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On the dissipative analysis and control of state-space symmetric systems

Gabriela Iuliana BARA

Abstract— **This paper addresses the quadratic dissipativity analysis and static output feedback control of linear time-invariant systems which are state-space symmetric. By considering particular weighting matrices, we present a necessary and sufficient inequality condition for checking asymptotic stability and quadratic dissipativity for this class of systems. Our analysis condition involves only system state matrices and known weighting matrices. Therefore, this condition is easy to check numerically and particularly suitable in the case of large-scale symmetric systems. The application of our analysis result to symmetric static output feedback (SSOF) control design is also reported in this paper. An easily tractable numerically, necessary and sufficient condition for the existence of a SSOF control law and an explicit parametrization of all SSOF controllers guaranteeing the asymptotic stability and the quadratic dissipativity of the closed-loop system is given. Note that the results presented in this paper generalize some results already proposed in the literature to a more general case of quadratic dissipativity analysis and control.**

Index Terms— **Linear systems, state-space (internally) symmetric systems, dissipativity analysis, static output feedback, large-scale systems.**

I. INTRODUCTION

The symmetry, which characterizes various phenomena, naturally arises in many fields such as quantum mechanics, bifurcation theory, chemistry and crystallography. The role of symmetry has also been investigated in the field of dynamical systems and control theory. Examples such as the twin lifting concept [1], the discretized partial differential equations [1] and dynamical systems composed of interconnected subsystems [1]–[3], show that linear models for such systems exhibit certain group-theoretic symmetries. The structure involved by these symmetries is central to the system and must be preserved when developing synthesis methods. Many group-theoretic approaches, using the representation theory, have been proposed in the literature for linear systems. In [4] and [1] the realization problem has been addressed while in [5] and [6] the stability has been investigated based on the decomposition of a symmetric system into smaller uncoupled systems. The latter results have been extended, in [7], to $\mathcal{H}_\infty$ performance analysis by showing that the $\mathcal{H}_\infty$ norm of a symmetric system can be determined from the $\mathcal{H}_\infty$ norms of the uncoupled systems. The group-theoretic methods have been used in [8] for studying fault tolerance properties of arrays of symmetric systems. In [9], the group-theoretic symmetry has been exploited to reduce the computational effort required for control synthesis whenever the design specifications are expressible via semi-definite programming.

The author is with University of Strasbourg, LSIIT-UMR CNRS-UdS 7005, bd. Sébastien Brant, BP 10413, 67412 Illkirch Cedex France. E-mail: Iuliana.Bara@lsiit-cnrs.unistra.fr, Phone: +33 (0)368854862, Fax: +33 (0)368854480.

Representations by symmetric transfer functions / matrices, also called externally symmetric models, appear in many electrical engineering applications such as electrical and power networks [10] and large-space structures with collocated sensors and actuators [11], [12]. As a particular subclass of symmetric transfer matrices, systems with state-space symmetry, also called internally symmetric systems, are used for modeling systems with zeros interlacing the poles [13] or physical systems with only one type of energy storage capability [12], [14]. Conditions when symmetric transfer matrices admit a symmetric state-space realization have been presented in [12], [15]. Various problems for state-space symmetric systems have been addressed such as: model reduction by optimal Hankel norm approximation [16]; stabilizability by decentralized controllers [12]; control design by using colored Petri nets and symbolic reachability graphs [17]; stabilization by symmetric static output feedback (SSOF) control [15], [18]; $\mathcal{H}_\infty$ norm characterization, positive real analysis and SSOF control [18] as well as mixed $\mathcal{H}_\infty$ / positive real performance analysis and SSOF control [14]. Note that the obtained results show that exploiting the symmetry property allows to reduce the numerical complexity of the analysis and synthesis results for the class of state-space symmetric systems. This is particularly suitable in the case of very large-scale systems where the classical analysis and design methods using LMI formulation or Riccati equalities are computationally prohibitive. For instance, an explicit formula for computing $\mathcal{H}_\infty$ norm of the system and an explicit expression of the optimally achievable closed-loop $\mathcal{H}_\infty$ norm and of the optimal control gains has been presented in [18]. The extension of these results to mixed $\mathcal{H}_\infty$ / positive real performance analysis and SSOF control has been achieved in [14].

In this paper, we address the quadratic dissipativity analysis and control of state-space symmetric systems. By considering particular weighting matrices, we present a necessary and sufficient inequality condition for checking asymptotic stability and quadratic dissipativity of symmetric systems. This analysis condition requires only the computation of eigenvalues of a decision matrix involving system state matrices and weighting matrices. Therefore, this analysis condition is easy to verify numerically and is particularly appropriate for very large-scale symmetric systems. Then, an easily tractable numerically, necessary and sufficient condition for the existence of a SSOF control is derived and an explicit parametrization of all SSOF controllers guaranteeing the asymptotic stability and the quadratic dissipativity of the closed-loop system is given. Despite the nonlinearity of the SOF control design [19], exploiting the state-space symmetry allows to obtain an analytical solution to the SSOF synthesis problem. Note that our results generalize the ones proposed in [18], for $\mathcal{H}_\infty$ analysis and control, and in [14], for mixed $\mathcal{H}_\infty$ / positive real performance analysis and control, to the more general case of quadratic dissipativity analysis and control with respect to a particular class of weighting matrices.

Notations: The notations used throughout the paper are standard. The relation $A > B$ ($A < B$) means the matrix $A-B$ is positive (negative) definite. The superscript T stands for matrix transposition. The matrix I stands for the identity matrix of appropriate dimension. $\star$ is used for the blocks without any importance. $\lambda_{max}(A)$ denotes the maximum eigenvalue of the symmetric matrix A. $M^\perp$ represents the orthogonal complement of M; for a real matrix $M \in I\!R^{n\times m}$ of rank m, $M^\perp \in I\!R^{(n-m)\times n}$ is such that $M^\perp M = 0$. M^+ represents the Moore-Penrose generalized inverse of M. M^* is the conjugate transpose of M. $\mathrm{Herm}\{M\}$ is the Hermitian part of matrix M *i.e.* $\mathrm{Herm}\{M\} = 1/2(M+M^*)$. $\mathcal{L}_2^n$ is the space of square integrable functions on $I\!R^+$ with values in $I\!R^n$. $\mathcal{L}_{2e}^n$ is the extended $\mathcal{L}_2^n$ space of measurable functions on $I\!R^+$. $< u, v >_T = \int_0^T u^T v \, dt$ for u, $v \in \mathcal{L}_{2e}^n$.

II. Problem statement

Consider the class of linear time-invariant systems described by

$$\dot{x}(t) = Ax(t) + Bw(t), \quad x_0 = 0 \tag{1a}$$

$$z(t) = Cx(t) + Dw(t) \tag{1b}$$

where $x(t) \in I\!R^n$ is the state vector, $w(t) \in I\!R^m$ is the exogenous input vector and $z(t) \in I\!R^p$ is the controlled output vector. The matrices A, B, C and D are known constant matrices of appropriate dimensions and are assumed to be a minimal state-space realization of the system. In the following, we also assume that the system is internally or state-space symmetric *i.e.*

$$A = A^T, \; B = C^T \text{ and } D = D^T. \tag{2}$$

This means that the system is square *i.e.* the input and output vectors have the same dimension $m = p$. The internal symmetry implies the external one *i.e.* $G(s) = G(s)^T$, where $G(s) = C(sI - A)^{-1}B + D$ is the system transfer matrix, but the converse is not necessarily true.

Let us associate to the system a quadratic energy supply rate

$$q(w(t), z(t)) = \begin{bmatrix} z^T(t) & w^T(t) \end{bmatrix} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} z(t) \\ w(t) \end{bmatrix} \tag{3}$$

where Q and R are symmetric matrices. Then, the energy supply function associated to the system is

$$E(w(t), z(t), T) = \int_0^T q(w(t), z(t)) \, dt.$$

The notion of dissipativity with respect to the quadratic supply rate $q(w(t), z(t))$ or (Q, S, R)-dissipativity is defined as follows (see [20]–[22] for a detailed presentation of the dissipativity concept).

Definition 2.1: Given symmetric matrices Q, R and a general matrix S, a system with quadratic supply rate (3) is called (Q, S, R)-dissipative if there exists a nonnegative storage function $V : I\!R^n \to I\!R_+$, $V(0) = 0$ such that

$$V(x_0) + E(w(t), z(t), T) \geq V(x(T)) \quad \forall w(t) \in \mathcal{L}_{2e}^n, \forall T \geq 0.$$

When this dissipation inequality is strict, we say that the system is strictly (Q, S, R)-dissipative.

It is well known that a linear differential system is dissipative with respect to the quadratic supply rate $q(w(t), z(t))$ if and only if there exists a quadratic storage function $V(x(t))$ [23]. The time-domain dissipation inequality condition of Definition 2.1 is equivalent to the following frequency-domain condition:

$$\begin{bmatrix} G(j\omega)^* & I \end{bmatrix} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \geq 0 \quad \forall \omega \in I\!R$$

where $G(s)$ is the system transfer matrix.

Assumption 1: In the following, we consider that the weighting matrices are given by

$$Q = \beta_1 I, \; S = \beta_2 I \text{ and } R = \beta_3 I \tag{4}$$

and, together with system (1), satisfy the assumptions

$$\beta_1 \leq 0 \text{ and} \tag{5a}$$

$$\beta_3 I + 2\beta_2 D + \beta_1 D^2 > 0. \tag{5b}$$

Note that, for this particular structure of weighting matrices, the strict (Q, S, R)-dissipativity is still general enough to include as special cases:

- $\mathcal{H}_\infty$ norm constraint which corresponds to
$$\beta_1 = -\gamma^{-1}, \; \beta_2 = 0 \text{ and } \beta_3 = \gamma. \tag{6}$$
- Passivity or strict positive real performance obtained when
$$\beta_1 = 0, \; \beta_2 = 1 \text{ and } \beta_3 = 0. \tag{7}$$
The notion of passivity is a special property of square systems meaning that the system cannot produce energy. The equivalence between the passivity of a system and the positive realness of its transfer matrix is given by the well known Kalman-Yakubovich-Popov lemma.
- Mixed $\mathcal{H}_\infty$ and positive real performance for
$$\beta_1 = -\gamma^{-1}\theta, \; \beta_2 = 1 - \theta \text{ and } \beta_3 = \gamma\theta \tag{8}$$
where θ is a weighting parameter that represents the trade-off between $\mathcal{H}_\infty$ and positive real performance.
- Sector bounded constraint which corresponds to
$$\beta_1 = -1, \; \beta_2 = (a+b)/2, \; \beta_3 = -ab. \tag{9}$$
We recall that, as defined in [24], a system is strictly inside the sector $[a, \; b]$ if $\mathrm{Herm}\{[G(j\omega) - aI]^*[G(j\omega) - bI] < 0\}$ for all $\omega \in I\!R$.

In the following section, we investigate the quadratic dissipativity analysis for internally symmetric systems under Assumption 1.

III. DISSIPATIVITY ANALYSIS

A. New analysis result

Theorem 3.1: Consider the system (1) satisfying the symmetry property (2). Under Assumption 1, the system (1) is asymptotically stable and strictly (Q, S, R)-dissipative if and only if the following inequality condition is satisfied

$$A + B\big((\alpha - \beta_2)I - \beta_1 D\big)\big(\beta_3 I + 2\beta_2 D + \beta_1 D^2\big)^{-1} B^T < 0 \tag{10}$$

where $\alpha = \sqrt{\beta_2^2 - \beta_1\beta_3}$.

Proof: Based on the dissipative analysis result in [23] and using (2) as well as Assumption 1, the system (1) is asymptotically stable and strictly (Q, S, R)-dissipative if and only if there exists a symmetric Lyapunov matrix $P > 0$ such that

$$\begin{bmatrix} AP + PA & PB - \beta_2 B & \sqrt{-\beta_1}B \\ B^T P - \beta_2 B^T & -\beta_3 I - 2\beta_2 D & \sqrt{-\beta_1}D \\ \sqrt{-\beta_1}B^T & \sqrt{-\beta_1}D & -I \end{bmatrix} < 0. \tag{11}$$

By Schur complement, this is equivalent to

$$AP + PA + B\big(-\beta_1 I + M_1M_2M_1\big)B^T - BM_1M_2B^TP - PBM_2M_1B^T + PBM_2B^TP < 0 \tag{12}$$

where $M_2 = \big(\beta_3 I + 2\beta_2 D + \beta_1 D^2\big)^{-1} > 0$ and $M_1 = \beta_2 I + \beta_1 D$.

Note that matrices M_1 and M_2 are symmetric and that, based on basic matrix manipulations, the following relation holds:

$$M_1M_2 = M_2M_1. \tag{13}$$

Moreover, based on this equality, we obtain

$$\begin{aligned} -\beta_1 I + M_1M_2M_1 &= (-\beta_1 M_2^{-1} + M_1^2)M_2 \\ &= (\beta_2^2 - \beta_1\beta_3)M_2. \end{aligned}$$

Since any Lyapunov matrix P can be rewritten as αP_0 with the scalar $\alpha > 0$ and the matrix $P_0 > 0$, inequality (12) can be rewritten as

$$\alpha AP_0 + \alpha P_0 A + (\beta_2^2 - \beta_1\beta_3)BM_2B^T - \alpha BM_1M_2B^TP_0 - \alpha P_0BM_2M_1B^T + \alpha^2 P_0BM_2B^TP_0 < 0. \tag{14}$$

Setting $\alpha^2 = \beta_2^2 - \beta_1\beta_3$ and using (13), it follows from Lemma 1.1 presented in the appendix that this inequality is equivalent to

$$\alpha A - \alpha BM_1M_2B^T + \alpha^2 BM_2B^T < 0$$

which is exactly the condition (10). ■

Remark 3.1: From the proof of Theorem 3.1, we deduce that the storage function ensuring the asymptotic stability and guaranteeing the strict (Q, S, R)-dissipativity of system (1) with symmetry property (2) is given by

$$V(x) = x^T P x = \sqrt{\beta_2^2 - \beta_1\beta_3} \parallel x \parallel^2 .$$

This is an explicit formulation of the storage function which facilitates the analysis of dissipativity property. Indeed, as stated in Theorem 3.1, the problem of checking asymptotic stability and $(\beta_1 I, \beta_2 I, \beta_3 I)$-dissipativity of symmetric systems reduces to checking the negative definiteness of the decision matrix given by the left-hand side expression of inequality (10). This can be done by simply computing the decision matrix eigenvalues and checking the negativity of its maximum eigenvalue. Therefore, the analysis condition of Theorem 3.1 can easily be checked numerically and does not require solving a LMI condition or a Riccati equality as, in general, required for nonsymmetric systems.

B. Connections with previous results

In the context of internally symmetric systems, some analysis and synthesis results have already been proposed in [14], [18]. The $\mathcal{H}_\infty$ performance as well as positive real analysis and control design have been addressed in [18] while the mixed $\mathcal{H}_\infty$ / positive real performance analysis and control has been dealt with in [14]. In this section, we discuss our analysis result of Theorem 3.1 in connection with the ones in [14] and [18].

When analyzing the $\mathcal{H}_\infty$ *performance*, the weighting matrices are given by (6). It has been shown in [18] (see Lemma 2 in [18]) that an internally symmetric system has an $\mathcal{H}_\infty$ norm less than γ if and only if

$$\gamma^2 I - DD > 0 \quad \text{and} \tag{15a}$$

$$2\gamma A + (\gamma B + BD)(\gamma^2 I - DD)^{-1}(\gamma B + BD)^T + BB^T < 0. \tag{15b}$$

Using (6), condition (15a) is equivalent to our assumption (5b). Based on the commutativity of the product between $(\gamma^2 I - DD)^{-1}$ and $(\gamma I + D)$ and rewriting BB^T in (15b) as

$$BB^T = B(\gamma^2 I - DD)(\gamma^2 I - DD)^{-1}B^T,$$

we obtain that condition (15b) is equivalent to

$$A + B\big(\gamma I + D\big)\big(\gamma^2 I - D^2\big)^{-1}B^T < 0 \tag{16}$$

which is exactly the condition (10). Hence, our Theorem 3.1 covers Lemma 2 in [18]. Note that an explicit formula for computing the $\mathcal{H}_\infty$ norm has been given in [18] as:

$$\gamma_{opt} = \max\Big(\lambda_{max}\big(-D\big), \lambda_{max}\big(D - B^TA^{-1}B\big)\Big).$$

This norm value can also been obtained from our Theorem 3.1 by solving the following optimization problem:

$$\text{Min}\,\gamma \text{ such that (10), which is equivalent to (16), holds.}$$

Note that this optimization problem can easily be solved, despite the nonlinearity of (16) with respect to γ, by simply iteratively decreasing the value of γ until checking condition (16) fails.

In the case of *passivity analysis*, it has been proven in [18] that a symmetric system is strongly positive real if and only if $A < 0$ and $D > 0$. Using the weighting matrices given by (7), our assumption (5b) reduces to $D > 0$ while the condition (10) reduces to $A < 0$ since $\alpha = 1$. Therefore, our Theorem 3.1 covers also Theorem 15 in [18].

Now, let us discuss the *mixed $\mathcal{H}_\infty$ / positive real performance analysis*. Based on Theorem 3.1, the following result is obtained.

Corollary 3.2: For a given scalar $\theta \in (0,1)$ in (8), the mixed $\mathcal{H}_\infty$/PR norm of stable internally symmetric system (1) is given by

$$\gamma_{opt} = \max\Big(f(\theta)^{-1}\lambda_{max}\big(-D\big), f(\theta)\lambda_{max}\big(D - B^T A^{-1} B\big)\Big), \quad (17)$$

where $f(\theta) = \dfrac{\alpha+\theta-1}{\theta}$ and $\alpha = \sqrt{(1-\theta)^2+\theta^2}$, whenever Assumption 1 holds with β_1, β_2 and β_3 given by (8).

Proof: Using weighting matrices (8), it follows from the proof of Theorem 3.1 that the Lyapunov matrix in (11) is $P = \alpha I$ where $\alpha = \sqrt{(1-\theta)^2+\theta^2}$. Hence, the stability of internally symmetric system (1) is equivalent to

$$\begin{bmatrix} 2\alpha A & \theta f(\theta) B & B \\ \theta f(\theta) B^T & -\gamma\theta I - 2(1-\theta)D & D \\ B^T & D & -\frac{\gamma}{\theta} I \end{bmatrix} < 0$$

where $f(\theta) = \dfrac{\alpha+\theta-1}{\theta}$. Based on this definition of $f(\theta)$, the following relations can be proved:

$$2\frac{\theta-1}{\theta} + f(\theta)^{-1} = \frac{\alpha}{\theta}, \quad (18a)$$

$$\frac{\theta}{2\alpha}\big(1 + f(\theta)^2\big) = f(\theta). \quad (18b)$$

Applying Generalized Finsler's Lemma (see Lemma 1.2 in the appendix), we deduce that the optimal γ is given by

$$\gamma_{opt} = \lambda_{max}\begin{bmatrix} \frac{\theta}{2\alpha} f(\theta)^2 \Omega + 2\frac{\theta-1}{\theta} D & \frac{\theta}{2\alpha} f(\theta)\Omega + D \\ \frac{\theta}{2\alpha} f(\theta)\Omega + D & \frac{\theta}{2\alpha}\Omega \end{bmatrix}$$

where $\Omega = -B^T A^{-1} B$. Using the relation (18a), the latter matrix can be rewritten as

$$\begin{bmatrix} f(\theta) & 0 \\ I & I \end{bmatrix} \begin{bmatrix} \frac{\theta}{2\alpha} f(\theta)\Omega + 2\frac{\theta-1}{\theta} f(\theta)^{-1} D & \frac{\theta}{2\alpha}\Omega + f(\theta)^{-1} D \\ f(\theta)^{-2} D & -f(\theta)^{-1} D \end{bmatrix}.$$

Since the spectrum of $\Gamma_1\Gamma_2$ is identical to the spectrum of $\Gamma_2\Gamma_1$ for any square matrices Γ_1 and Γ_2, if follows based on (18) that

$$\gamma_{opt} = \lambda_{max}\begin{bmatrix} f(\theta)\Omega + f(\theta) D & \star \\ 0 & -f(\theta)^{-1} D \end{bmatrix}$$

which leads to (17). ■

Note that, based on relation (18a), our Corollary 3.2 covers the formula proposed in [14], Theorem 7.

As far as we know, there are no analysis results with regard to sector bounded constraints for state-space symmetric systems. Our Theorem 3.1 allows easily checking sector-bounds for a given symmetric system. The next two particular cases follow directly. Given a positive scalar b, the symmetric internally system (1) is

- strictly inside the sector $[-b,\ b]$ if and only if

$$b \geq \max\Big(\lambda_{max}\big(-D\big), \lambda_{max}\big(D - B^T A^{-1} B\big)\Big)$$

- strictly inside the sector $[0,\ b]$ if and only if it is strongly positive real, *i.e.* $A < 0$ and $D > 0$, and

$$b \geq \lambda_{max}\big(D - B^T A^{-1} B\big).$$

IV. Dissipative output feedback control design

Consider the system with the following representation

$$\dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t) \quad (19a)$$

$$z(t) = C_1 x(t) + D w(t) \quad (19b)$$

$$y(t) = C_2 x(t) \quad (19c)$$

where $x(t) \in I\!R^n$ is the state vector, $w(t) \in I\!R^{m_1}$ is the exogenous input vector, $u(t) \in I\!R^{m_2}$ is the control input vector, $z(t) \in I\!R^{p_1}$ is the controlled output vector and $y(t) \in I\!R^{p_2}$ is the measured output vector. We assume that the system satisfies the symmetry conditions:

$$A = A^T,\ B_1 = C_1^T,\ B_2 = C_2^T \text{ and } D = D^T. \quad (20)$$

The symmetric static output feedback (SSOF) dissipative control design problem consists in synthesizing a symmetric gain K such that the static output feedback control law

$$u(t) = Ky(t)$$

guarantees the asymptotic stability and the strict (Q,S,R)-dissipativity of the closed-loop system. The state-space representation of the closed-loop system is given by:

$$\dot{x}(t) = (A + B_2 K B_2^T)x(t) + B_1 w(t) \quad (21a)$$

$$z(t) = C_1 x(t) + D w(t). \quad (21b)$$

Note that, based on (20), the closed-loop system is state-space symmetric.

Theorem 4.1: For the symmetric system (19), there exists a symmetric static output feedback controller that guarantees the asymptotic stability and the strict (Q,S,R)-dissipativity of the closed-loop system if and only if condition (22) is satisfied. When this condition holds, the set of all symmetric output feedback gains K is given by (23).

Proof: Applying the result of Theorem 3.1 to the closed-loop system (21) and using Generalized Finsler's Lemma (see Lemma 1.2 in the appendix), we obtain the necessary and sufficient condition (22) for the existence of a dissipative SSOF controller as well as the characterization (23) of all SSOF controllers. ■

Remark 4.1: Our Remark 3.1 also applies to the SSOF existence condition of Theorem 4.1. Note that Theorem 4.1 extends the results presented in [14] by firstly, considering a larger class of weighting matrices Q, S and R, and secondly, by considering a nonzero D matrix. Indeed, in the case of mixed $\mathcal{H}_\infty$ / positive real control (see (8) for the weighting matrices) and $D = 0$, Theorem 4.1 reduces to Theorem 8 in [14]. When $D \neq 0$, using condition (22), Lemma 1.3 and similar arguments to ones of Corollary 3.2's proof, we can show that the level of mixed $\mathcal{H}_\infty$ /PR performance optimally achievable by symmetric output feedback is given by

$$\gamma_{CL} = \max\Big(f(\theta)^{-1}\lambda_{max}\big(-D\big), f(\theta)\lambda_{max}\big(D + B_1^T B_2^{\perp T}(-B_2^{\perp} A B_2^{\perp T})^{-1} B_2^{\perp} B_1\big)\Big) \quad (24)$$

when the trade-off parameter θ between $\mathcal{H}_\infty$ and positive real performances is given. This, together with Theorem 4.1,

$$B_2^{\perp}\Big(A + B_1\big((\alpha-\beta_2)I_p - \beta_1 D\big)\big(\beta_3 I_p + 2\beta_2 D + \beta_1 D^2\big)^{-1} B_1^T\Big) B_2^{\perp\, T} < 0 \text{ where } \alpha = \sqrt{\beta_2^2 - \beta_1\beta_3} \tag{22}$$

$$K \le -B_2^{+}\Big(\Sigma - \Sigma B_2^{\perp\, T}\big(B_2^{\perp}\Sigma B_2^{\perp\, T}\big)^{-1} B_2^{\perp}\Sigma\Big) B_2^{+\, T} \text{ where}$$
$$\Sigma = A + B_1\big((\alpha-\beta_2)I_p - \beta_1 D\big)\big(\beta_3 I_p + 2\beta_2 D + \beta_1 D^2\big)^{-1} B_1^T \text{ with } \alpha = \sqrt{\beta_2^2 - \beta_1\beta_3} \tag{23}$$

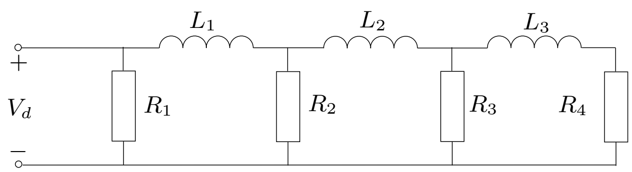

Fig. 1. RL circuit network

generalizes the results of Theorem 8 in [14] to non-null matrices D.

V. Numerical example

Consider a RL circuit network similar to the one in [14] and shown in Figure 1. In addition, let us consider that the inductor L_3 is submitted to a magnetic field perturbation which generates a perturbation $w(t)$ in its current. By considering the currents of inductors L_1, L_2 and L_3 as the state variables and the voltage V_d as the control input, we obtain the state space equation (19a) where:

$$A = \begin{bmatrix} -R_2/L_1 & R_2/L_1 & 0 \\ R_2/L_2 & -(R_2+R_3)/L_2 & R_3/L_2 \\ 0 & R_3/L_3 & -(R_3+R_4)/L_3 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \text{ and } B_2 = \begin{bmatrix} 1/L_1 & 0 & 0 \end{bmatrix}^T.$$

The measured output is the current of resistor R_1 while the controlled output is the voltage of resistor R_4. Therefore, $C_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$, $C_1 = \begin{bmatrix} 0 & 0 & R_4 \end{bmatrix}$ and $D = R_4$. Assume $L_1 = L_2 = L_3 = 1H$ and $R_1 = 0.5$ ohm, $R_2 = 8$ ohms, $R_3 = 5$ ohms, $R_4 = 1$ ohm. Then, the open-loop system is a symmetric state-space system as in (19)-(20).

Let us study mixed $\mathcal{H}_\infty$ / positive real performance analysis and control of this system with a trade-off parameter $\theta = 0.5$. Then, $f(\theta) = 0.4142$ and the optimal γ is computed, using Corollary 3.2 or [14], as being 1.2426. The control input can be used in order to improve the value of γ. Since the matrix $D \neq 0$, the control synthesis result proposed in [14] can not be used. Based on our Theorem 4.1, a symmetric output feedback controller can be synthesized. Note that the optimally achievable level of mixed $\mathcal{H}_\infty$/PR performance for the closed-loop system is given by (24) *i.e.* $\bar{\gamma}_{CL} = 0.93$. This represents an improvement of 25%. For any $\gamma \ge \bar{\gamma}_{CL}$, all feedback gains rendering the closed-loop system stable with a mixed $\mathcal{H}_\infty$/PR parameter less than γ are given by (23). For instance, when γ equals the optimal level $\bar{\gamma}_{CL} = 0.93$, the control gain corresponding to this optimal level is $K = \begin{bmatrix} -2.3593e+004 \end{bmatrix}$.

Now, let us assume that the weighting matrices are given by $Q = -I$, $S = 2I$ and $R = 3I$. Then, the assumptions (5) are satisfied. Using Theorem 3.1, it is easy to check that the system is strictly (Q,S,R)-dissipative. When the weighting matrices are $Q = -I$, $S = 0.8I$ and $R = 2I$, Assumption 1 is satisfied but the system is not strictly (Q,S,R)-dissipative since condition (10) is not verified. As condition (22) is verified, we can compute a SSOF gain rendering the closed-loop system strictly (Q,S,R)-dissipative based on Theorem 4.1. For instance, $K = \begin{bmatrix} -2.4181 \end{bmatrix}$ is a possible choice for the control gain which guarantees the closed-loop strict (Q,S,R)-dissipativity.

VI. Conclusion

In this paper, we have proposed explicit solutions for dissipativity analysis and SOF control problems for state-space symmetric systems. Our necessary and sufficient analysis condition reduces to checking the negative definiteness of a decision matrix involving only system state matrices and weighting matrices. This allows deriving a necessary and sufficient existence condition for a SSOF controller. An explicit parametrization of all SSOF controllers has also been presented. The proposed solutions do not require solving LMI conditions or Riccati equations as, in general, required for nonsymmetric systems. Hence, they are applicable for the case of very large-scale symmetric systems by exploiting current developments in the area of largest eigenvalue determination for large matrices. The results presented in this paper generalize some results already proposed in the literature to a more general case of quadratic dissipativity analysis and control.

Appendix

Lemma 1.1: Consider the following quadratic matrix inequality with respect to the symmetric matrix parameter P

$$\Omega\Omega^T + P\Xi + \Xi P + P\Omega\Omega^T P < 0 \tag{25}$$

where the symmetric matrix Ξ and the general matrix Ω are given. Then, there exists a symmetric positive-definite matrix P_0 solution to inequality (25) if and only if

$$\Omega\Omega^T + \Xi < 0.$$

Proof: *Sufficiency:* It is straightforward by choosing $P_0 = I$.

Necessity: Assume that $P_0 > 0$ is a solution of (25). Pre- and post-multiplying this inequality by P_0^{-1}, we obtain the same inequality (25) with respect to P_0^{-1}. Therefore, P_0 and its inverse P_0^{-1} are both solutions of this inequality. By Schur complement, (25) is equivalent to

$$\begin{bmatrix} \Omega\Omega^T + P_0\Xi + \Xi P_0 & P_0\Omega \\ \Omega^T P_0 & -I \end{bmatrix} < 0. \tag{26}$$

Now, the proof mainly follows the procedure employed in the proof of Lemma 2 in [18]. The eigenvalues decomposition of P_0 allows to rewrite this matrix as

$$P_0 = U\Delta_0 U^T,\ U^T = U^{-1},\ \Delta_0 = \mathrm{Diag}(\sigma_1, \ldots, \sigma_n) > 0.$$

Note that $P_0^{-1} = U\Delta_0^{-1}U^T$. As $\sigma_1 > 0$, there exists $0 \leq \lambda_1 \leq 1$ such that $\lambda_1\sigma_1 + (1-\lambda_1)\sigma_1^{-1} = 1$. Therefore, the convex combination between (26) with respect to P_0 and (26) with respect to P_0^{-1} provides

$$\begin{bmatrix} \Omega\Omega^T + P_1\Xi + \Xi P_1 & P_1\Omega \\ \Omega^T P_1 & -I \end{bmatrix} < 0 \tag{27}$$

where $P_1 = \lambda_1 P_0 + (1-\lambda_1)P_0^{-1} = U\,\mathrm{Diag}(1, \bar{\sigma}_2, \ldots, \bar{\sigma}_n)U^T$ with $\bar{\sigma}_i = \lambda_1\sigma_i + (1-\lambda_1)\sigma_i^{-1}$ for $i = 2, \ldots, n$. Using Schur complement for (27) and then, pre- and post-multiplying the result by P_1^{-1}, we obtain by Schur complement the inequality (27) this time in P_1^{-1}. Hence, P_1 and its inverse P_1^{-1} are both solutions of (27). As $\sigma_2 > 0$, there exists $0 \leq \lambda_2 \leq 1$ such that $\lambda_2\sigma_2 + (1-\lambda_2)\sigma_2^{-1} = 1$. The convex combination between (27) with respect to P_1 and (27) with respect to P_1^{-1} provides

$$\begin{bmatrix} \Omega\Omega^T + P_2\Xi + \Xi P_2 & P_2\Omega \\ \Omega^T P_2 & -I \end{bmatrix} < 0$$

where $P_2 = \lambda_2 P_1 + (1-\lambda_2)P_1^{-1} = \mathrm{Diag}(1, 1, \tilde{\sigma}_3, \ldots, \tilde{\sigma}_n)$ with $\bar{\sigma}_i = \lambda_2\sigma_i + (1-\lambda_2)\sigma_i^{-1}$ for $i = 3, \ldots, n$. By repeating this method, we obtain that $P_n = UU^T = I$ is a solution of (25). ∎

Lemma 1.2 (Generalized Finsler's Lemma [18]): Consider matrices M and Q such that M has full column rank and $Q = Q^T$. Then, the following statements are equivalent:

- There exists a symmetric matrix X such that
$$MXM^T - Q > 0.$$
- The following condition holds: $M^{\perp}QM^{\perp T} < 0$.

If the above statements hold, then all matrices X satisfying the first statement are given by

$$X > M^+\big(Q - QM^{\perp T}(M^{\perp}QM^{\perp T})^{-1}M^{\perp}Q\big)M^{+T}.$$

When X is a scalar μ then the latter relation reduces to

$$\mu > \mu_{min} = \lambda_{max}\Big(M^+\big(Q - QM^{\perp T}(M^{\perp}QM^{\perp T})^{-1}M^{\perp}Q\big)M^{+T}\Big).$$

Lemma 1.3 ([25]): Consider the symmetric positive-definite matrix Δ and the full column rank matrix Γ. Then, $\Delta \geq \Gamma\Gamma^T$ if and only if $\lambda_{max}(\Gamma^T\Delta^{-1}\Gamma) \leq 1$.

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