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Distributed H_∞ Optimal Control of Networked Uncertain Nonlinear Euler-Lagrange Systems with Switching Communication Network Topologies

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Abstract— **This paper is concerned with the design of distributed state synchronization and trajectory tracking control laws for nonlinear Euler-Lagrange (EL) systems in presence of parameter uncertainty and external disturbances with fixed and switching communication network topologies. Specifically, H_∞ optimal control techniques are employed to formally design controllers which address the state synchronization and trajectory tracking of a team of multi-agent nonlinear EL systems while the agents have access to only local information. It is shown that the state synchronization (or consensus) protocol and trajectory tracking controllers can be formally derived by employing our proposed analysis. In addition, we formally show that our proposed distributed state synchronization and tracking control algorithms for EL systems is input-to-state stable (ISS) where the input is taken as parameter uncertainty as well as external disturbances. Our results are obtained for *both* fixed and switching communication network topologies. Simulation results for attitude control of a network of spacecraft demonstrate the effectiveness and capabilities of our proposed distributed control algorithms.**

I. INTRODUCTION

Multi-agent systems and networked control of robotic systems have been identified as one of the major research challenges in the field of robotics and control systems [1]. Formation control of networked multi-agent systems has several civilian applications including intelligent transportation systems, space explorations, as well as military applications in intelligence, surveillance and reconnaissance (ISR) missions in presence of vehicle failures and in battlefield environments subject to uncertainties [2], [3]. Consequently, synchronization and formation control of multi-agent systems have been studied extensively in the past few years. A large number of work in the literature consider formation control of single and double integrators or linear systems [2], [4], [5], [6]. Formation control and consensus seeking for networked nonlinear Euler-Lagrange (EL) systems have also been considered in the literature. Specifically, in [7] consensus seeking for a class of networked EL systems, namely robot manipulators is studied under a fixed, that is time-invariant communication network topology. Formation control of robot manipulators is also considered in [8]. Furthermore, in [9] state/output synchronization of networked passive systems has been studied. Formation control of EL systems under switching communication network topology has been studied in [10]. Distributed optimal synchronization and formation control of networked EL systems has been considered in [3]. In addition, in [11], [12] we also have considered synchronization control of networked EL systems in presence of actuator faults with switching in the communication network topology.

The approach in the present work, unlike most of the references in the literature which only rely on the analysis [4], [5], [7], [8], [9], is based on design of a controller by using H_∞ control techniques. Specifically, H_∞ optimal control techniques are employed to formally design a controller which addresses state synchronization and trajectory tracking of a team of multi-agent nonlinear EL systems while the agents have access to only local information.

The H_∞ control of nonlinear systems have been studied in [13], [14]. An adaptive H_∞ control approach for single robotic manipulators has appeared in [15], and in [16] the authors employ a game theoretic approach for H_∞ control design of robotic systems. The inverse-H_∞ control of EL systems have been considered in [17]. Furthermore, H_∞ control of underactuated robotic manipulators is considered in [18] based on quasi-linear parameter varying (quasi-LPV) representation as well as game theory.

In this study, we formulate the problem of synchronization and trajectory tracking control of multi-agent EL systems as an H_∞ optimal control problem in presence of parameter uncertainty and external disturbances. We show that the state synchronization (or consensus) protocol and tracking controllers can be formally derived by employing our proposed analysis in presence of parametric uncertainty and external disturbances. This implies that the proposed method is indeed a formal approach to derive the trajectory tracking control and state synchronization (or consensus) protocol for networked nonlinear EL systems. This can be considered as one of the key features of the present work when compared to other approaches that are reported in the literature. In addition, we formally show that our proposed distributed state synchronization and trajectory tracking control algorithm for EL systems is input-to-state stable (ISS) where the input is considered to be the parameter uncertainty as well as external disturbances for *both* fixed and switching communication network topologies. Simulation results for attitude control of a network of spacecraft demonstrate the effectiveness and capabilities of our proposed distributed control algorithm.

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II. Background and Preliminaries

A. Euler-Lagrange (EL) Systems

In this work, we consider $m > 1$ Euler-Lagrange (EL) systems, where the j-th *nominal* system is governed by the following *nonlinear* dynamic equation, namely,

$$\hat{\mathbf{D}}_j(\boldsymbol{q}_j)\ddot{\boldsymbol{q}}_j + \hat{\mathbf{C}}_j(\boldsymbol{q}_j,\dot{\boldsymbol{q}}_j)\dot{\boldsymbol{q}}_j + \hat{g}_j(\boldsymbol{q}_j) + \hat{\bar{\mathscr{F}}}_j = \mathrm{u}_j \quad (1)$$

where $j \in \{1,\ldots,m\}$, $\boldsymbol{q}_j = \{q_{1,j},\ldots,q_{k,j}\} \in \Re^k$ is the generalized coordinates vector. Let the *actual* j-th EL system be governed by the following *nonlinear* dynamic equation, namely,

$$\mathbf{D}_j(\boldsymbol{q}_j)\ddot{\boldsymbol{q}}_j + \mathbf{C}_j(\boldsymbol{q}_j,\dot{\boldsymbol{q}}_j)\dot{\boldsymbol{q}}_j + g_j(\boldsymbol{q}_j) + \bar{\mathscr{F}}_j = \mathrm{u}_j + d(t) \quad (2)$$

where $\mathbf{D}_j(\boldsymbol{q}_j) \in \Re^{k\times k}$ and $\hat{\mathbf{D}}_j(\boldsymbol{q}_j) \in \Re^{k\times k}$ are symmetric positive definite matrices known as the general inertia matrices for the actual and the nominal systems, respectively, $\mathbf{C}_j(\boldsymbol{q}_j,\dot{\boldsymbol{q}}_j)$ and $\hat{\mathbf{C}}_j(\boldsymbol{q}_j,\dot{\boldsymbol{q}}_j) \in \Re^{k\times k}$ are the matrices of Coriolis and centrifugal forces for the actual and the nominal systems, respectively, $g_j(\boldsymbol{q}_j)$ and $\hat{g}_j(\boldsymbol{q}_j)$, are the gravitational force vectors (GFV) for the actual and the nominal systems, respectively, and $\bar{\mathscr{F}}_j$ and $\hat{\bar{\mathscr{F}}}_j$ are the damping functions for the actual and the nominal systems, respectively. Furthermore, $d(t)$ represents the external time-varying disturbance on the actual system.

The dynamic model (1) has the following properties [19], [20], [21], [22], namely, **P1:** The general inertia matrices for both the actual and the nominal systems are bounded, specifically, $\exists \underline{k}_j, \overline{k}_j$ such that: $\underline{k}_j\ \Im_k < \mathbf{D}_j(\boldsymbol{q}_j) < \overline{k}_j\ \Im_k,\ \forall \boldsymbol{q}_j$ and $\underline{k}_j\ \Im_k < \hat{\mathbf{D}}_j(\boldsymbol{q}_j) < \overline{k}_j\ \Im_k,\ \forall \boldsymbol{q}_j$, where $\Im_k$ is an $k\times k$ identity matrix. **P2:** GFVs are assumed to be upper bounded for both the actual and the nominal systems, that is, $0 \le \sup_{\boldsymbol{q}_j\in\Re^k}\{|g_{i,j}(\boldsymbol{q}_j)|\} \le \bar{g}_{i,j},\ \forall i \in \{1,\ldots,k\}$ and $0 \le \sup_{\boldsymbol{q}_j\in\Re^k}\{|\hat{g}_{i,j}(\boldsymbol{q}_j)|\} \le \bar{g}_{i,j}$, where $\hat{g}_{i,j}(\boldsymbol{q}_j)$ denotes the elements of $\hat{g}_j(\boldsymbol{q}_j)$ and $g_{i,j}(\boldsymbol{q}_j)$ denotes the elements of $g_j(\boldsymbol{q}_j)$, and **P3:** $\dot{\mathbf{D}}_j(\boldsymbol{q}_j) - 2\mathbf{C}_j(\boldsymbol{q}_j,\dot{\boldsymbol{q}}_j)$ and $\dot{\hat{\mathbf{D}}}_j(\boldsymbol{q}_j) - 2\hat{\mathbf{C}}_j(\boldsymbol{q}_j,\dot{\boldsymbol{q}}_j)$ are skew-symmetric matrices.

We now make the following assumption explicit.

Assumption 1: The generalized coordinates vector $\boldsymbol{q}_j$ and its time derivative $\dot{\boldsymbol{q}}_j$ are available for feedback and exchange among the agents.

B. Graph Theory and Communication Topology

In this work, it is assumed that information exchanges among the m EL systems can be represented by a graph. Graph $\mathscr{G}$ consists of a node set $\mathscr{V} = \{1,\ldots,m\}$, an edge set $\mathscr{E} \subseteq \mathscr{V}\times\mathscr{V}$, and a weighted adjacency matrix $\Lambda = [\lambda_{jn}] \in \Re^{m\times m}$. The m agents in the network are considered as nodes of a graph. The communication links among the agents are considered as the graph edge set.

The weighted adjacency matrix Λ is defined such that $\lambda_{jn} = \lambda_{nj}$ is a positive weight if $(j,n) \in \mathscr{E}$, while $\lambda_{jn} = \lambda_{nj} = 0$, otherwise. Associated with Λ, we introduce a symmetric positive semi-definite matrix known as the Laplacian matrix $\mathscr{L} = [l_{jn}] \in \Re^{m\times m}$ such that $l_{jj} = \sum_{n=1,n\neq j}^{m} \lambda_{jn}$ and $l_{jn} = -\lambda_{jn}$, where $k \neq j$. Furthermore, if the graph is connected, $\mathscr{L}$ has a simple eigenvalue 0 with an associated eigenvector of $\mathbf{1}_m$, where $\mathbf{1}_m$ is an $m\times 1$ column vector of ones. All the other eigenvalues of $\mathscr{L}$ are positive if and only if the graph $\mathscr{G}$ is connected. For a given node j in the communication network the set of agents from which it can receive information is called a neighboring set $\mathscr{N}_j$, that is $\forall j = 1,\ldots,n : \mathscr{N}_j = \{n = 1,\ldots,m | (j,n) \in \mathscr{E}\}$. In addition, the number of neighbors of the j-th agent is denoted by $\left|\mathscr{N}_j\right|$. We now state our first definition.

Definition 1: We define a finite set of h communication graphs by $\bar{\mathscr{G}} = \{\mathscr{G}_1,\ldots,\mathscr{G}_h\}$ that are characterized by having the same node set, i.e. $\mathscr{V}_1 = \ldots = \mathscr{V}_h = \mathscr{V}$. Furthermore, the edge set for f ($f \le h$) communication graphs are different from the others, i.e. $\mathscr{E}_1 \neq \ldots \neq \mathscr{E}_f$, which result in a different neighboring set for the j-th agent. In addition, the $h - f \ge 0$ communication graphs with the same node set and edge set(s) have different weighted adjacency matrices. This results in a different weighted adjacency matrix for each communication graph, specifically, $\Lambda_1 \neq \ldots \neq \Lambda_h$. Consequently, the Laplacian matrix associated with each $i \in \{1,\ldots,h\}$ communication graph, denoted by $\mathscr{L}_i$, will also be different.

It should be noted that all the h communication graphs are assumed to be *connected*, therefore, $\mathscr{L}_i$ is a positive semi-definite matrix $\forall i \in \{1,\ldots,h\}$.

C. The $\mathfrak{L}_2$-Gain of General Networked Nonlinear Systems

Definition 2: [13] Consider the following nonlinear system

$$\begin{aligned} \dot{x_j} &= f_j(x_j) + g_j(x_j)u_j + \bar{g}_j(x_j)w_j \\ y_j &= h_j(x_j) \end{aligned} \quad (3)$$

where $x_j \in \Re^{\bar{n}}$, $u_j \in \Re^{\bar{m}}$, $y_j \in \Re^p$, $w_j \in \Re^l$, $g_j(x_j) \in \Re^{\bar{n}\times\bar{m}}$, and $\bar{g}_j(x_j) \in \Re^{\bar{n}\times l}$. Let $\gamma_j \ge 0$ and $w_j(t) = 0, \forall t \ge 0$. The above nonlinear system is said to have $\mathfrak{L}_2$-gain from the input $u_j(t)$ to the output $y_j(t)$ less than or equal to γ if

$$\int_0^T \left\|y_j(t)\right\|^2 dt \le \gamma^2 \int_0^T \left\|u_j(t)\right\|^2 dt$$

is satisfied for all the initial conditions $T \ge 0$ and all $y_j(t), u_j(t) \in [0,T)$.

Definition 3: Consider a network of 'm' multiple heterogeneous nonlinear systems where the dynamics of the j-th agent can be expressed by (3). The nonlinear state-feedback H_∞ control problem is to find the control $u_j = K_j(x_j) + K_{jn}(x_{jn})$, where $x_{jn} = x_j - x_n$, for the j-th nonlinear system with $K_j(0) = 0$ and $K_{jn}(0) = 0$, such that the $\mathfrak{L}_2$ gain from the disturbance $w_j(t)$ to the block vector of outputs $y_j(t)$, $y_{jn}(t)$, where $n \in \mathscr{N}_j$ and $y_{jn}(t) = y_j(t) - y_n(t)$, and the input $u_j(t)$ is less than $\gamma_j \ge 0$. In other words, there exists functions $K_j(x_j) \succeq 0$ and $K_{jn}(x_{jn}) \succeq 0$ such that,

$$\begin{aligned} &\int_0^\infty \left(k_1 \left\|y_j(t)\right\|^2 + k_2 \sum_{n\in\mathscr{N}_j} \left\|y_{jn}(t)\right\|^2 + k_3 \left\|u_j(t)\right\|^2 \right) dt \\ &\quad \le k_4 \gamma^2 \int_0^\infty \left\|w_j(t)\right\|^2 dt,\ j \in \mathscr{V}, n \in \mathscr{N}_j \end{aligned} \quad (4)$$

is satisfied for some weighting parameters $k_i > 0$, $i = \{1,\ldots,4\}$, all initial conditions and all $y_j(t), u_j(t), w_j(t) \in [0,\infty)$. The H_∞ *optimal* control problem is to find, if it exists, the smallest value $\gamma_j^\star$ of the $\mathfrak{L}_2$ gains γ_j.

D. Input-to-State Stability of General Networked Nonlinear Systems

In this subsection, we extend the standard definition (Definition 4.7 in [23]) of the input-to-state stability (ISS) of general nonlinear systems to general *networked* nonlinear systems.

Definition 4: Consider a network of 'm' multiple heterogeneous nonlinear systems where the dynamics of the j-th agent can be expressed by (3). A nonlinear state-feedback control law $u_j = K_j(x_j) + K_{jn}(x_{jn})$ for the j-th nonlinear system, with $x_{jn} = x_j - x_n,\ j \in \mathscr{V}, n \in \mathscr{N}_j,\ K_j(0) = 0$ and $K_{jn}(0) = 0$, is said to be ISS if for the closed-loop system there exists a class $\mathscr{K}\mathscr{L}$ function $\bar{\beta}_j$ and a class $\mathscr{K}$ function [1] $\bar{\gamma}_j$ such that for any initial conditions $x_j(0)$ and $x_{jn}(0)$, where $n \in \mathscr{N}_j$, and any bounded input $w_j(t)$, the solutions $x_j(t)$ and $x_{jn}(t)$ exist for all $t \geq 0$ and satisfy,

$$\begin{aligned}\left\|x_j(t)\right\| + \left\|x_{jn}(t)\right\| \leq \bar{\beta}_j\left(\left\|x_j(0)\right\| + \left\|x_{jn}(0)\right\|, t\right) \\ + \bar{\gamma}_j\left(\sup_{0\leq\xi\leq t}\left\|w_j(\xi)\right\|\right)\end{aligned} \tag{5}$$

The above inequality guarantees that for any bounded disturbance $w_j(t)$, the states $x_j(t)$ and $x_{jn}(t)$ will remain bounded. In addition, as time evolves (t increases) the states $x_j(t)$ and $x_{jn}(t)$ will remain ultimately bounded by a class $\mathscr{K}$ function of $\sup_{0\leq\xi\leq t}\left\|w_j(\xi)\right\|$. One can further show that if $w_j(t) \to 0$ as $t \to \infty$, then, $x_j(t) \to 0$ and $x_{jn}(t) \to 0$ as $t \to \infty$.

The ISS can be shown by using a Lyapunov-like theorem as discussed below.

Lemma 1: Consider a network of 'm' multiple heterogeneous nonlinear systems where the dynamics of the j-th agent can be expressed by (3). Suppose there exists a nonlinear state-feedback control law $u_j = K_j(x_j) + K_{jn}(x_{jn})$ for the j-th nonlinear system, with $K_j(0) = 0$ and $K_{jn}(0) = 0$, and a continuously differentiable positive definite radially unbounded Lyapunov function $\mathscr{W}$ for the networked heterogeneous nonlinear system such that for the closed-loop system we have,

$$\begin{aligned}\dot{\mathscr{W}} &\leq -\bar{\bar{\gamma}}(\left\|x_j(t)\right\| + \left\|x_{jn}(t)\right\|) + \underline{\gamma}\left\|w_j(t)\right\|, \text{ and} \\ \dot{\mathscr{W}} &\leq -\underline{\underline{\gamma}}(\left\|x_j(t)\right\| + \left\|x_{jn}(t)\right\|) \\ &\Leftrightarrow \left\|x_j(t)\right\| + \left\|x_{jn}(t)\right\| \geq \rho(\left\|w_j(t)\right\|)\end{aligned} \tag{6}$$

for all $x_j(t)$, $x_{jn}(t)$, and $w_j(t)$, where $\bar{\bar{\gamma}}$, $\underline{\gamma}$, and $\underline{\underline{\gamma}}$ are class $\mathscr{K}_\infty$ functions and ρ is a class $\mathscr{K}$ function. Then the system is ISS.

Proof: The proof is similar to the proof of Theorem 4.19 in [23] and it is omitted due to space limitations. ■

Definition 5: Any positive definite radially unbounded Lyapunov function $\mathscr{W}$ which satisfies (6) is denoted as the *ISS-Lyapunov function*.

[1] See e.g. page 144 in [23] for the definitions of class $\mathscr{K}\mathscr{L}$, $\mathscr{K}$ and $\mathscr{K}_\infty$ functions.

III. Distributed H_∞-Optimal State Synchronization and Trajectory Control of a Class of Nonlinear Systems

Our first result is concerned with a network of general nonlinear systems whose dynamic equations can be written in the form (3).

Lemma 2: Consider a network of 'm' multiple heterogeneous nonlinear systems (3). Let $\gamma_j > 0$. Suppose there exist smooth functions $\mathscr{Y}_j(x_j,t)$ belonging to class $\mathscr{K}\mathscr{L}$ with $\mathscr{Y}_j(0,t) = 0$, $\forall j \in \mathscr{V}$ such that the following Hamilton-Jacobi-Isaacs (HJI) partial differential inequality is satisfied

$$\begin{aligned}&\frac{\partial \mathscr{Y}_j(x_j,t)}{\partial t} + \frac{\partial \mathscr{Y}_j(x_j,t)}{\partial x_j} f_j(x_j) x_j \\ &\quad + \frac{1}{2}\frac{\partial \mathscr{Y}_j^T(x_j,t)}{\partial x_j}\left[\frac{1}{\gamma_j^2}\bar{g}_j^T(x_j)\bar{g}_j(x_j) \right. \\ &\quad \left. - \frac{1}{2} g_j^T(x_j)\mathbf{R}_j^{-1} g_j(x_j)\right]\frac{\partial \mathscr{Y}_j(x_j,t)}{\partial x_j} \\ &\quad + \frac{1}{2}x_j^T\left(\mathbf{Q}_j + \sum_{n\in\mathscr{N}_j}\mathbf{Q}_{jn}\right)x_j \leq 0\end{aligned} \tag{7}$$

Now consider the following distributed control law for the j-th system

$$u_j \triangleq \underbrace{-\frac{1}{2}\mathbf{R}_j^{-1}\bar{\tau}_j^\star}_{\bar{\tau}_j} + \underbrace{\frac{1}{2}\sum_{n\in\mathscr{N}_j}\Upsilon_j\mathbf{Q}_{jn}x_n}_{\sum_{n\in\mathscr{N}_j}\mathbf{F}_{jn}} \tag{8}$$

where $\bar{\tau}_j^\star = g_j^T(x_j)\frac{\partial \mathscr{Y}_j^T(x_j,t)}{\partial x_j}$ and Υ_j is chosen such that $\frac{\partial \mathscr{Y}_j(x_j,t)}{\partial x_j} g_j(x_j)\Upsilon_j = \Im_{\bar{n}}$. In addition, $\mathbf{F}_{jn}$ represents interaction among the agents and $\bar{\tau}_j$ represents the dependence of the control input of the agent j on its local information. Then by choosing the distributed control law (8) for the j-th nonlinear system it is guaranteed that the expression (9) is satisfied for $t \geq 0$

$$\begin{aligned}\sum_{j=1}^m\int_0^\infty\left[\frac{1}{2}x_j^T\mathbf{Q}_jx_j + \bar{\tau}_j^T\mathbf{R}_j\bar{\tau}_j + \frac{1}{4}\sum_{n\in\mathscr{N}_j}x_{jn}^T\mathbf{Q}_{jn}x_{jn}\right]dt \\ \leq \frac{1}{2}\sum_{j=1}^m\gamma_j^2\int_0^\infty w_j^Tw_jdt\end{aligned} \tag{9}$$

where $\mathbf{Q}_j \succeq 0$, $\mathbf{R}_j \succ 0$ and $\mathbf{Q}_{jn} \succeq 0$ are diagonal matrices. We further assume that $\mathbf{Q}_{jn}$ is chosen such that $\sum_{n\in\mathscr{N}_j}\mathbf{Q}_{jn} = \sum_{j\in\mathscr{N}_n}\mathbf{Q}_{nj},\ j,n \in \mathscr{V},\ j \neq n$.

Proof: Let us consider $\mathscr{Y}_j(x_j,t)$ as the value function for the j-th nonlinear system (3). Consequently, from (3), (7) and (8) we have

$$\begin{aligned}\frac{d}{dt}\mathscr{Y}_j =& \frac{\partial \mathscr{Y}_j}{\partial x_j}f_j(x_j)x_j + \frac{\partial \mathscr{Y}_j}{\partial x_j}g_j(x_j)u_j + \frac{\partial \mathscr{Y}_j}{\partial x_j}\bar{g}_j(x_j)w_j \\ \leq& -\frac{1}{2}x_j^T\mathbf{Q}_jx_j - \frac{1}{2}x_j^T\sum_{n\in\mathscr{N}_j}\mathbf{Q}_{jn}x_{jn} \\ &+ \frac{1}{2}\gamma_j^2\left\|w_j\right\|^2 - \bar{\tau}_j^T\mathbf{R}_j\bar{\tau}_j\end{aligned}$$

therefore,

$$\frac{d}{dt}\mathscr{Y} \leq \sum_{j=1}^{m}\left[-\frac{1}{2}x_j^T\mathbf{Q}_jx_j - \frac{1}{4}\sum_{n\in\mathscr{N}_j} x_{jn}^T\mathbf{Q}_{jn}x_{jn} + \frac{1}{2}\gamma_j^2\left\|w_j\right\|^2 - \bar{\tau}_j^T\mathbf{R}_j\bar{\tau}_j\right] \tag{10}$$

where $\mathscr{Y} \triangleq \sum_{j=1}^{m}\mathscr{Y}_j$. By integrating we obtain,

$$\sum_{j=1}^{m}\int_0^{\infty}\left[\frac{1}{2}x_j^T\mathbf{Q}_jx_j + \bar{\tau}_j^T\mathbf{R}_j\bar{\tau}_j + \frac{1}{4}\sum_{n\in\mathscr{N}_j} x_{jn}^T\mathbf{Q}_{jn}x_{jn}\right]dt \leq \frac{1}{2}\sum_{j=1}^{m}\gamma_j^2\int_0^{\infty}\left\|w_j\right\|^2dt + \mathscr{Y}(0) - \mathscr{Y}(\infty) \tag{11}$$

Since $\mathscr{Y}$ is a non-increasing function of time, the expression (9) follows. This completes the proof of the lemma. ■

Remark 1: By satisfying the inequality (9) for a given γ_j it is guaranteed that the control law $\bar{\tau}_j$ is an H_∞-optimal controller (refer to Definition 3). Also note that by satisfying the above inequality one may guarantee that all the agents in a neighboring set would synchronize their states and have the trajectory tracking error vector converges to a neighborhood of the origin in the steady state.

Remark 2: When γ_j is very large, i.e. $\gamma_j \to \infty$, the distributed control law (8) is transformed into an H_2 optimal distributed controller that is studied in [3]. Consequently, no disturbance attenuation is expected for large values of γ_j.

IV. Distributed H_∞-Optimal State Synchronization and Trajectory Tracking Control of Uncertain Euler-Lagrange Systems

Denote the desired position, velocity and acceleration coordinates vector of all the EL systems in the network by $\boldsymbol{q}^\star(t)$, $\dot{\boldsymbol{q}}^\star(t)$, and $\ddot{\boldsymbol{q}}^\star(t)$, respectively, where they are smooth functions of time. Let the biased desired position for the j-th EL system be denoted as $\mathfrak{e}_j(t) = \boldsymbol{q}^\star(t) + \boldsymbol{q}_j^\flat$, where $\boldsymbol{q}_j^\flat$ is added to guarantee the EL systems do not collide at the steady state. Also let $\tilde{\boldsymbol{q}}_j(t) = \boldsymbol{q}_j(t) - \mathfrak{e}_j(t)$, and $\boldsymbol{q}_{jn} = \tilde{\boldsymbol{q}}_j - \tilde{\boldsymbol{q}}_n$. Our goal, in this section is to introduce a distributed control law which guarantees synchronization and trajectory tracking of the EL system coordinates, i.e. $\boldsymbol{q}_{jn} \to 0$, $\dot{\boldsymbol{q}}_{jn} \to 0$, $\tilde{\boldsymbol{q}}_j \to 0$, $\dot{\tilde{\boldsymbol{q}}}_j \to 0$ as $t \to \infty$.

Based on the information we have on the j-th nominal EL system (1), we employ the following modified computed-torque control input, i.e.,

$$\mathfrak{u}_j = \hat{\mathbf{D}}_j(\boldsymbol{q}_j)\dot{\boldsymbol{r}}_j + \hat{\mathbf{C}}_j(\boldsymbol{q}_j,\dot{\boldsymbol{q}}_j)\boldsymbol{r}_j + \hat{g}_j(\boldsymbol{q}_j) + \hat{\mathscr{F}}_j + \tau_j \tag{12}$$

where τ_j is an auxiliary control input vector and $\boldsymbol{r}_j = \dot{\boldsymbol{q}}^\star - \bar{\mathbf{K}}_j\,\tilde{\boldsymbol{q}}_j - \bar{\bar{\mathbf{K}}}_j\int_0^t\tilde{\boldsymbol{q}}(\xi)_jd\xi$, where $\bar{\mathbf{K}}_j$ and $\bar{\bar{\mathbf{K}}}_j$ are positive definite diagonal matrices. Note that the modified computed-torque control only requires measurements from the generalized coordinates vector $\boldsymbol{q}_j$ and its time derivative $\dot{\boldsymbol{q}}_j$ (refer to Assumption 1). The dynamics of the *actual* EL system (2) is reduced to

$$\mathbf{D}_j(\boldsymbol{q}_j)(\ddot{\tilde{\boldsymbol{q}}}_j + \bar{\mathbf{K}}_j\dot{\tilde{\boldsymbol{q}}}_j + \bar{\bar{\mathbf{K}}}_j\tilde{\boldsymbol{q}}) + \mathbf{C}_j(\boldsymbol{q}_j,\dot{\boldsymbol{q}}_j)(\dot{\tilde{\boldsymbol{q}}}_j + \bar{\mathbf{K}}_j\tilde{\boldsymbol{q}}_j + \bar{\bar{\mathbf{K}}}_j\int_0^t\tilde{\boldsymbol{q}}(\xi)d\xi) = \tau_j + w_j(t) \tag{13}$$

where $w_j(t)$ is now considered as a new auxiliary disturbance that is applied to the system and is defined according to

$$w_j = \tilde{\mathbf{D}}_j(\boldsymbol{q}_j)(\ddot{\tilde{\boldsymbol{q}}}_j + \bar{\mathbf{K}}_j\dot{\tilde{\boldsymbol{q}}}_j + \bar{\bar{\mathbf{K}}}_j\tilde{\boldsymbol{q}}) + \tilde{\mathbf{C}}_j(\boldsymbol{q}_j,\dot{\boldsymbol{q}}_j)(\dot{\tilde{\boldsymbol{q}}}_j + \bar{\mathbf{K}}_j\tilde{\boldsymbol{q}}_j + \bar{\bar{\mathbf{K}}}_j\int_0^t\tilde{\boldsymbol{q}}(\xi)d\xi) + \tilde{g}_j(\boldsymbol{q}_j) + \tilde{\mathscr{F}}_j + d(t) \tag{14}$$

where $\tilde{\mathbf{D}}_j = \mathbf{D}_j - \hat{\mathbf{D}}_j$, $\tilde{\mathbf{C}}_j = \mathbf{C}_j - \hat{\mathbf{C}}_j$, $\tilde{g}_j = g_j - \hat{g}_j$, and $\tilde{\mathscr{F}}_j = \mathscr{F}_j - \hat{\mathscr{F}}_j$.

We make the following assumption about the uncertainties in the system which will be used subsequently.

Assumption 2: Define the operator $H_1(\boldsymbol{s}_j,\dot{\boldsymbol{s}}_j) = \tilde{\mathbf{D}}_j(\boldsymbol{q}_j)\dot{\boldsymbol{s}}_j + \tilde{\mathbf{C}}_j(\boldsymbol{q}_j,\dot{\boldsymbol{q}}_j)\boldsymbol{s}_j + \tilde{g}_j(\boldsymbol{q}_j) + \tilde{\mathscr{F}}_j$, where $\boldsymbol{s}_j = \dot{\tilde{\boldsymbol{q}}}_j + \bar{\mathbf{K}}_j\tilde{\boldsymbol{q}}_j + \bar{\bar{\mathbf{K}}}_j\int_0^t\tilde{\boldsymbol{q}}(\xi)d\xi$. From (14) when $d(t) = 0$ one has $w_j = H_1(\boldsymbol{s}_j,\dot{\boldsymbol{s}}_j)$. We assume the operator $H_1(\boldsymbol{s}_j,\dot{\boldsymbol{s}}_j)$ is finite-gain $\mathfrak{L}_2$ stable with the gain $\check{\gamma}_j$ for all $j \in \mathscr{V}$.

The dynamics of system (13) can now be written in the following state-space form,

$$\dot{x}_j = \mathbf{A}_j(x_j)x_j + \mathbf{B}_j(x_j)\tau_j + \mathbf{B}_j(x_j)w_j \tag{15}$$

where $x_j = [\int_0^t\tilde{\boldsymbol{q}}_j^Td\xi, \tilde{\boldsymbol{q}}_j^T, \dot{\tilde{\boldsymbol{q}}}_j^T]^T \in \Re^{3k}$, and

$$\mathbf{A}_j(x_j) = \begin{bmatrix} 0 & \mathfrak{I}_k & 0 \\ 0 & 0 & \mathfrak{I}_k \\ -\mathbf{D}_j^{-1}\mathbf{C}_j\bar{\bar{\mathbf{K}}}_j & -\mathbf{D}_j^{-1}\mathbf{C}_j\bar{\mathbf{K}}_j - \bar{\bar{\mathbf{K}}}_j & -\mathbf{D}_j^{-1}\mathbf{C}_j - \bar{\mathbf{K}}_j \end{bmatrix}$$

$$\mathbf{B}_j(x_j) = \begin{bmatrix} 0 \\ 0 \\ \mathbf{D}_j^{-1} \end{bmatrix}$$

The auxiliary control input vector τ_j can be decomposed as:

$$\tau_j = \bar{\tau}_j + \sum_{n\in\mathscr{N}_j}\mathbf{F}_{jn}x_n \tag{16}$$

Our objective is to design $\bar{\tau}_j$ such that the inequality (9) is always satisfied for the networked EL systems.

Lemma 3: Consider a network of 'm' multiple heterogeneous EL systems with the dynamics governed by (15). Let us select the following value function for the j-th system

$$\mathscr{Y}_j(x_j) = \frac{1}{2}x_j^T\mathbf{P}_j(x_j)x_j \tag{17}$$

with,

$$\mathbf{P}_j(x_j) = \begin{bmatrix} \bar{\bar{\mathbf{K}}}_j\mathbf{D}_j\bar{\bar{\mathbf{K}}}_j + \bar{\bar{\mathbf{K}}}_j\bar{\mathbf{K}}_j\mathbf{K}_j & \bar{\bar{\mathbf{K}}}_j\mathbf{D}_j\bar{\mathbf{K}}_j + \bar{\bar{\mathbf{K}}}_j\mathbf{K}_j & \bar{\bar{\mathbf{K}}}_j\mathbf{D}_j \\ \bar{\mathbf{K}}_j\mathbf{D}_j\bar{\bar{\mathbf{K}}}_j + \bar{\bar{\mathbf{K}}}_j\mathbf{K}_j & \bar{\mathbf{K}}_j\mathbf{D}_j\bar{\mathbf{K}}_j + \bar{\mathbf{K}}_j\mathbf{K}_j & \bar{\mathbf{K}}_j\mathbf{D}_j \\ \mathbf{D}_j\bar{\bar{\mathbf{K}}}_j & \mathbf{D}_j\bar{\mathbf{K}}_j & \mathbf{D}_j \end{bmatrix} \tag{18}$$

where $\mathbf{P}_j(x_j) = \mathbf{P}_j^T(x_j)$, $\mathbf{K}_j \in \Re^{k\times k}$ is a positive definite symmetric matrix and $0 \prec \bar{\mathbf{K}}_j^2 - 2\bar{\bar{\mathbf{K}}}_j$. This selection of the matrices guarantees positive definiteness of the matrix $\mathbf{P}_j(x_j)$ and the HJI inequality (7) is satisfied if the following Riccati equation is satisfied, namely,

$$\dot{\mathbf{P}}_j(x_j) + \mathbf{P}_j(x_j)\mathbf{A}_j(x_j) + \mathbf{A}_j^T(x_j)\mathbf{P}_j(x_j) + \sum_{n\in\mathscr{N}_j}\mathbf{Q}_{jn} - \mathbf{P}_j(x_j)\mathbf{B}_j(x_j)\mathbf{R}_j^{-1}\mathbf{B}_j^T(x_j)\mathbf{P}_j(x_j) + \mathbf{Q}_j + \frac{1}{\gamma_j^2}\mathbf{P}_j(x_j)\mathbf{B}_j(x_j)\mathbf{B}_j^T(x_j)\mathbf{P}_j(x_j) = 0 \tag{19}$$

.

Proof: It follows that $\frac{\partial \mathscr{Y}_j}{\partial t} + \frac{\partial \mathscr{Y}_j}{\partial x_j}\mathbf{A}_j(x_j)x_j = \frac{1}{2}\Big(\dot{\mathbf{P}}_j(x_j) + \mathbf{P}_j(x_j)\mathbf{A}_j(x_j) + \mathbf{A}_j^T(x_j)\mathbf{P}_j(x_j)\Big)$. One can also show that $\frac{\partial \mathscr{Y}_j}{\partial x_j}\mathbf{B}_j(x_j) = x_j^T\mathbf{P}_j(x_j)\mathbf{B}_j(x_j)$ [24]. Consequently, the HJI equation (7) can be written as in (19). This completes the proof of the lemma. ■

It is not generally straight-forward to solve the Riccati equation (19) for an arbitrary selection of the weighting matrices $\mathbf{Q}_j$, $\mathbf{Q}_{jn}$ and $\mathbf{R}_j$. In our next result, inspired from [17], we provide a guideline for selecting the weighting matrices in order to guarantee existence of a solution for the Riccati equation (19).

Lemma 4: For a given $\gamma_j > 0$ let us choose $\mathbf{K}_j$ such that $\mathbf{K}_j - \frac{1}{\gamma_j^2}\mathfrak{I}_3 \succ 0$. Let the weighting matrix $\mathbf{R}_j$ be selected as follows, namely,

$$\mathbf{R}_j = \left(\mathbf{K}_j - \frac{1}{\gamma_j^2}\mathfrak{I}_3\right)^{-1} \tag{20}$$

and the weighting matrices $\mathbf{Q}_j$ and $\mathbf{Q}_{jn}$ be selected as follows, namely,

$$\mathbf{Q}_j + \sum_{n\in\mathscr{N}_j}\mathbf{Q}_{jn} = \begin{bmatrix} \bar{\bar{\mathbf{K}}}_j^2\mathbf{K}_j & 0 & 0 \\ 0 & \mathbf{K}_j(\bar{\mathbf{K}}_j^2 - 2\bar{\bar{\mathbf{K}}}_j) & 0 \\ 0 & 0 & \mathbf{K}_j \end{bmatrix} \tag{21}$$

Then the Riccati equation (19) is satisfied by taking into account (18).

Proof: By noting (20) one can simplify (19) as follows,

$$\begin{aligned} \dot{\mathbf{P}}_j + &\mathbf{P}_j\mathbf{A}_j + \mathbf{A}_j^T\mathbf{P}_j + \mathbf{Q}_j + \sum_{n\in\mathscr{N}_j}\mathbf{Q}_{jn} \\ &- \mathbf{P}_j\mathbf{B}_j\mathbf{K}_j\mathbf{B}_j^T\mathbf{P}_j = 0 \end{aligned} \tag{22}$$

It follows from the property **P3** that

$$\dot{\mathbf{P}}_j + \mathbf{P}_j\mathbf{A}_j + \mathbf{A}_j^T\mathbf{P}_j = \begin{bmatrix} 0 & \bar{\bar{\mathbf{K}}}_j\bar{\mathbf{K}}_j\mathbf{K}_j & \bar{\bar{\mathbf{K}}}_j\mathbf{K}_j \\ \bar{\bar{\mathbf{K}}}_j\bar{\mathbf{K}}_j\mathbf{K}_j & 2\bar{\bar{\mathbf{K}}}_j\mathbf{K}_j & \bar{\bar{\mathbf{K}}}_j\mathbf{K}_j \\ \bar{\bar{\mathbf{K}}}_j\mathbf{K}_j & \bar{\mathbf{K}}_j\mathbf{K}_j & 0 \end{bmatrix} \tag{23}$$

In addition, one can show,

$$\mathbf{P}_j\mathbf{B}_j\mathbf{K}_j\mathbf{B}_j^T\mathbf{P}_j = \begin{bmatrix} \bar{\bar{\mathbf{K}}}_j^2\mathbf{K}_j & \bar{\bar{\mathbf{K}}}_j\bar{\mathbf{K}}_j\mathbf{K}_j & \bar{\bar{\mathbf{K}}}_j\mathbf{K}_j \\ \bar{\bar{\mathbf{K}}}_j\bar{\mathbf{K}}_j\mathbf{K}_j & \bar{\mathbf{K}}_j^2\mathbf{K}_j & \bar{\mathbf{K}}_j\mathbf{K}_j \\ \bar{\bar{\mathbf{K}}}_j\mathbf{K}_j & \bar{\mathbf{K}}_j\mathbf{K}_j & \mathbf{K}_j \end{bmatrix} \tag{24}$$

Consequently, by adding (23) and (24) and in view of (22) one obtains (21). This completes the proof of the lemma. ■

Let us now define $0 < \alpha_j < 1$ such that $(1-\alpha_j)\bar{\mathbf{K}}_j^2 - 2\bar{\bar{\mathbf{K}}}_j \succ 0$. Consequently, one obtains,

$$\sum_{n\in\mathscr{N}_j}\mathbf{Q}_{jn} = \alpha_j\begin{bmatrix} \bar{\bar{\mathbf{K}}}_j^2\mathbf{K}_j & 0 & 0 \\ 0 & \bar{\mathbf{K}}_j^2\mathbf{K}_j & 0 \\ 0 & 0 & \mathbf{K}_j \end{bmatrix} \tag{25}$$

Therefore, one gets

$$\begin{aligned} \mathbf{Q}_j = &(1-\alpha_j)\begin{bmatrix} \bar{\bar{\mathbf{K}}}_j^2\mathbf{K}_j & 0 & 0 \\ 0 & \bar{\mathbf{K}}_j^2\mathbf{K}_j & 0 \\ 0 & 0 & \mathbf{K}_j \end{bmatrix} \\ &- \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2\bar{\bar{\mathbf{K}}}_j\mathbf{K}_j & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{26}$$

The parameter α_j plays an important weighting rule. Specifically, smaller values of α_j put more weight on the trajectory tracking control law over the state synchronization control law. On the other hand, by selecting higher values for α_j one can put more emphasis on the state synchronization of the agents and less emphasis on the trajectory tracking.

We also assume that $\mathbf{Q}_{jn} = \mathbf{Q}_{jk}$ where $n \neq k$ and $n,k \in \mathscr{N}_j$. It can be shown that in view of (8) and (20) and with the parameterizations that are provided above one obtains the following control law for the j-th EL system:

$$\begin{aligned} \tau_j \triangleq &-\frac{1}{2}\left(\mathbf{K}_j - \frac{1}{\gamma_j^2}\mathfrak{I}_3\right)(\dot{\tilde{\boldsymbol{q}}}_j + \bar{\mathbf{K}}_j\tilde{\boldsymbol{q}}_j + \bar{\bar{\mathbf{K}}}_j\int_0^t \tilde{\boldsymbol{q}}_j d\xi) \\ &+ \frac{\alpha_j}{2}\mathbf{K}_j\sum_{n\in\mathscr{N}_j}\frac{1}{|\mathscr{N}_j|}(\dot{\tilde{\boldsymbol{q}}}_n + \bar{\mathbf{K}}_j\tilde{\boldsymbol{q}}_n + \bar{\bar{\mathbf{K}}}_j\int_0^t \tilde{\boldsymbol{q}}_n d\xi) \end{aligned} \tag{27}$$

The complete control law is obtained from (12) and (27).

A. Stability Analysis of the Networked Euler-Lagrange Systems

Out first objective in this subsection is to demonstrate the global asymptotic stability of the networked nonlinear EL system (2) under the distributed control law (12) and (27) in absence of external disturbances and parametric uncertainties. Next we show that the closed-loop networked EL system is finite-gain $\mathfrak{L}_2$ stable under Assumption 2 in presence of parametric uncertainties and without external disturbances by using the small-gain theorem [23].

Theorem 1: Consider a network of 'm' multiple heterogeneous EL systems that is governed by the dynamics (2) and subject to the distributed control law (12) and (27) for the j-th system. Suppose for a given $\gamma_j > 0$ the controller gain, $\bar{\bar{\mathbf{K}}}_j$, $\bar{\mathbf{K}}_j$, $\mathbf{K}_j$, and α_j are selected such that the following conditions are satisfied, namely,

$$\bar{\mathbf{K}}_j \succ 0,\ \bar{\bar{\mathbf{K}}}_j \succ 0,\ \mathbf{K}_j - \frac{1}{\gamma_j^2}\mathfrak{I}_3 \succ 0, \tag{28}$$

$$1 > \alpha_j > 0 \tag{29}$$

$$(1-\alpha_j)\bar{\mathbf{K}}_j^2 - 2\bar{\bar{\mathbf{K}}}_j \succ 0 \tag{30}$$

Consequently, $\mathbf{P}_j(x_j)$, $\mathbf{Q}_j$, $\mathbf{R}_j$, and $\mathbf{Q}_{jn}$ are positive definite matrices $\forall j \in \mathscr{V},\ j \neq n$. Then, in absence external disturbances and assuming that the nominal and the actual systems are exactly the same then the closed-loop system (2), (12), (15) and (27) is globally asymptotically stable and the networked EL system synchronizes its states and follows the desired trajectory, i.e. $\int_0^t \boldsymbol{q}_{jn}(\xi)d\xi \to 0$, $\boldsymbol{q}_{jn} \to 0$, and $\dot{\boldsymbol{q}}_{jn} \to 0$ as $t \to \infty$, and $\int_0^t \boldsymbol{q}_j(\xi)d\xi \to 0$, $\boldsymbol{q}_j \to 0$, and $\dot{\boldsymbol{q}}_j \to 0$ as $t \to \infty$.

Proof: Consider the following function as a positive definite, radially unbounded, Lyapunov function candidate for the networked nonlinear EL systems, namely,

$$\mathscr{W} = \frac{1}{2}\sum_{j=1}^{m} \boldsymbol{s}_j^T \mathbf{D}_j \boldsymbol{s}_j \tag{31}$$

The time derivative of the Lyapunov function candidate along the trajectories of the closed-loop system (2), (12) and (27) is given by $\dot{\mathscr{W}} = \sum_{j=1}^{m} \frac{1}{2}\boldsymbol{s}_j^T \dot{\mathbf{D}}_j \boldsymbol{s}_j + \sum_{j=1}^{m} \boldsymbol{s}_j^T \mathbf{D}_j \dot{\boldsymbol{s}}_j$. This by noting (13) can be written as:

$$\begin{aligned}\dot{\mathscr{W}} &= \sum_{j=1}^{m} \frac{1}{2}\boldsymbol{s}_j^T(\dot{\mathbf{D}}_j - 2\mathbf{C}_j)\boldsymbol{s}_j + \sum_{j=1}^{m} \boldsymbol{s}_j^T \tau_j \\ &\leq -\frac{1}{2}\sum_{j=1}^{m} \boldsymbol{s}_j^T \left(\mathbf{K}_j - \frac{\alpha_j}{\gamma_j^2}\mathfrak{I}_3\right)\boldsymbol{s}_j - \frac{\alpha_j}{4}\boldsymbol{s}_{jn}^T \mathbf{K}_j \left[\sum_{n\in\mathscr{N}_j} \frac{1}{|\mathscr{N}_j|}\boldsymbol{s}_{jn}\right]\end{aligned} \tag{32}$$

where $\boldsymbol{s}_{jn} = \boldsymbol{s}_j - \boldsymbol{s}_n$. By noting (28) and (29) one can conclude that (32) is a negative definite function. Since the Lyapunov function $\mathscr{W}$ is radially unbounded, all the signals remain globally bounded. By invoking Lyapunov stability theory [23] one can conclude that the closed-loop system is globally asymptotically stable in absence of the external disturbances, i.e. $\boldsymbol{s}_j \to 0$ and $\boldsymbol{s}_{jn} \to 0$ as $t \to \infty$. Furthermore, by invoking Lemma A.12 in [19], one can conclude that $\int_0^t \boldsymbol{q}_{jn}(\xi)d\xi \to 0$, $\boldsymbol{q}_{jn} \to 0$, and $\dot{\boldsymbol{q}}_{jn} \to 0$ as $t \to \infty$ and $\int_0^t \boldsymbol{q}_j(\xi)d\xi \to 0$, $\boldsymbol{q}_j \to 0$, and $\dot{\boldsymbol{q}}_j \to 0$ as $t \to \infty$. This completes the proof of the theorem. ■

The next theorem considers stability of the networked nonlinear EL systems in presence of parametric uncertainties.

Theorem 2: Consider a network of 'm' multiple heterogeneous EL system that is governed by the dynamics (15) and subject to the distributed control law (27) for the j-th system and let Assumption 2 also holds. Suppose γ_j in chosen such that $0 < \gamma_j\check{\gamma}_j < 1$ for all $j \in \mathscr{V}$. For this $\gamma_j > 0$ the controller gains $\bar{\bar{\mathbf{K}}}_j$, $\bar{\mathbf{K}}_j$, $\mathbf{K}_j$, and α_j are selected such that the conditions (28), (29), and (28) are all satisfied. This essentially implies that $\mathbf{P}_j(x_j)$, $\mathbf{Q}_j$, $\mathbf{R}_j$, and $\mathbf{Q}_{jn}$ are positive definite matrices $\forall j,n \in \mathscr{V},\ j \neq n$. Then the j-th closed-loop nonlinear EL system is finite-gain $\mathfrak{L}_2$ stable.

Proof: According to Assumption 2 we have $d_j(t) = 0$. Application of the distributed control law (27) to the j-th system guarantees that it is finite-gain $\mathfrak{L}_2$ stable with the gain $0 < \gamma_j < 1/\check{\gamma}_j$ in presence of uncertainties. From Assumption 2 it follows that the operator $H_1(\boldsymbol{s}_j, \dot{\boldsymbol{s}}_j)$ is finite-gain $\mathfrak{L}_2$ stable with the gain $\check{\gamma}_j$. Consequently, when $\gamma_j\check{\gamma}_j < 1$ for all $j \in \mathscr{V}$, by invoking Theorem 5.6 in [23] (which is known as the *small-gain theorem*), it follows that the feedback connection is finite-gain $\mathfrak{L}_2$ stable. This completes the proof of the theorem. ■

B. Input-to-State Stability of the Networked Euler-Lagrange Systems

The purpose of this subsection is to demonstrate that the networked EL systems (15) under the distributed control law (27) and in presence of the modeling uncertainty and external disturbances is ISS. We first consider a fixed communication network topology. Time-varying (switching) communication topology is considered subsequently.

Theorem 3: Consider a network of 'm' multiple heterogeneous EL systems that is governed by the dynamics (15) and subject to the distributed control law (27) for the j-th system. Suppose for a given $\gamma_j > 0$ the controller gains $\bar{\bar{\mathbf{K}}}_j$, $\bar{\mathbf{K}}_j$, $\mathbf{K}_j$, and α_j are selected such that the conditions (28), (29), and (30) are satisfied. Consequently, $\mathbf{P}_j(x_j)$, $\mathbf{Q}_j$, $\mathbf{R}_j$, and $\mathbf{Q}_{jn}$ are positive definite matrices $\forall j \in \mathscr{V},\ j \neq n$. It the follows that in presence of modeling uncertainty and external disturbances (nonzero $w_j(t)$) the closed-loop EL system is ISS stable (refer to Definition 4) and the synchronization and the tracking trajectory errors remain globally ultimately bounded.

Proof: Consider the function in (17) as a positive definite, radially unbounded, ISS-Lyapunov function candidate (as per Definition 5) for the j-th system. Let $\mathscr{Y} \triangleq \sum_{j=1}^{m} \mathscr{Y}_j$ be the ISS-Lyapunov function candidate for the networked EL systems. One can show, similar to (10), that the time derivative of the Lyapunov function candidate $\mathscr{Y}$ along the trajectories of the closed-loop system (15), (20), (25), (26) and (27) can be written as:

$$\begin{aligned}\dot{\mathscr{Y}} \leq &-\frac{1}{4}\sum_{j=1}^{m} x_j^T\left[\mathbf{P}_j(x_j)\mathbf{B}_j(x_j)\mathbf{R}_j\mathbf{B}_j^T(x_j)\mathbf{P}_j(x_j)\right]x_j \\ &-\frac{1}{2}\sum_{j=1}^{m} x_j^T\mathbf{Q}_j x_j - \frac{1}{4}\sum_{j=1}^{m}\sum_{n\in\mathscr{N}_j} x_{jn}^T\mathbf{Q}_{jn}x_{jn} \\ &+\frac{1}{2}\sum_{j=1}^{m}\gamma_j^2\left\|w_j\right\|^2\end{aligned} \tag{33}$$

Positive definite matrices $\mathbf{Q}_j$, $\mathbf{R}_j$, and $\mathbf{Q}_{jn}$ $\forall j \in \mathscr{V},\ n \in \mathscr{N}_j$ imply that the first two terms in the right hand side of the inequality (33) are $\mathscr{K}_\infty$ function of x_j and x_{jn}, respectively. Define the bounded region $\mathfrak{B}_r$ that includes the origin, that is $\mathfrak{B}_r = \Big\{x_j, (x_j - x_n) \mid \frac{1}{2}x_j^T\left[\mathbf{Q}_j + \frac{1}{2}\mathbf{P}_j(x_j)\mathbf{B}_j(x_j)\mathbf{R}_j\mathbf{B}_j^T(x_j)\mathbf{P}_j(x_j)\right]x_j + \frac{1}{4}\sum_{n\in\mathscr{N}_j} x_{jn}^T\mathbf{Q}_{jn}x_{jn} \leq \frac{1}{2}\gamma_j^2\left\|w_j\right\|^2\Big\}$. For all x_j and x_{jn} outside this region, we have $\frac{d}{dt}\mathscr{Y} < 0$. Consequently, by invoking Lemma 1 one can conclude that the closed-loop networked EL systems under the distributed control law (27) for the j-th system is ISS and the synchronization and the tracking trajectory errors remain globally ultimately bounded. ■

In our last result, we consider switching in the communication network topology.

Lemma 5: Consider a network of 'm' multiple heterogeneous EL systems that is governed by the dynamics (15) and subject to the following distributed control law for the j-th system and the ith communication network topology (refer to Definition 1), namely,

$$\begin{aligned}\tau_{j,i} \triangleq &-\frac{1}{2}\left(\mathbf{K}_j - \frac{1}{\gamma_{j,i}^2}\mathfrak{I}_3\right)(\dot{\tilde{\boldsymbol{q}}}_j + \bar{\mathbf{K}}_j\tilde{\boldsymbol{q}}_j + \bar{\bar{\mathbf{K}}}_j\int_0^t \tilde{\boldsymbol{q}}_j d\xi) \\ &+\frac{\alpha_{j,i}}{2}\mathbf{K}_j\sum_{n\in\mathscr{N}_{j,i}}\frac{1}{|\mathscr{N}_{j,i}|}(\dot{\tilde{\boldsymbol{q}}}_n + \bar{\mathbf{K}}_j\tilde{\boldsymbol{q}}_n + \bar{\bar{\mathbf{K}}}_j\int_0^t \tilde{\boldsymbol{q}}_n d\xi)\end{aligned} \tag{34}$$

Suppose for given $\gamma_{j,i} > 0$ the controller gains $\bar{\bar{\mathbf{K}}}_j$, $\bar{\mathbf{K}}_j$, $\mathbf{K}_j$, and $\alpha_{j,i}$ are selected such that the conditions (28), (29), and (30) are satisfied for all $i \in \{1,\ldots,h\}$. Consequently, $\mathbf{P}_j(x_j)$, $\mathbf{Q}_{j,i}$, $\mathbf{R}_{j,i}$, and $\mathbf{Q}_{jn,i}$ are positive definite matrices $\forall j \in \mathscr{V},\ j \neq n$ and for all $i \in \{1,\ldots,h\}$. It then follows that in presence of modeling uncertainty and external disturbances the closed-loop system is ISS stable (refer to Definition 4) and the synchronization and tracking trajectory errors remain globally ultimately bounded for *arbitrary* switching in the communication network topology.

Proof: The proof is based on the existence of a common ISS-Lyapunov function for the considered switched system. Let $\mathscr{Y} \triangleq \sum_{j=1}^{m} \mathscr{Y}_j$ be the Lyapunov function candidate for the network. It follows from Theorem 3 that the closed-loop networked EL systems is ISS for the ith communication network topology that is given in Definition 1. In addition, note that $\mathbf{P}_j(x_j)$ is the same for all the communication networks. Consequently, the function $\mathscr{Y}$ is a common ISS-Lyapunov function for all the communication network topologies. By invoking Theorem 3.1 in [25] one can conclude that the closed-loop system is ISS under *arbitrary* switching in the communication network topology. This completes the proof of the lemma. ■

V. Simulation Studies: Distributed Control of Networked Spacecraft

In this section, our proposed distributed control strategy is applied to spacecraft formation flying problem, which is an application area of significant strategic interest. The 3-degree of freedom (DOF) attitude dynamics of a spacecraft can be written in the form of (2) with $g_j(\boldsymbol{q}_j) = \frac{\partial \mathscr{F}_j(\dot{\boldsymbol{q}}_j)}{\partial \dot{\boldsymbol{q}}_j} = 0$, where we specifically have [3], [26],

$$\mathbf{D}_j(\boldsymbol{q}_j) = \bar{\mathbf{R}}_j^{-T}\mathbf{J}_j\bar{\mathbf{R}}_j^{-1} \tag{35}$$

and

$$\mathbf{C}_j(\boldsymbol{q}_j,\dot{\boldsymbol{q}}_j) = -\bar{\mathbf{R}}_j^{-T}\mathbf{S}(\mathbf{J}_j\bar{\mathbf{R}}_j^{-1}\dot{\boldsymbol{q}}_j)\bar{\mathbf{R}}_j^{-1} + \bar{\mathbf{R}}_j^{-T}\mathbf{J}_j\frac{d}{dt}\bar{\mathbf{R}}_j^{-1} \tag{36}$$

where $\boldsymbol{q}_j = [\theta_j, \phi_j, \psi_j]^T$ is the vector of the Euler angles, $\mathbf{J}_j = \mathbf{J}_j^T$ is the j-th spacecraft positive definite moment of inertia matrix, and $\bar{\mathbf{R}}_j$ is defined as:

$$\bar{\mathbf{R}}_j = \frac{1}{c_{\theta_j}}\begin{bmatrix} c_{\theta_j} & s_{\phi_j}s_{\theta_j} & c_{\phi_j}s_{\theta_j} \\ 0 & c_{\phi_j}c_{\theta_j} & -s_{\phi_j}c_{\theta_j} \\ 0 & s_{\phi_j} & c_{\phi_j} \end{bmatrix}$$

where c_{θ_j} stands for $\cos(\theta_j)$, s_{θ_j} stands for $\sin(\theta_j)$, s_{ϕ_j} stands for $\sin(\phi_j)$, and c_{ϕ_j} stands for $\cos(\phi_j)$. In addition, $\mathbf{S}(x)$ is the skew-symmetric operator, given by,

$$\mathbf{S}(x) = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

In simulations, we consider a network of 8 spacecraft. We consider three communication topologies in simulations. The communication network graphs are depicted in Fig. 1. One can observe from this figure that all the three networks are strongly connected and the connections are bi-directional.

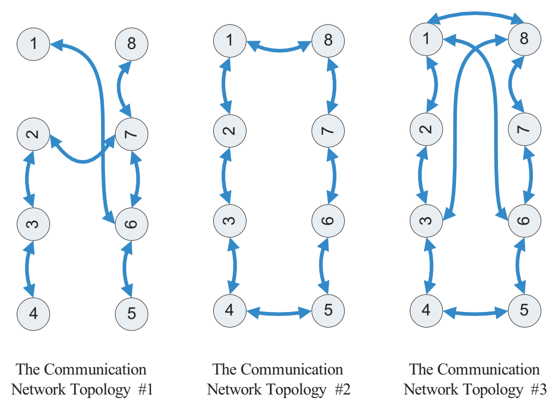

Fig. 1. The three communication network topologies considered in simulations.

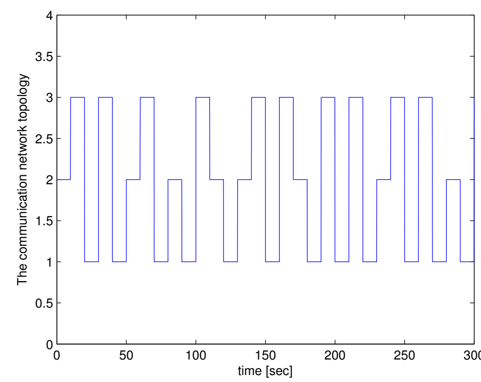

Fig. 2. The random switching among the communication network topologies.

Furthermore, we randomly switch among the three communication graphs every 10 seconds. The switchings in the communication topologies are depicted in Fig. 2.

For simulations we set $\gamma_{j,i} = 0.6$, $\alpha_{j,i} = 0.86$, $\forall j \in \{1,\ldots,8\}, i \in \{1,2,3\}$. In addition, in view of (28), (29), and (30) the distributed controller (34) gains are selected as: $\mathbf{K}_j = 20\mathfrak{I}_3$, $\bar{\mathbf{K}}_j = 0.16\mathfrak{I}_3$, and $\bar{\bar{\mathbf{K}}}_j = 0.001\mathfrak{I}_3$. This results in the following parameters for the j-th EL system, namely, $\mathbf{R}_{j,i} = 17.22\mathfrak{I}_3$, $\sum_{n\in\mathscr{N}_j}\mathbf{Q}_{jn,i} = \mathrm{diag}([0.17e-4,\ 0.17e-4,\ 0.17e-4,\ 0.44,\ 0.44,\ 0.44,\ 17.22,\ 17.22,\ 17.22])$, and $\mathbf{Q}_{j,i} = \mathrm{diag}([0.27e-5,\ 0.27e-5,\ 0.27e-5,\ 0.031e-3,\ 0.031e-3,\ 0.031e-3,\ 2.78,\ 2.78,\ 2.78])$. The above selection of the controller gains imposes more emphasis on the synchronization of the spacecraft attitudes and their attitude rates and considerably less emphasis on the state regulation. Our desired objective is to keep the spacecraft attitude states in the neighborhood of origin.

The inertia matrix of a deployed spacecraft without a propulsion system does not change during its mission. For spacecraft with a propulsion system, the fuel tanks are placed usually close to the center of mass of the spacecraft so that as the fuel is consumed, the center of mass and inertia do not change significantly. Therefore, we assume that the inertia matrix of the spacecraft in the network is known within a

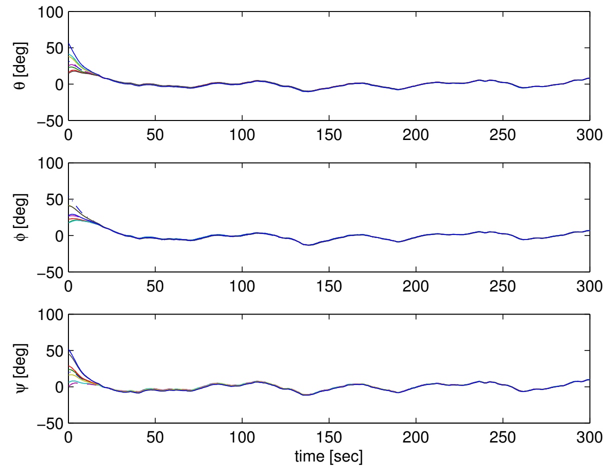

Fig. 3. Spacecraft attitudes of a network of 8 agents under the distributed control law (34) for the first 300 seconds.

$\pm 10\%$ accuracy, i.e. $\mathbf{J}_j = \hat{\mathbf{J}}_j \pm 0.10\hat{\mathbf{J}}_j$, where $\mathbf{J}_j$ is the actual spacecraft inertia matrix and $\hat{\mathbf{J}}_j$ is it's nominal value. The disturbance $d(t)$ is considered to be a Gaussian distributed noise with the mean value of zero and variance of 0.001. The initial attitudes of the spacecraft are selected randomly between zero to 60 degrees.

The attitudes of the 8 spacecraft in the network under the distributed control law (12) and (34) with the parameters selected above are shown in Fig. 3 for the first 300 seconds. This figure shows that the spacecraft synchronize their attitudes quickly despite the uncertainties and topological switchings.

VI. Conclusion

The main contribution of this paper is a formal development of distributed state synchronization and trajectory tracking control laws for nonlinear Euler-Lagrange (EL) systems by employing H_∞ control techniques. Specifically, in presence of parametric uncertainty and external disturbances, H_∞ optimal control techniques are utilized to formally design a distributed control law which addresses the state synchronization and trajectory tracking of a team of multi-agent nonlinear EL systems given that the agents have access to only local information. In addition, we formally show that our proposed distributed state synchronization and trajectory tracking control algorithm for EL systems is input-to-state stable (ISS) when the input is considered as the parameter uncertainty and external disturbances for *both* fixed and switching communication network topologies. Simulation results for attitude control of a network of eight spacecraft demonstrate the effectiveness and capabilities of our proposed distributed control algorithms.

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