulation relations is a consequence of the results in [5]. Using ideas from [5], we will furthermore add suitable subspaces to obtain the relations $(S'_i + \bar{S}'_i)^{\text{sym}}$, where

$$\begin{aligned} \bar{S}'_I &= \{(x_L, x_{P_1}, \bar{x}_{Q_2}, x_R, x_L, x_{Q_1}, -x_{Q_2}, x_L) \mid \bar{x}_{Q_2} \in \ker H_{Q_2}, \\ &\quad x_i \in \ker H_i, i \in \{L, P_1, Q_2, Q_1, R\}, \\ &\quad (x_L, x_{P_1}, x_{Q_2}, x_R, x_L, x_{Q_1}, \bar{x}_{Q_2}, x_L) \in S'_I\} \\ \bar{S}'_{II} &= \{(x_L, \bar{x}_{Q_1}, x_{P_2}, x_R, x_L, -x_{Q_1}, x_{Q_2}, x_L) \mid \bar{x}_{Q_1} \in \ker H_{Q_1}, \\ &\quad x_i \in \ker H_i, i \in \{L, P_2, Q_1, Q_2, R\}, \\ &\quad (x_L, x_{Q_1}, x_{P_2}, x_R, x_L, \bar{x}_{Q_1}, x_{Q_2}, x_L) \in S'_{II}\} \end{aligned}$$

and

$$\begin{aligned} (S'_I + \bar{S}'_I)^{\text{sym}} &= \{(x_L, x_{P_1}, \bar{x}_{Q_2}, x_R, x_L, x_{Q_1}, x_{Q_2}, x_L) \mid \\ &\quad (x_L, x_{P_1}, x_{Q_2}, x_R, x_L, x_{Q_1}, \bar{x}_{Q_2}, x_L) \in (S'_I + \bar{S}'_I)\}, \\ (S'_{II} + \bar{S}'_{II})^{\text{sym}} &= \{(x_L, \bar{x}_{Q_1}, x_{Q_2}, x_R, x_L, x_{Q_1}, x_{Q_2}, x_L) \mid \\ &\quad (x_L, x_{Q_1}, x_{P_2}, x_R, x_L, \bar{x}_{Q_1}, x_{Q_2}, x_L) \in (S'_{II} + \bar{S}'_{II})\} \end{aligned}$$

Similarly as before, construct S as

$$\begin{aligned} S &= \{(x_L, x_{P_1}, x_{P_2}, x_R, x_L, x_{Q_1}, x_{Q_2}, x_R) \mid \exists \bar{x}_{Q_1}, \bar{x}_{Q_2} : \\ &\quad (x_L, x_{P_1}, \bar{x}_{Q_2}, x_R, x_L, x_{Q_1}, x_{Q_2}, x_R) \in (S'_I + \bar{S}'_I)^{\text{sym}}, \\ &\quad (x_L, \bar{x}_{Q_1}, x_{P_2}, x_R, x_L, x_{Q_1}, x_{Q_2}, x_R) \in (S'_{II} + \bar{S}'_{II})^{\text{sym}}\} \end{aligned}$$

- The proofs that $S'_i + \bar{S}'_i, i = I, II$, and $(S'_i)^{\text{sym}}$ are full simulation relations of $\Sigma_L \parallel \Sigma_{P_1} \parallel \Sigma_{Q_2} \parallel \Sigma_R$ and of $\Sigma_L \parallel \Sigma_{Q_1} \parallel \Sigma_{P_2} \parallel \Sigma_R$ by $\Sigma_L \parallel \Sigma_{Q_1} \parallel \Sigma_{Q_2} \parallel \Sigma_R$, respectively, are analogous to the proofs of Lemma 1 and Lemma 2 in [5], respectively.
- The proof that for every $x \in \ker H_{Q_2}, y \in \ker H_{Q_1}$, there exist elements $(0, 0, x, 0, 0, 0, x, 0) \in (S'_I + \bar{S}'_I)^{\text{sym}}$ and $(0, y, 0, 0, 0, 0, y, 0) \in (S'_{II} + \bar{S}'_{II})^{\text{sym}}$ is similar to the proof of Lemma 3 in [5].

Finally, we have to prove that S as constructed in (19) is indeed a full simulation relation of $\Sigma_L \parallel \Sigma_{P_1} \parallel \Sigma_{P_2} \parallel \Sigma_R$ by $\Sigma_L \parallel \Sigma_{Q_1} \parallel \Sigma_{Q_2} \parallel \Sigma_R$. To do so, the proof of Theorem 7 can be repeated in all its steps. ■

We are now in the position to prove the validity of circular assume-guarantee reasoning for feedback interconnections of arbitrarily many linear systems. *Proof:* [Proof of Theorem 10] The idea of this proof is to successively combine the k conditions (16) and apply Lemma 11 at every step. Let k simulation relations $S_i, i = 1, \dots, k$, as in (16) be given, i.e.,

$$S_i : \Sigma_{Q_1} \parallel \dots \parallel \Sigma_{Q_{i-1}} \parallel \Sigma_{T_i} \parallel \Sigma_{Q_{i+1}} \parallel \dots \parallel \Sigma_{Q_k} \preceq \Sigma_Q$$

where for compactness of notation we have defined $\Sigma_{T_i} := \Sigma_{P_i} \parallel \Sigma_{C_i}, i = 1, \dots, k$. Starting with $i = 1$, consider $\lfloor \frac{k}{2} \rfloor$ pairs of two relations S_i and $S_{i+1}, i = 1, \dots, k-1$ where $\lfloor x \rfloor$ is the greatest natural number less or equal to x . Apply Lemma 11 to each of the $\lfloor \frac{k}{2} \rfloor$ pairs to obtain simulation relations $S_{i,i+1}$ of the form

$$S_{i,i+1} : \Sigma_{Q_1} \parallel \dots \parallel \Sigma_{Q_{i-1}} \parallel \Sigma_{T_i} \parallel \Sigma_{T_{i+1}} \parallel \dots \parallel \Sigma_{Q_k} \preceq \Sigma_Q$$

After this first step, $\lceil \frac{k}{2} \rceil$ simulation relations are left where $\lceil x \rceil$ is the smallest natural number greater or equal to x ,

$$\begin{cases} \{S_{i,i+1}, i = 1, \dots, k-1\}, & \text{if } \lfloor \frac{k}{2} \rfloor = \frac{k}{2} \\ \{\{S_{i,i+1}, i = 1, \dots, k-2\}, S_k\}, & \text{otherwise} \end{cases} \quad (19)$$

Continue by forming $\lceil \frac{k}{2} \rceil$ pairs of two simulation relations $S_j, S_j \in (19)$ to apply Lemma 11 on them. Repeating this procedure $\lceil \frac{k}{2} \rceil$ -times in total, the desired result follows in the last step. We formalize this approach in the following

Algorithm 12: Compute S from k simulation relations $S_i, i = 1, \dots, k$ of the form (16)

```

 $k = \lfloor \frac{N}{2} \rfloor$ 
 $R = \{S_i, i = 1, \dots, k\}$ 
for  $i = 1$  to  $\lceil \frac{k}{2} \rceil$  do
   $k = |R|$ 
  for  $j = 1$  to  $\lfloor \frac{k}{2} \rfloor$  do
    apply Lemma 11 to  $S_{2j-1}, S_{2j}, S_j \in R$  to obtain the
    relations  $S_{2j-1,2j}$  as given by (19)
  end for
  if  $\frac{k}{2} == \lfloor \frac{k}{2} \rfloor$  then
     $R = \{S_{2j-1,2j}, j = 1, \dots, \lfloor \frac{k}{2} \rfloor\}$ 
  else
     $R = \{\{S_{2j-1,2j}, j = 1, \dots, \lfloor \frac{k}{2} \rfloor\}, S_k\}$ 
  end if
end for
 $S = R$ 

```

We illustrate Theorem 10 and Algorithm 12 with the following example. ■

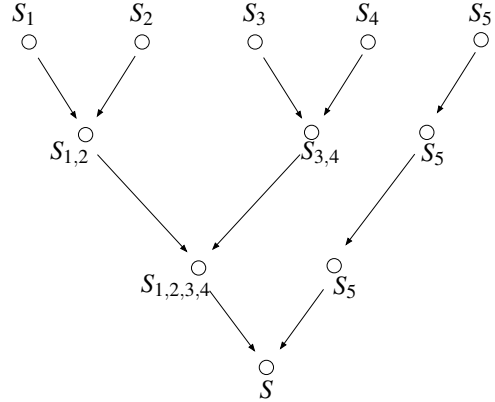


Fig. 5. Algorithm 12 applied to Example 13.

Example 13: Consider a plant system $\Sigma_P = \Sigma_{P_1} \parallel \dots \parallel \Sigma_{P_5}$ and a corresponding global specification $\Sigma_Q = \Sigma_{Q_1} \parallel \dots \parallel \Sigma_{Q_5}$ where all components $\Sigma_{P_i}, \Sigma_{Q_i}$ are of the form (2) and (4), respectively. Assume that there exist full simulation relations $S_i, i \in \{I, \dots, V\}$, as follows:

$$\begin{aligned} S_I &: \Sigma_{T_1} \parallel \Sigma_{Q_2} \parallel (\Sigma_{Q_3} \parallel \Sigma_{Q_4} \parallel \Sigma_{Q_5}) \preceq \Sigma_{Q_1} \parallel \Sigma_{Q_2} \parallel (\Sigma_{Q_3} \parallel \Sigma_{Q_4} \parallel \Sigma_{Q_5}) \\ S_{II} &: \Sigma_{Q_1} \parallel \Sigma_{T_2} \parallel (\Sigma_{Q_3} \parallel \Sigma_{Q_4} \parallel \Sigma_{Q_5}) \preceq \Sigma_{Q_1} \parallel \Sigma_{Q_2} \parallel (\Sigma_{Q_3} \parallel \Sigma_{Q_4} \parallel \Sigma_{Q_5}) \\ S_{III} &: (\Sigma_{Q_1} \parallel \Sigma_{Q_2}) \parallel \Sigma_{T_3} \parallel \Sigma_{Q_4} \parallel \Sigma_{Q_5} \preceq (\Sigma_{Q_1} \parallel \Sigma_{Q_2}) \parallel \Sigma_{Q_3} \parallel \Sigma_{Q_4} \parallel \Sigma_{Q_5} \\ S_{IV} &: (\Sigma_{Q_1} \parallel \Sigma_{Q_2}) \parallel \Sigma_{Q_3} \parallel \Sigma_{T_4} \parallel \Sigma_{Q_5} \preceq (\Sigma_{Q_1} \parallel \Sigma_{Q_2}) \parallel \Sigma_{Q_3} \parallel \Sigma_{Q_4} \parallel \Sigma_{Q_5} \\ S_V &: (\Sigma_{Q_1} \parallel \Sigma_{Q_2} \parallel \Sigma_{Q_3} \parallel \Sigma_{Q_4}) \parallel \Sigma_{T_5} \preceq (\Sigma_{Q_1} \parallel \Sigma_{Q_2} \parallel \Sigma_{Q_3} \parallel \Sigma_{Q_4}) \parallel \Sigma_{Q_5} \end{aligned}$$

In the first step, we pair the relations S_I, S_{II} and S_{III}, S_{IV} . Applying Lemma 11 to each pair, we conclude that there exist full simulation relations $S_{I,II}$ and $S_{III,IV}$ such that

$S_{I,II}$:

$$(\Sigma_{T_1} \parallel \Sigma_{T_2}) \parallel (\Sigma_{Q_3} \parallel \Sigma_{Q_4}) \parallel \Sigma_{Q_5} \preceq (\Sigma_{Q_1} \parallel \Sigma_{Q_2}) \parallel (\Sigma_{Q_3} \parallel \Sigma_{Q_4}) \parallel \Sigma_{Q_5}$$

$S_{III,IV}$:

$$(\Sigma_{Q_1} \parallel \Sigma_{Q_2}) \parallel (\Sigma_{T_3} \parallel \Sigma_{T_4}) \parallel \Sigma_{Q_5} \preceq (\Sigma_{Q_1} \parallel \Sigma_{Q_2}) \parallel (\Sigma_{Q_3} \parallel \Sigma_{Q_4}) \parallel \Sigma_{Q_5}$$

In the second step, Lemma 11 can now be applied to $S_{I,II}$ and $S_{III,IV}$ to obtain

$S_{I,II,III,IV}$:

$$(\Sigma_{T_1} \parallel \Sigma_{T_2} \parallel \Sigma_{T_3} \parallel \Sigma_{T_4}) \parallel \Sigma_{Q_5} \preceq (\Sigma_{Q_1} \parallel \Sigma_{Q_2} \parallel \Sigma_{Q_3} \parallel \Sigma_{Q_4}) \parallel \Sigma_{Q_5}$$

In the third step, consider the relations $S_{I,II,III,IV}$ and S_V . As a special case of Lemma 11, they fulfill the circular assume-guarantee rule of Theorem 7. Hence, there indeed exists a full simulation relation

$$S : \Sigma_{T_1} \parallel \Sigma_{T_2} \parallel \Sigma_{T_3} \parallel \Sigma_{T_4} \parallel \Sigma_{T_5} \preceq \Sigma_{Q_1} \parallel \Sigma_{Q_2} \parallel \Sigma_{Q_3} \parallel \Sigma_{Q_4} \parallel \Sigma_{Q_5} .$$

Finally, it is worth pointing out that conditions of the form (11) and (16) can be combined in a triangular proof rule to obtain a decentralized control scheme based on *non-circular* assume-guarantee reasoning. Like for interconnections for two systems only, non-circular assume-guarantee reasoning is always valid for more than two systems due to compositionality of series interconnections and transitivity of simulation, see [5] for more details. Not stating this formally, we provide a simple example instead to illustrate this point.

Example 14: Consider three plant systems $\Sigma_{P_i}, i = 1, 2, 3$, and three specifications Σ_{Q_i} . Let local controllers $\Sigma_{C_i}, i = 1, 2, 3$, be given such that the following conditions hold:

$$S_1 : \Sigma_{P_1} \parallel_{u,y} \Sigma_{C_1} \preceq \Sigma_{Q_1}$$

$$S_{II} : \Sigma_{Q_1} \parallel (\Sigma_{P_2} \parallel_{u,y} \Sigma_{C_2}) \preceq \Sigma_{Q_1} \parallel \Sigma_{Q_2}$$

$$S_{III} : \Sigma_{Q_1} \parallel \Sigma_{Q_2} \parallel (\Sigma_{P_3} \parallel_{u,y} \Sigma_{C_3}) \preceq \Sigma_{Q_1} \parallel \Sigma_{Q_2} \parallel \Sigma_{Q_3}$$

Combining S_1 and S_{II} by interconnecting the systems involved in S_1 with Σ_{S_2} yields

$$S_{I,II} : (\Sigma_{P_1} \parallel_{u,y}^{\Pi_1} \Sigma_{C_1}) \parallel (\Sigma_{P_2} \parallel_{u,y}^{\Pi_2} \Sigma_{C_2}) \preceq \Sigma_{Q_1} \parallel \Sigma_{Q_2}$$

while by the same reasoning, $S_{I,II}$ and S_{III} result in

$$S : (\Sigma_{P_1} \parallel_{u,y}^{\Pi_1} \Sigma_{C_1}) \parallel (\Sigma_{P_2} \parallel_{u,y}^{\Pi_2} \Sigma_{C_2}) \parallel (\Sigma_{P_3} \parallel_{u,y}^{\Pi_3} \Sigma_{C_3}) \preceq \Sigma_{Q_1} \parallel \Sigma_{Q_2} \parallel \Sigma_{Q_3}$$

V. CONCLUSIONS

The aim of this paper was to demonstrate that compositional analysis techniques can be successfully applied to solve decentralized control problems. In order to apply compositional analysis techniques based on simulation theory we extended previous results on compositional reasoning and (circular) assume-guarantee reasoning for two feedback interconnected systems to series feedback interconnections of arbitrarily many systems. A crucial assumption in our approach is that the global specification should be decomposable according to the same interconnection structure as the global plant.

A following paper will be devoted to Problem 4, by deriving necessary and sufficient conditions for the existence of local controllers satisfying the obtained conditions for the solution of Problem 3.

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