

A Structured Model Reduction Method for Large Scale Networks

Bing Chu, Stephen Duncan and Antonis Papachristodoulou

Abstract—Mathematical models of networked systems often take the form of a set of complex large-scale differential equations. Model reduction is a commonly used technique of producing a simplified, yet accurate, description of these systems. Most available model reduction techniques require state transformations, which can cause the structural information of the system to be lost. In this paper, a systematic methodology is proposed for reducing linear network system models without employing state transformations. The proposed method is based on minimising the Hankel error norm between the original system and the reduced order model while ensuring that the state vector in the reduced model is a subset of the original state vector, which preserves the model structure. An error bound between the original and reduced models is ensured and the steady-state behaviour of the system is also preserved. The methodology can be automated so that it be applied to large scale networks. The proposed method can be extended to uncertain systems described by linear parameter varying models. The effectiveness of the proposed methods is demonstrated through simulation examples.

I. INTRODUCTION

Mathematical modelling approaches are widely used in a range of disciplines for understanding and making predictions about system behaviour under different operating conditions. For many networked systems (e.g. biological networks and power networks), the resulting mathematical models often take the form of a large set of differential equations. Direct systems analysis and design on the full model requires considerable computation effort, which in extreme cases can become unmanageable. For this reason, model order reduction techniques are often used to obtain a lower order, simplified model, which captures the important system architecture and dynamic characteristics of the original large-scale model.

Model reduction techniques have received significant attention in different research areas [1]–[5]. Traditional approaches can be roughly classified into lumping techniques, time-scale analysis approaches, optimisation based methods, balanced truncation techniques and Krylov subspace reduction methods. Lumping techniques have been extensively used in biological/chemical reaction network systems [6]–[8], where several states are combined (linearly or non-linearly) into a ‘superstate’ and hence the dimension of the state vector can be reduced. However, this implies that the states in the reduced order model consists of combinations of the original states, which usually do not have physical

meaning; this destroys the structure of the original system and limits the practical use of the reduced model for analysis and design.

Time-scale analysis based methods and in particular, singular perturbation methods [9]–[14], have also been used extensively in biological/chemical systems. These methods take advantage of the fact that frequently, these systems contain states that evolve over different time scales; fast states evolve to quasi-steady state faster than other states. Collapsing the dynamics of these fast states retains the low-frequency behavior of the system by replacing the differential equations associated with the fast states by a set of algebraic equations. Eliminating these fast states from the remaining equations results in a reduced order model. Time-scale analysis methods are capable of preserving the physical meaning of the states, but identifying and separating fast/slow states is not always obvious and these methods are not particularly suitable for automation. Moreover, deriving an error bound between the original system and the reduced model is not straightforward.

Considerable research effort has concentrated on optimisation based model reduction techniques. These methods aim to obtain a reduced order model that minimises the difference (usually measured by system norms) between the original model and the reduced model, while preserving the main system dynamics, such as stability. Various approaches have been proposed [15]–[18], among which, optimal Hankel norm approximation is a popular approach [19], [20]. In the optimal Hankel norm approximation method, the Hankel norm is used to measure the difference between the original and reduced order models, and this quantity is minimised during the model reduction process. Optimisation based model reduction methods have the advantage that an error bound between the original and reduced order models can be guaranteed. However, the structural information of the system is lost during the optimisation process.

The balanced truncation method is one of the most successful model order reduction approaches in systems and control theory [21]–[23]. The model is firstly transformed into standard balanced realisation form using a state transformation, from which a reduced order model is obtained by applying state space truncation. The most important feature of balanced truncation is that the error between the original and the reduced order models can be characterised by a given upper bound, which is very desirable in practical applications. However, because a state transformation is required during the model reduction process and unless the transformation preserves the structure (e.g. when the state transformation matrix is diagonal), the structural information

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of the system is lost.

Krylov subspace reduction methods are based on moment matching [24]–[26]. The idea is to find a projection from a high order to a lower order space in order to obtain a reduced order model, such that the moments and/or Markov parameters of the original and reduced order models are matched up to a given frequency. The desired projection is constructed using bases of particular Krylov subspaces. The main advantage of Krylov methods is that they involve simple calculations and thus allow applications to relatively large scale models. However, the stability of the reduced order model cannot be guaranteed and characterising the error bound between the original and reduced order system is not an easy task.

Recently, methods that obtain a reduced order model when the original system possesses certain structured properties that need to be preserved [27]–[29] have been developed. In [27], a model reduction technique was proposed to reduce the system according to a particular partition of the system states. The problem was formulated into the coprime factor framework and the solution was expressed as a set of Linear Matrix Inequalities (LMIs). In [28], the authors considered the problem of model reduction with certain interconnected structure based on the Linear Fractional Transformation (LFT) framework. Two different model reduction methods were proposed and compared. These approaches provided useful results for structured model reduction, but they are based on balanced truncation, and state transformations are still involved in the model reduction process. In these methods, the transformation is required to have a special structure (consistent with the structure of the system) to preserve the structural information. The difficulty is that constructing the particular transformation is not always possible and within the substructure of the system, the state meaning is lost. It is also not easy to determine an error bound between the original and reduced models.

In recent work [30], [31], a structured model reduction technique for nonlinear autonomous network models was proposed. The proposed method is similar to singular perturbation, but does not explicitly separate the slow/fast modes, and an error bound between the original and reduced models can be ensured. Inspired by this approach, this paper proposes a systematic methodology for reducing large scale network system models with control inputs (rather than autonomous systems), which avoids the use of state transformation. The resulting state vector in the reduced order model is a subset of the original states and hence the system structural information is preserved. It also ensures that steady-state properties are preserved. The method provides an error bound between the original and reduced order models. The method can also be extended to uncertain systems described by linear parameter varying (LPV) models.

The remainder of the paper is organised as follows. Section II presents the main results of the paper. The model reduction algorithm is described and the error between the original and reduced models is characterised. A systematic structured model reduction approach is then proposed using

a greedy algorithm. Its possible extension to LPV systems to solve the uncertain model reduction problem is also discussed briefly at the end of this section. In Section III, the proposed method is firstly applied to a mass-spring-damper network, and then to a more practical system, the UK 4see model (a model for the UK economy) to demonstrate its effectiveness. Finally, concluding remarks are given in Section IV.

II. STRUCTURED MODEL REDUCTION ALGORITHM

Consider the following stable linear time invariant (LTI) system $G(s)$,

$$\dot{x} = Ax + Bu, y = Cx + Du, x(0) = x_0$$

where $y(t) \in \mathbb{R}^q$, $u(t) \in \mathbb{R}^p$, $x(t) \in \mathbb{R}^n$. The objective is to find a reduced order system $\tilde{G}(s)$

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u, \tilde{y} = \tilde{C}\tilde{x} + \tilde{D}u, \tilde{x}(0) = \tilde{x}_0$$

where $\tilde{y}(t) \in \mathbb{R}^q$, $\tilde{x}(t) \in \mathbb{R}^{\tilde{n}}$, $\tilde{n} < n$ and the set of elements of \tilde{x} is a strict subset of those in x . The aim is to choose \tilde{x} so that the difference between the original and the reduced order system models, $G(s) - \tilde{G}(s)$ is as small as possible in some norm. In this paper, the Hankel norm is used as a measure of system size, so that $\|G(s) - \tilde{G}(s)\|_H$ is minimised under the constraint that \tilde{x} is a strict subset of x , in order to preserve structural information.

The model reduction procedure we follow in this paper is based on a standard singular perturbation approach, as in [30]. Suppose the states to be collapsed have been identified and the original state vector has been reordered as

$$x = \begin{bmatrix} \tilde{x} \\ \hat{x} \end{bmatrix}$$

where \tilde{x} is the state vector to be retained and \hat{x} are the states to be collapsed to algebraic relations using singular perturbation. Assuming A_{22} is invertible, the reduced order system $\tilde{G}(s)$ can then be obtained as follows by back substituting in the expressions of the collapsed states

$$\begin{aligned} \dot{\tilde{x}} &= \underbrace{(A_{11} - A_{12}A_{22}^{-1}A_{21})}_{\tilde{A}} \tilde{x} + \underbrace{(B_1 - A_{12}A_{22}^{-1}B_2)}_{\tilde{B}} u \\ \tilde{y} &= \underbrace{(C_1 - C_2A_{22}^{-1}A_{21})}_{\tilde{C}} \tilde{x} + \underbrace{(D - C_2A_{22}^{-1}B_2)}_{\tilde{D}} u \end{aligned} \quad (1)$$

The first question to answer is how to characterise the difference between the original system $G(s)$ and the reduced order system $\tilde{G}(s)$.

To characterise this difference, the error system $G(s) - \tilde{G}(s)$ is constructed in state space (A_e, B_e, C_e, D_e) as

$$\begin{aligned} \dot{z} &= \underbrace{\begin{bmatrix} A & 0 \\ 0 & \tilde{A} \end{bmatrix}}_{A_e} z + \underbrace{\begin{bmatrix} B \\ \tilde{B} \end{bmatrix}}_{B_e} u \\ e &= \underbrace{\begin{bmatrix} C & -\tilde{C} \end{bmatrix}}_{C_e} z + \underbrace{(D - \tilde{D})}_{D_e} u \end{aligned} \quad (2)$$

with initial condition $z(0) = z_0 = \begin{bmatrix} x_0 \\ \hat{x}_0 \end{bmatrix} = \begin{bmatrix} \tilde{x}_0 \\ \hat{x}_0 \\ \tilde{x}_0 \end{bmatrix}$. We

now have the following theorem giving the approximation error bound in terms of the Hankel norm of the error system.

Theorem 1: Consider the error system (2). Suppose that there exist $\Phi \geq 0$ and $\Psi \geq 0$ such that

$$\begin{aligned} A_e \Phi + \Phi A_e^T + B_e B_e^T &= 0 \\ A_e^T \Psi + \Psi A_e + C_e^T C_e &= 0. \end{aligned}$$

Denote

$$M = \begin{bmatrix} I & 0 \\ 0 & I \\ I & 0 \end{bmatrix}$$

conforming to the structure of z_0 . Now define

$$P = (M^T \Phi^{-1} M)^{-1}, \quad Q = M^T \Psi M.$$

Then the Hankel norm of the error system is

$$\|G(s) - \tilde{G}(s)\|_H = \sqrt{\lambda_{\max}(PQ)}.$$

Proof. Note that the error system has a specific structure, i.e. the initial condition has to satisfy

$$z_0 \in \text{Struct } z_0 = \left\{ z_0 : z_0 = \begin{bmatrix} x_0 \\ \hat{x}_0 \\ \tilde{x}_0 \end{bmatrix} = \begin{bmatrix} \tilde{x}_0 \\ \hat{x}_0 \\ \tilde{x}_0 \end{bmatrix} \right\}$$

The Hankel norm of the error system is

$$\|G(s) - \tilde{G}(s)\|_H^2 = \sup_{u \in L_2(-\infty, 0], z_0 \in \text{Struct } z_0} \left(\int_0^\infty y^T y dt \int_{-\infty}^0 u^T u dt \right)$$

Suppose $u \in L_2(-\infty, 0]$ results in the current state $z(0) = z_0$, then

$$\int_0^\infty y^T y dt = z_0^T \Psi z_0$$

and

$$\min_{u \in L_2(-\infty, 0]} \int_0^\infty u^T u dt = z_0^T \Phi^{-1} z_0.$$

Hence

$$\|G(s) - \tilde{G}(s)\|_H^2 = \sup_{z_0 \in \text{Struct } z_0} \frac{z_0^T \Psi z_0}{z_0^T \Phi^{-1} z_0}.$$

Following from $x_0 = \begin{bmatrix} \tilde{x}_0 \\ \hat{x}_0 \end{bmatrix}$, we have

$$z_0 = \begin{bmatrix} \tilde{x}_0 \\ \hat{x}_0 \\ \tilde{x}_0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_0 \\ \hat{x}_0 \end{bmatrix} = M x_0.$$

The above expression of Hankel norm can then be written as

$$\begin{aligned} \|G(s) - \tilde{G}(s)\|_H^2 &= \sup_{x_0} \frac{x_0^T M^T \Psi M x_0}{x_0^T M^T \Phi^{-1} M x_0} \\ &= \lambda_{\max}((M^T \Phi^{-1} M)^{-1} M^T \Psi M), \end{aligned}$$

where the second equation results from the fact that x_0 is unstructured. Defining

$$P = (M^T \Phi^{-1} M)^{-1}, \quad Q = M^T \Psi M$$

completes the proof. \blacksquare

Remark 1: In Theorem 1, two matrices P and Q need to be computed. The computation of Q is straightforward. Suppose Ψ is partitioned conforming to z_0 as

$$\Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} \\ \Psi_{12}^T & \Psi_{22} & \Psi_{23} \\ \Psi_{13}^T & \Psi_{23}^T & \Psi_{33} \end{bmatrix}$$

then Q can be obtained

$$Q = M^T \Psi M = \begin{bmatrix} \Psi_{11} + \Psi_{13} + \Psi_{13}^T + \Psi_{33} & \Psi_{12} + \Psi_{23}^T \\ \Psi_{12}^T + \Psi_{23} & \Psi_{22} \end{bmatrix}.$$

The computation of P , however, needs more effort. Suppose that

$$\begin{bmatrix} I & 0 & -I \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^T \Phi \begin{bmatrix} I & 0 & -I \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} \tilde{\Phi}_{11} & \tilde{\Phi}_{12} & \tilde{\Phi}_{13} \\ \tilde{\Phi}_{12}^T & \tilde{\Phi}_{22} & \tilde{\Phi}_{23} \\ \tilde{\Phi}_{13}^T & \tilde{\Phi}_{23}^T & \tilde{\Phi}_{33} \end{bmatrix} \quad (3)$$

and assuming that $\tilde{\Phi}_{33}$ is invertible, P can be computed as (refer to [32] for a detailed proof)

$$P = \begin{bmatrix} \tilde{\Phi}_{11} & \tilde{\Phi}_{12} \\ \tilde{\Phi}_{12}^T & \tilde{\Phi}_{22} \end{bmatrix} - \begin{bmatrix} \tilde{\Phi}_{13} \\ \tilde{\Phi}_{23} \end{bmatrix} \tilde{\Phi}_{33}^{-1} \begin{bmatrix} \tilde{\Phi}_{13}^T & \tilde{\Phi}_{23}^T \end{bmatrix}. \quad (4)$$

Theorem 1 gives the Hankel norm of the error system between the original and \tilde{n} th order reduced models given a particular choice of state vector \hat{x} to be collapsed. To obtain a \tilde{n} th order reduced system, $n - \tilde{n}$ states will have to be removed from the original state vector. The second question about the model reduction process is which $n - \tilde{n}$ states should be collapsed such that the Hankel norm of the error system between the original and \tilde{n} th order reduced models is minimised.

One obvious answer to the above question is to try all the possible combinations of these $n - \tilde{n}$ states and choose the one producing the smallest approximation error (in terms of Hankel norm), which gives the optimal solution. However, optimising over all the possible combinations is computationally expensive and when the system order is large, the computational requirement will be considerable. An alternative solution uses a greedy algorithm, which requires much less computation and can be used to remove the states from the original system iteratively. It is noted that the greedy approach may produce suboptimal solutions, see [1], [33] for more details. The pseudo-code of the algorithm is shown in Table I and the input to the algorithm is the system model to be reduced. The output is the reduced order model and if there is no reduced model order specified, it will produce an ordered list of states to be removed with associated Hankel error norm. It will stop if no stable reduced order models could be found. The algorithm will be used in examples to demonstrate its effectiveness.

Because the proposed structured model reduction method is based on a singular perturbation approach, the reduced order model obtained has zero steady state approximation error, which is often desirable in practical applications. The proposed structured model reduction method can also be

TABLE I
GREEDY LINEAR REDUCTION ALGORITHM

Greedy Reduction Algorithm	
1	$\tilde{x} \leftarrow x, \hat{x} \leftarrow []$
2	For $j = 1 : n$
3	For $i = 1 : \text{length}(\tilde{x})$
4	$x_c[j] \leftarrow \hat{x}[i]$
5	Compute $A, \tilde{B}, \tilde{C}, \tilde{D}$
6	Construct A_e, B_e, C_e, D_e
7	Solve Φ, Ψ
8	Compute P, Q
9	$\text{errornorm}[i] \leftarrow \sqrt{\lambda_{\max}(PQ)}$
10	End for
11	$\text{minerrornorm}[j] \leftarrow \min(\text{errornorm})$
12	$x_c[j] \leftarrow$ species corresponding to $\text{minerrornorm}[j]$
13	Clear errornorm
14	Remove $x_c[j]$ from \hat{x}
15	End for

extended to uncertain systems described by linear parameter varying (LPV) models using the result from [34] and Schur complement, following a similar approach as in [31], the details of which are omitted here for brevity.

III. EXAMPLES

In this section, two examples are presented to demonstrate the effectiveness of the proposed model reduction method.

A. A Simple Mass-Spring-Damper Network

Consider the 9-mass physical system shown in Figure 1 which is similar to, but more complex than the network described in [27]. The states are chosen to be the position and the velocity of each mass. There is a force input and position output at each mass. The dynamics of each mass $M_i, i = 1, \dots, 9$ satisfy

$$\begin{aligned}
 \dot{x}_i &= v_i, \text{ for } i = 1, \dots, 9 \\
 M_i \dot{v}_i &= K_i(x_{i-1} - x_i) + K_{i+1}(x_{i+1} - x_i) \\
 &\quad + d_1(x_{i+4} - x_i) - d_0 v_i + u_i, \text{ for } i = 1, 2, 3, 4 \\
 M_i \dot{v}_i &= K_i(x_{i-1} - x_i) + K_{i+1}(x_{i+1} - x_i) \\
 &\quad - d_1(x_i - x_{i-4}) - d_0 v_i + u_i, \text{ for } i = 5, 6, 7, 8 \\
 M_i \dot{v}_i &= K_i(x_{i-1} - x_i) + K_{i+1}(x_{i+1} - x_i) \\
 &\quad - d_0 v_i + u_i, \text{ for } i = 9
 \end{aligned}$$

where $x_0 = x_{10} = 0$. Note that the full system is an 18th order dynamical system.

In this system, all the masses are taken to be of unit size $M_i = 1$, and the damping parameters are chosen to be $d_0 = 0.1$ and $d_1 = 0.5$. Spring parameters are $K = [50, 40, 3, 2, 1, 2, 3, 4, 5, 6], i = 1, \dots, 9$, respectively. Compared to the example in [27], the additional dampers between masses M_1 and M_5 , M_2 and M_6 , M_3 and M_7 , M_4 and M_8 make the mass interactions more complex. We seek to obtain a 16th order reduced system by removing two states from the system.

For this mass-spring-damper network, partitioning the system into different subsystems is not straightforward, which may cause difficulties in using the model reduction methods in [27], [28] that require a priori structural information. Our

proposed method, on the other hand, can still be used in this case to obtain a reduced order model. Applying the proposed method to obtain a 16th reduced order model suggests two states, velocity v_1 and position x_1 , should be collapsed, which is because the spring parameters K_1, K_2 are large and the additional damping connections are strong such that the movements of mass M_1 can be neglected. The approximation error is given by $\|G - G_{red}\|_{\infty} = 0.4003$. For comparative purposes, a standard balanced truncation method is applied and the approximation error is given by $\|G - G_{red}^{bal}\|_{\infty} = 0.1579$. The most appealing feature of the proposed algorithm is that the structural information of the model is preserved, so the meanings of the states in the reduced order systems are kept unchanged, i.e. positions and velocities of masses. However, as seen, this is achieved at the expense of an increase in the H_{∞} approximation error.

B. The UK 4see Economic Model

Consider the UK 4see model, which describes the relationship of energy use polices and the dynamic evolution of CO₂ levels emitted by the UK economy, developed in [35]. A linearised state space model was obtained as

$$\begin{aligned}
 \dot{x} &= Ax + Bu \\
 y &= Cx + Du
 \end{aligned} \tag{5}$$

where $x \in \mathbb{R}^{41}$ is the state vector with economic or physical meanings (e.g. *capital stock of dwellings*); $u \in \mathbb{R}^{61}$, the system inputs, are national energy use polices (e.g. *fraction of nuclear power* and *investment in energy efficiency ratio in dwelling sector*); $y \in \mathbb{R}^2$, the system outputs are the variables that are of interest, *CO₂ generated index* and *index material standard of living per capita* (an index of the economic growth); A, B, C, D are system matrices of appropriate dimensions. Detailed data are omitted here for brevity [35].

Model (5) has 41 states, corresponding to the states of 12 sectors (subsystems), namely, industry and growth, services, dwellings, standard of living, employment, resource and mining, electricity generation, transport, agriculture, water, global and sectoral coefficients, carbon dioxide. As shown in Figure 2, a graph illustrating the interactions between the states of different sectors can be obtained, by exploring the structure of the system matrix A , which demonstrates the complex feedback mechanisms in the UK economic system. Analysis on the model shows that although all the 12 subsystems are stable, the whole system (5) has an unstable pole at $s = 0.014$. Further analysis reveals that this instability is caused by the strong feedback interconnection between service and dwelling sectors, which needs careful consideration in system analysis and design.

Remark 2: It is interesting to see that the transport sector does not have a direct effect on the carbon dioxide sector. This is because in the 4see model, the information about conventional transport methods (e.g. cars) which affect the use of fuel (e.g. oil) and in turn affects the carbon dioxide emissions, are not included in the states, which is a reflec-

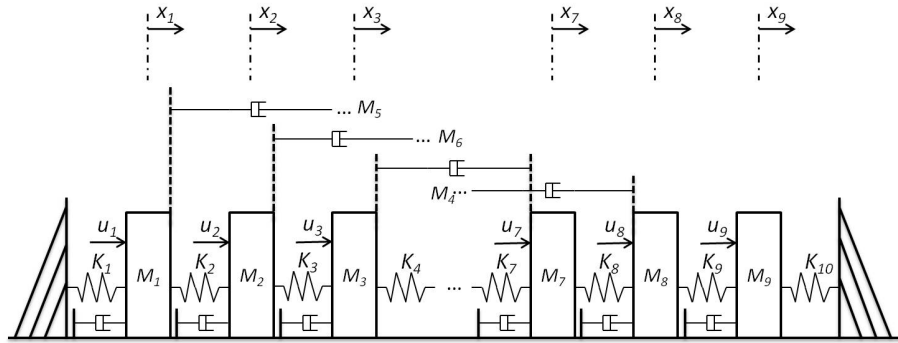


Fig. 1. Example A: a more complex mass-spring-damper network.

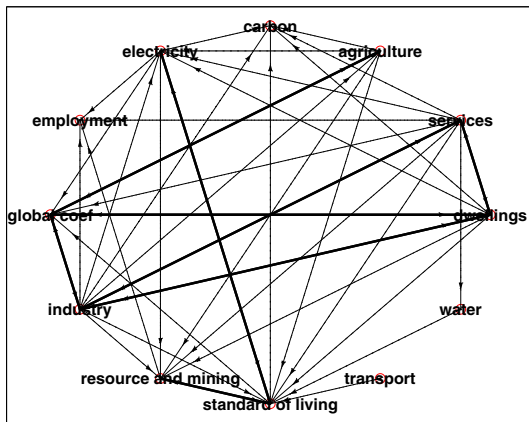


Fig. 2. Example B: UK 4see Model. This figure shows the interconnections between different sectors of the UK 4see model. Thin line arrows represent single direction effects and bold lines represent bi-directional effects. It can be seen there are strong interconnections between different subsystems in the UK economic system.

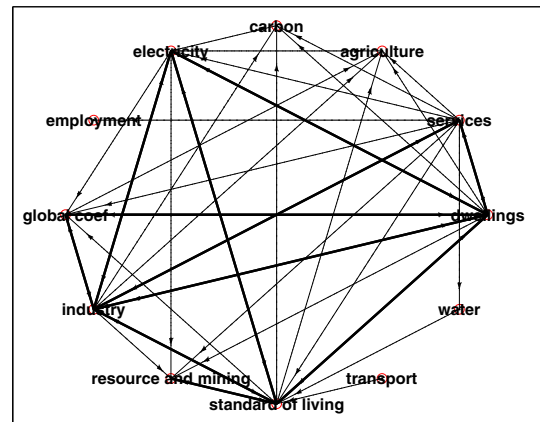


Fig. 3. Example B: UK 4see Model: A 35th order reduced model.

tion on the ECCO (Evolution of Capital Creation Options) modelling methodology that the 4see model uses.

To facilitate further system analysis and design, a reduced order model is needed, which preserves the structural information of the system and ideally, keeps the physical meaning of the states. Balanced truncation based methods will fail for this purpose as state transformation is involved and unless the transformation matrix is diagonal (which generally will not be satisfied), the states' physical meanings is lost. On the other hand, our proposed method can meet this requirement. Since the model is unstable, a shifting technique [36] is used when computing the gramians in Theorem 1. Using the method proposed in Section 2, a 35th order reduced model is firstly obtained. The topological structure of the 35th order model is shown in Figure 3. From the figure, it can be seen that using the proposed methods to obtain a 35th order reduced model, some of the interconnections are removed and some new interconnections are created. The strengths of these interconnections are also updated. It is important to note that the meanings of the states are kept unchanged,

which satisfies our requirement.

A 20th order reduced model is then obtained to reveal the 'supporting structure' of the UK 4see model, the topological graph of which is shown in Figure 4. From the figure, it can be seen that the four sectors, industry, dwellings, services and electricity, are the main sectors and among these four sectors, the first three, i.e. industry, dwellings and services are the dominant sectors. It is also seen that there are bi-directional strong feedback interactions between service and dwellings. As mentioned before, this causes the instability of the whole system. It is interesting to note that these observations are consistent with the analysis results in [35] using hierarchy structure analysis, which demonstrates another potential advantage of the proposed model reduction method.

IV. CONCLUSIONS AND FUTURE WORK

In this paper, a systematic methodology has been proposed for reducing networked system models that avoids the use of state transformations. The proposed method is based on minimising the Hankel error norm between the original system and the reduced order model while ensuring that the state vector in the reduced model is a subset of the original

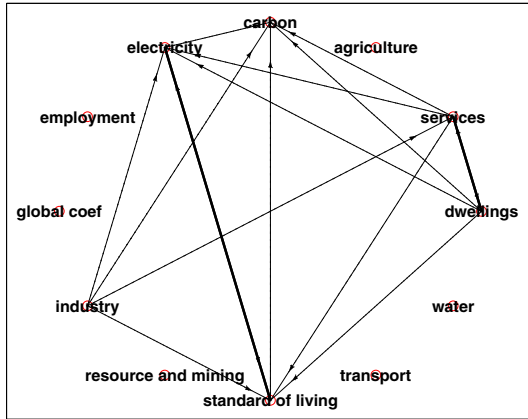


Fig. 4. Example B: UK 4see Model: A 20th order reduced model. From the figure, the ‘supporting structure’ of the UK 4see model can be identified.

states. Compared to other model reduction methods, the proposed method preserves the model structure and an error bound between the original and reduced models is ensured. The steady-state property of the system is also preserved. The methodology can be automated and can be applied to large scale networks. The proposed method can also be extended to LPV systems to solve the uncertain model reduction problem. Two examples are given to demonstrate the effectiveness of the proposed method. Future research includes extension of this approach to nonlinear systems, which will be reported separately.

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