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Nuclear Magnetic Resonance: the Contrast Imaging Problem

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***Abstract*— Starting as a tool for characterization of organic molecules, the use of NMR has spread to areas as diverse as pharmacology, medical diagnostics (medical resonance imaging) and structural biology. Recent advancements on the study of spin dynamics strongly suggest the efficiency of geometric control theory to analyze the optimal synthesis. This paper focuses on a new approach to the contrast imaging problem using tools from geometric optimal control. It concerns the study of an uncoupled two-spin system and the problem is to bring one spin to the origin of the Bloch ball while maximizing the modulus of the magnetization vector of the second spin. It can be stated as a Mayer-type optimal problem and the Pontryagin Maximum Principle is used to select the optimal trajectories among the extremal solutions. Correlation between the contrast problem and the optimal transfer time problem is demonstrated. Further, we develop some analysis of the singular extremals and apply the results to examples of cerebrospinal fluid/water and grey/white matter of the cerebrum.**

I. INTRODUCTION

The primary objective of this paper is to apply techniques of geometric optimal control theory to the control of the spin dynamics by magnetic fields in Nuclear Magnetic Resonance (NMR). Through interaction with a magnetic field, NMR involves the manipulation of nuclear spins. It has many potential applications extending from the determination of molecular structures (NMR spectroscopy) and quantum computing, where NMR remains one of the most promising roads in the construction of a scalable quantum computer [8], to medical imagery (MRI). As a first example the saturation problem, which consists in vanishing the magnetization vector of a given sample, has recently been studied. Saturating a spin 1/2 particle removes its contribution from the NMR spectrum and therefore increases the resolution of this spectrum. This type of control could also have applications in medicine where this non-magnetized sample can be used as a tracer for following the motion of the blood in the human brain. It was shown that a gain of 50% in the control duration with respect to existing techniques was obtained using geometric control theory [10]. This control law has also been experimentally implemented.

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In [3], [4], [13], this study has been generalized and the time optimal problem of one spin 1/2 particle in a dissipative environment is considered. Classification of the optimal synthesis with respect to the relevant experimental parameters has been obtained and it shows the preponderant role of singular extremals in the optimal solution. We propose here to extend these works to the study of the simultaneous control of two non-interacting spins with different relaxation parameters. Very recently, in [1], the authors analyzed the time optimal problem of uncoupled two-spin systems with different resonance offsets and no relaxation parameters. The problem that we propose to address here differs as follows. Rather than minimizing the time, we consider the contrast imaging problem [5], [6]. The basic goal is to bring the magnetization vector of the first spin toward the center of the Bloch ball while maximizing the modulus of the magnetization vector of the other substance. Roughly speaking, the substance with a zero magnetization will appear dark, while the other substance with a maximum modulus of the magnetization vector will be white.

We introduce in the following a simple model reproducing the main features of this control problem. We describe the general structure of the optimal solution and we compute them for two particular examples.

Each spin 1/2 particle is governed by the Bloch equation

$$\begin{aligned}\frac{dM_x}{dt} &= -\frac{M_x}{T_2} + \omega_y M_z \\ \frac{dM_y}{dt} &= -\frac{M_y}{T_2} - \omega_x M_z \\ \frac{dM_z}{dt} &= \frac{(M_0 - M_z)}{T_1} + \omega_x M_y - \omega_y M_x\end{aligned}$$

where the state variable is the magnetization vector and T_1, T_2 are the relaxation times. The control is the magnetic field $\omega = (\omega_x, \omega_y)$ which is bounded by $|\omega| \leq \omega_{max}$. We use the normalization introduced in [10]. The normalized coordinates are $q = (x, y, z) = (M_x, M_y, M_z)/M_0$. In these coordinates, the equilibrium point is the north pole $(0, 0, 1)$ and the normalized control is $u = (u_x, u_y) = \frac{2\pi}{\omega_{max}}(\omega_x, \omega_y)$, $|u| \leq 2\pi$, while the normalized time is given by $\tau = \omega_{max} t/(2\pi)$. Hence the system takes the form

$$\begin{aligned}\dot{x} &= -\Gamma x + u_y z \\ \dot{y} &= -\Gamma y - u_x z \\ \dot{z} &= \gamma(1 - z) + (u_x y - u_y x)\end{aligned} \tag{1}$$

where $\Gamma = 2\pi/(\omega_{max} T_2)$ and $\gamma = 2\pi/(\omega_{max} T_1)$. In the experiments, ω_{max} can be chosen up to 15,000 Hz but the value $2\pi \times 32.3$ Hz will be considered in this paper.

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The experiments are performed for the contrast problems of cerebrospinal fluid/water [14] and grey/white matter of cerebrum [7]. In the cerebrospinal fluid/water situation, the relaxation parameters for the first spin describing the fluid are $T_1 = 2000$ ms and $T_2 = 200$ ms, while for the second spin $T_1 = T_2 = 2500$ ms. In the second example, the rates of the grey matter are taken to be $T_1 = 920$ ms and $T_2 = 100$ ms, the rates for the white matter are $T_1 = 780$ ms and $T_2 = 90$ ms.

For the contrast problem, we consider one pair of spin systems, each of them solutions of the Bloch equation (1), with respective damping coefficients (γ_1, Γ_1), (γ_2, Γ_2) and controlled by the same control field. Denoting each system by $\frac{dq_i}{dt} = F_i(q_i, \Lambda_i, u)$, $\Lambda_i = (\gamma_i, \Gamma_i)$ and $q_i = (x_i, y_i, z_i)$ is the magnetization vector representing each spin. This leads to consider the system

$$\frac{dq_1}{dt} = F_1(q_1, \Lambda_1, u), \quad \frac{dq_2}{dt} = F_2(q_2, \Lambda_2, u) \tag{2}$$

which is written shortly as $\frac{dx}{dt} = F(x, u)$, where $x = (q_1, q_2)$. The associated optimal control problem is the following: *starting from the equilibrium point* $x_0 = ((0,0,1),(0,0,1))$ *reach in a given transfer time* T *the final state* $q_1(T) = 0$ *of the first spin (corresponding to zero magnetization) while maximizing* $|q_2(T)|^2$.

Remark 1: A subcase of this problem is to restrict the system to $x_1 = x_2 = 0$, while only the real component u_1 of the control field $u = u_1 + iu_2$ is used.

In both cases, the optimal control is given by the following criteria:

1) A system $\frac{dx}{dt} = F(x, u)$, $x \in \mathbb{R}^n$ with fixed initial state $x(0) = x_0$ where the control belongs to a control domain U.
2) A terminal manifold M defined by $f(x) = 0$, where $f : \mathbb{R}^n \to \mathbb{R}^k$.
3) A cost to be minimized of the form $\min_{u(\cdot)} c(x(T))$ where $c : \mathbb{R}^n \to \mathbb{R}$.

We shall consider the two cases: the full control case of $\frac{dx}{dt} = F_0(x) + u_1 F_1(x) + u_2 F_2(x)$, where $x = (q_1, q_2) \in \mathbb{R}^6 \bigcap \{|q_1| \leq 1,\ |q_2| \leq 1\}$ and $|u| \leq M$ and the subcase where the control field is restricted to the real field, i.e. $u_2 = 0$, while $x \in \mathbb{R}^4 \bigcap \{|q_1| \leq 1,\ |q_2| \leq 1\}$.

II. GEOMETRIC OPTIMAL CONTROL

Consider the system $\frac{dx}{dt} = F(x, u)$, $x(0) = x_0$, $x \in \mathbb{R}^n$, with the terminal manifold M defined by $f(x) = 0$, and the problem $\min_u c(x(T))$. Fixing the level set to $c(x) = m$, this with the terminal condition $f(x(T)) = 0$, leads us to introduce a family of manifolds denoted M_m. We denote $A(x_0, T) = \bigcup_{u \in U} x(T, x_0, u)$ the accessibility set union of terminal points of trajectories emanating at $t = 0$ from x_0 for each admissible control $u(\cdot) \in L^\infty[0, T] \bigcap U$ such that the trajectory $x(\cdot, x_0, u)$ is defined on the whole interval. Clearly, according to the maximum principle, an optimal control u_* is such that the corresponding terminal point $x_*(T)$ belongs to the boundary of the accessibility set $A(x_0, T)$ and corresponds to a terminal manifold M_m such that m is minimum, see Fig. 1.

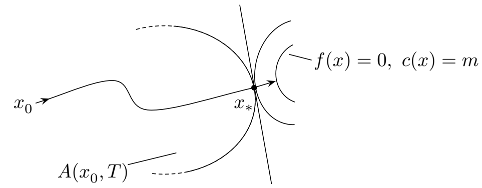

Fig. 1. Geometric interpretation of the contrast problem.

This can be restated as the following proposition [11].

Proposition 1 (Maximum Principle): Define the pseudo-Hamilonian $H(x, p, u) = \langle p, F(x, u) \rangle$. An optimal control has to satisfy the following necessary optimality conditions

1) $\frac{dx}{dt} = \frac{\partial H}{\partial p}(x, p, u)$, $\frac{dp}{dt} = -\frac{\partial H}{\partial x}(x, p, u)$
2) $H(x, p, u) = \max_{v \in U} H(x, p, v)$
3) $f(x(T)) = 0$
4) $p(T) = p_0 \frac{\partial c}{\partial x}(x(T) + \langle \sigma, \frac{\partial f}{\partial x}(x(T)) \rangle$, $\sigma \in \mathbb{R}^n$, $p_0 \leq 0$.

The final condition corresponds to a transversality condition with the standard orientation of the adjoint vector.

Definition 1: The solutions of the first two conditions of the maximum principle are called extremals, and if they satisfy the boundary conditions they are called BC-extremals.

For the contrast problem, we have that the boundary condition is $q_1(T) = 0$ and the cost is $|q_2(T)|^2$. Splitting the adjoint vector into $p = (p_1, p_2)$, we deduce the transversality condition $p_2(T) = -2p_0 q_2(T)$, $p_0 \leq 0$. The case $p_0 = 0$ gives $p_2(T) = 0$. Since the system splits into

$$\dot{q}_1 = F_1(q_1, \Lambda_1, u),\ \dot{q}_2 = F_2(q_2, \Lambda_2, u),$$

the adjoint system decomposes into

$$\dot{p}_1 = -p_1 \frac{\partial F_1}{\partial q_1},\ \dot{p}_2 = -p_2 \frac{\partial F_2}{\partial q_2}$$

where $p = (p_1, p_2)$ is written as a row vector. The condition $p_2(T) = 0$ corresponds to a second spin which is not controlled. In the non-trivial case, p_0 is nonzero and it can be normalized to $p_0 = -1/2$.

From the necessary conditions of the Maximum Principle [11], the two following propositions can be easily shown.

Proposition 2: The time minimizing solutions of the first spin 1/2 particle can be embedded as extremals of the contrast problem, with $p_0 = 0$.

Proposition 3: In the contrast problem, the extremals of the real input case are embedded in the extremal flow of the complex input case.

As a consequence of the next proposition, the optimal solution to the contrast problem for the cerebrospinal fluid/water situation, when the transfer time is not fixed, is to bring the magnetization vector of the fluid to the origin in minimum time.

Proposition 4: In the real case assume that $\gamma_2 = \Gamma_2 < 1/2$ and the transfer time is not fixed. Then the solution to the contrast problem is given by the time minimizing solutions of the first spin particle.

Proof: The cost to be maximized is defined as $c(T) = q_2^T(T)q_2(T)$ where T is the duration of the trajectory. Let us introduce the function $c(t) = q_2^T(t)q_2(t) = y_2^2(t)+z_2^2(t)$ and differentiate it along the trajectory solution of our system. We obtain:

$$\begin{aligned}\dot{c}(t) &= 2q_2^T(t)q_2(t)\\ &= 2(-\Gamma_2 y_2^2 - u_1 y_2 z_2 + z_2\Gamma_2 - \Gamma_2 z_2^2 + u_1 y_2 z_2)\\ &= -2\Gamma_2 c(t) + 2\Gamma_2 z_2 \leq -2\Gamma_2 c(t) + 2\Gamma_2\end{aligned}$$

using $\Gamma_2 = \gamma_2$ and $|z(t)| \leq 1$.

Solving the system

$$\begin{cases}\dot{g}(t) = -2\Gamma_2 g(t) + 2\Gamma_2\\ g(0) = 1\end{cases}$$

gives $g \equiv 1$. In conclusion, since $\dot{c}(t) \leq \dot{g}(t)$ and $c(0) = g(0) = 1$ we have obtained that $c(t)$ is bounded above by a constant function, hence to maximize $c(T)$ we need to minimize the duration of the trajectory T. ■

III. PRELIMINARY RESULTS IN THE CONTRAST PROBLEM

From Prop. 3 we see that the real input case plays an important role in the general synthesis. Thus we can restrict the analysis to the situation where the control field has only one component and the contrast problem is governed by the differential system $\dot{x} = F_0(x) + u_1F_1(x)$, $x = (y_1, z_1, y_2, z_2)$:

$$F_0 = \sum_{i=1}^{2}\left[-\Gamma_i y_i \frac{\partial}{\partial y_i} + \gamma_i(1-z_i)\frac{\partial}{\partial z_i}\right]$$

$$F_1 = \sum_{i=1}^{2}(-z_i\frac{\partial}{\partial y_i} + y_i\frac{\partial}{\partial z_i}).$$

A. Collinear Set

The collinear set is contained in the intersection of the two collinear sets associated to each spin. More precisely, the parameters of the spin i being Γ_i and γ_i, each set is given by the equation

$$-\Gamma_i y_i^2 + \gamma_i z_i(1-z_i) = 0,$$

which is the intersection of the two parabolas $y_i = \pm\sqrt{\frac{\gamma_i}{\Gamma_i}(1-z_i)z_i}$, where $0 \leq z_i \leq 1$. This set contains the origin and the north pole. Moreover, the compatibility relation $y_1\Gamma_1 z_2 = y_2\Gamma_2 z_1$ must be satisfied, and the collinear set is a curve.

B. Singular set and singular flow

Denoting $\delta_i = \gamma_i - \Gamma_i$, $i = 1, 2$ one has:

$$[F_1, F_0] = \sum_{i=1}^{2}(-\gamma_i + \delta_i z_i)\frac{\partial}{\partial y_i} + \delta_i y_i\frac{\partial}{\partial z_i}$$

$$[[F_1, F_0], F_0] = \sum_{i=1}^{2}\left[\gamma_i(\gamma_i - 2\Gamma_i) - \delta_i^2 z_i\right]\frac{\partial}{\partial y_i} + \delta_i^2 y_i\frac{\partial}{\partial z_i}$$

$$[[F_1, F_0], F_1] = \sum_{i=1}^{2} 2\delta_i y_i\frac{\partial}{\partial y_i} + (\gamma_i - 2\delta_i z_i)\frac{\partial}{\partial z_i}$$

and the corresponding singular flow is defined by

$$\begin{aligned}0 = H_1 &= \{H_1, H_0\}\\ &= \{\{H_1, H_0\}, H_0\} + u_{1s}\{\{H_1, H_0\}, H_1\}\end{aligned}$$

where $H_i = \langle p, F_i(q)\rangle$ are the Hamiltonian lifts and the singular control is computed by deriving $\frac{\partial H}{\partial u_1} = H_1 = 0$ along an extremal solution of the vector field with Hamiltonian H using the computation rule $\frac{dG}{dt} = \{G, H\}$ for any function G.

Since the equations are linear with respect to p, for each initial condition q_0, this defines a two-dimensional surface $S(q_0)$ in the state space. An additional condition is provided by the generalized Legendre-Clebsch condition: $\{\{H_1, H_0\}, H_1\} \geq 0$. The structure of this surface is related to the relaxation parameters (Γ_i, γ_i).

If the transfer time is not fixed, this leads to the additional constraint $H_0 = 0$. In this case, the singular flow defines a single vector field in the state space, since the adjoint vector can be eliminated and the restricted singular control is given by

$$u_{1s} = -\frac{D'(x)}{D(x)}$$

where

$$\begin{aligned}D'(x) &= \det(F_0, F_1, [F_1, F_0], [[F_1, F_0], F_0])\\ D(x) &= \det(F_0, F_1, [F_1, F_0], [[F_1, F_0], F_1])\end{aligned}$$

with the corresponding vector field

$$\frac{dx}{dt} = F_0(x) - \frac{D'(x)}{D(x)}F_1(x)$$

which can be analyzed using the time reparameterization $d\tau = dt/D(q(\tau))$. In this framework, singular trajectories are used to classify the systems.

In the general case, a similar computation shows that the singular trajectories are solutions of an equation of the form

$$\frac{dx}{dt} = F_0(x) - \frac{D'(x,\lambda)}{D(x,\lambda)}F_1(x) \qquad (3)$$

where λ is a one-dimensional time dependent parameter whose dynamics are deduced from the adjoint equation. The solutions of (3) emanating from q_0 will form $S(q_0)$.

C. Numerical simulations

Here, we present some numerical simulations concerning the singular trajectories. The projection of $S(q_0)$ on the planes (y_1, z_1), (y_2, z_2) shows the effect of the relaxation parameters on the contrast. This point is illustrated by Figs. 2 and 3 for the cerebrospinal fluid/water and grey/white matter of cerebrum cases, respectively. In each example, we assume that a bang pulse of large amplitude has been first applied to the system, the initial point of the singular flow has coordinates $((-\sqrt{1-z_0^2}, z_0), (-\sqrt{1-z_0^2}, z_0))$ where $z = z_0$ is the horizontal singular line of the first spin. This first bang is necessary so that the singular trajectory of the first spin can reach the center of the Bloch ball. One clearly sees in Fig. 3 the similar structure of the different singular trajectories of the two spins. The situation is completely different in Fig. 2 for the first example. This explains the excellent and weak contrasts that can be reached in the first and second examples with an optimal sequence of the form bang-singular. Note that some singular control fields diverge as displayed in Figs. 2 and 3. The conjugate points [2] have been computed for each singular extremal as shown in Fig. 4. Similar results have been obtained for the second spin and for the grey/white matter case. This shows that the structure bang-singular is not optimal since the first conjugate point occurs before the saturation of the spin. A more complicated pulse sequence such as bang-singular-bang-singular has therefore to be used.

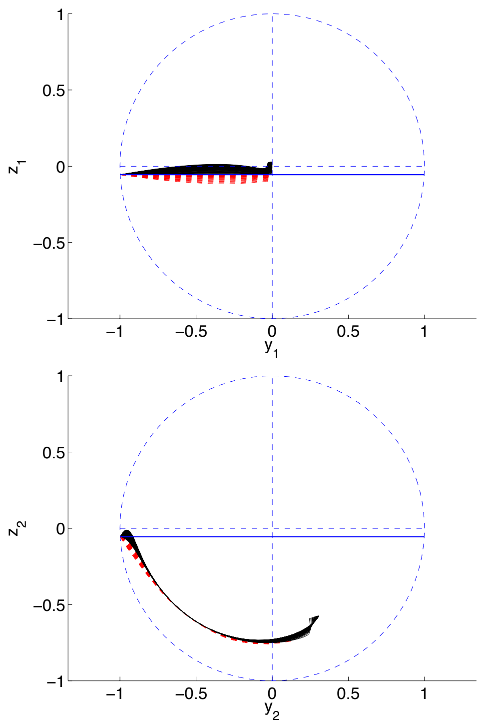

Fig. 2. Structure of the projection of the singular flow onto the planes (y_1, z_1) and (y_2, z_2) in the cerebrospinal fluid/water case. The trajectories are plotted in black (solid line) and in red (dashed line). The control fields of the dashed extremals diverge. The trajectories have been plotted up to the explosion of the field (the absolute value of the field is larger than 10^5). The horizontal solid line is a singular line of the first spin.

Due to the numerical difficulty of the computation of the bang-singular-bang-singular optimal sequence, we have used a regularized cost by adding an L^2 (or an $L^{2-\lambda}$) penalty on the control. To produce the numerical simulations, we have used a differential continuation method of the Hampath code [9]. Such results can be compared with the GRAPE algorithm [12] which is a standard approach in NMR to solve the optimization equations.

We illustrate our numerical results on a simulated contrast experiment. We consider two surfaces as displayed in Fig. 5 filled in with spins 1 or 2 in a homogeneous manner. We apply the optimal control field and we associate a color to the final modulus of the magnetization vector of the second spin. This color is white if the modulus is equal to 1, black if it is zero and a grey variant between. One clearly sees in Fig. 5 the excellent and weak contrasts that can be obtained in the first and second examples.

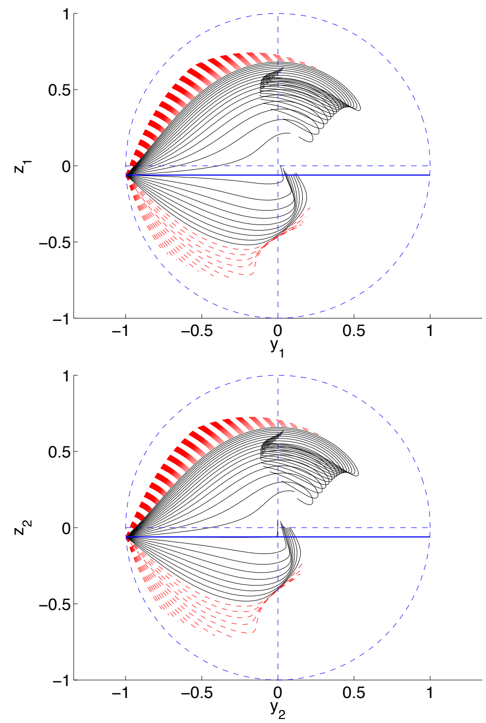

Fig. 3. Same as Fig. 2 but for the grey/white matter case. An explosion of the control field is observed for the red (dashed) trajectories.

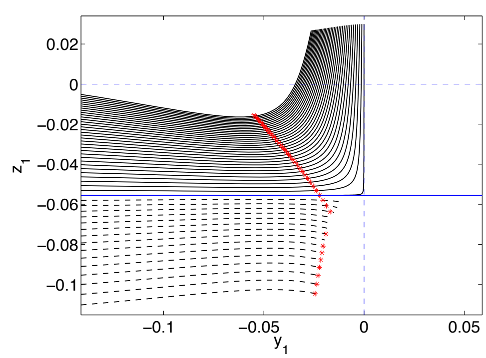

Fig. 4. Zoom of the results of Fig. 2 for the first spin near the origin. The red crosses indicate the position of the first conjugate point. The dashed lines represent the singular trajectories for which the control field diverges.

In addition, numerical continuation methods are used to further analyze the contrast problem. The time-minimal control for the first spin (without regard for the second spin) is used as an initial point, and the problem is continuously deformed to allow an increased time duration. This is illustrated in Figs. 6 and 7.

IV. ACKNOWLEDGMENTS

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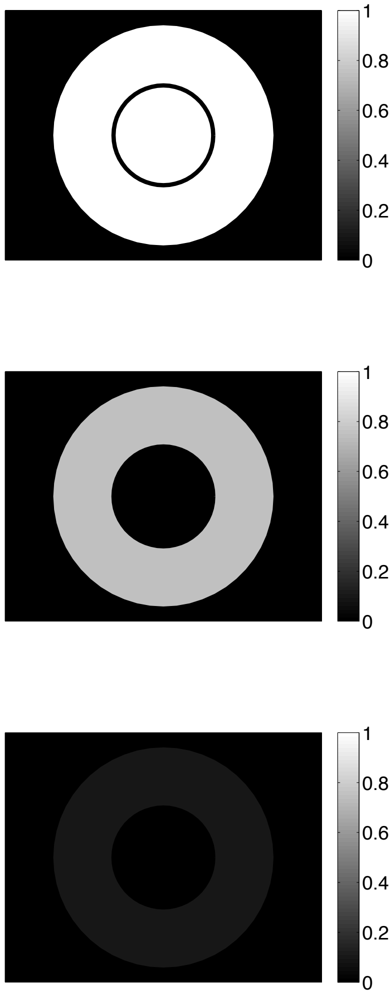
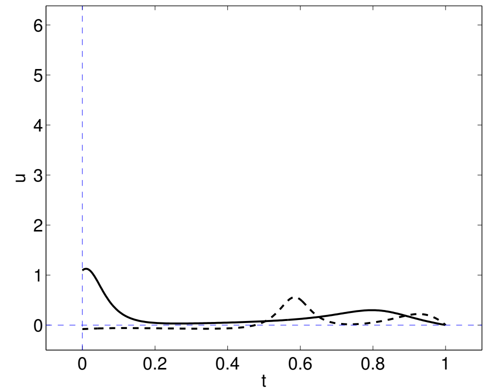
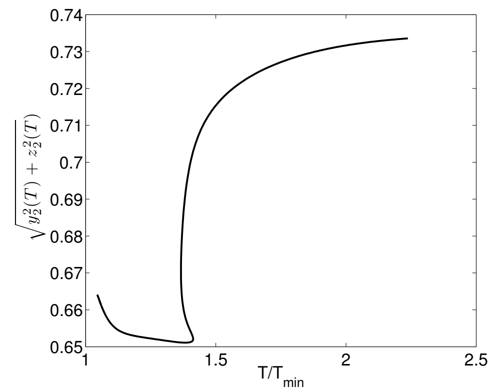


Fig. 5. Numerical experimental results on the contrast problems of the cerebrospinal fluid/water (middle) and the grey/white matter of cerebrum (bottom) examples. The inner disk represents the first spin, while the outside ring represents the second spin. The two surfaces are separated by a thin black circle. The top figure is a reference image (for both cases) before the application of the control field when the two spins are at the north pole of the Bloch sphere. The middle and bottom images are a representation of the final contrast as could be achieved in a real experiment. A color has been associated to each value of the contrast between 0 (black) and 1 (white).



(a) Evolution of the control field for $T_{\min}+\varepsilon$ and $2T_{\min}$ in solid and dashed lines, respectively. The time T has been normalized to 1 to plot the two control fields on the same figure.



(b) Evolution of the contrast parameter $\sqrt{y_2(T)^2+z_2(T)^2}$ as a function of the control duration, where $T_{\min}$ is the minimum time to drive the first spin to zero.

Fig. 6. Control field and resulting contrast for in the cerebrospinal fluid/water case.

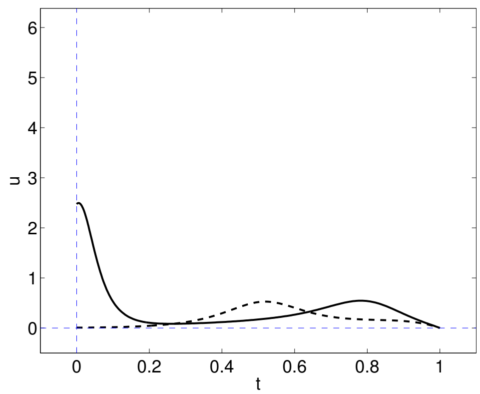
REFERENCES

[1] E. Assémat, M. Lapert, Y. Zhang, M. Braun, S. J. Glaser and D. Sugny, *Simultaneous time-optimal control of the inversion of two spin 1/2 particles*, Phys. Rev. A, **82**, 013415 (2010).

[2] B. Bonnard and M. Chyba, *Singular trajectories and their role in control theory*, Math. and Applications 40, Springer-Verlag, Berlin (2003)

[3] B. Bonnard, M. Chyba and D. Sugny, *Time-minimal control of dissipative two-level quantum systems: The generic case*, IEEE Transactions A. C., **54**, 2598 (2009).

[4] B. Bonnard and D. Sugny, *Time-minimal control of dissipative two-*



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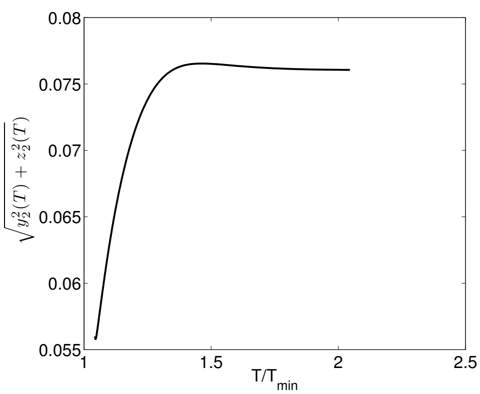

(b) Evolution of the contrast parameter $\sqrt{y_2(T)^2+z_2(T)^2}$ as a function of the control duration.

Fig. 7. Control field and resulting contrast for in the grey/white matter of cerebrum case.

level quantum systems: The integrable case, SIAM J. Control Optim., **48**, 1289 (2009).

[5] G. M. Bydder, J. V. Hajnal and I. R. Young, *MRI: Use of the inversion recovery pulse sequence*, Clinical Radiology, **53**, 159 (1998)

[6] M. Carl, M. Bydder, J. Du, A. Takahashi and E. Han, *Optimization of RF excitation to maximize signal and T_2 contrast of tissues with rapid transverse relaxation*, Magnetic Resonance in Medicine, **64**, 481 (2010)

[7] K. V. R. Chary and G. Govil, *NMR in biological systems, from molecules to human*, Focus on structural biology, vol. 6, Springer (2008)

[8] L. M. K. Vandersypen and I. L. Chuang, *NMR techniques for quantum control and computation*, Rev. Mod. Phys. 76, 1037 (2005).

[9] http://apo.enseeiht.fr/hampath

[10] M. Lapert, Y. Zhang, M. Braun, S. J. Glaser and D. Sugny, *Singular extremals for the time-optimal control of dissipative spin 1/2 particles*, Phys. Rev. Lett., **104**, 083001 (2010)

[11] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko, "The mathematical theory of optimal processes," Translated from the Russian by K. N. Trirogoff; edited by L. W. Neustadt. Interscience Publishers John Wiley & Sons, Inc. New York-London, 1962.

[12] T. E. Skinner, T. O. Reiss, B. Luy, N. Khaneja and S. J. Glaser, *Reducing the duration of broadband excitation pulses using optimal control with limited RF amplitude*, J. Magn. Reson., **167**, 68 (2004); T. E. Skinner, T. O. Reiss, B. Luy, N. Khaneja and S. J. Glaser, *Tailoring the optimal control cost function to a desired output: Application to minimizing phase errors in short broadband excitation pulse*, J. Magn. Reson., **172**, 17 (2005); N. Khaneja, T. Reiss, C. Kehlet, T. Schulte-Herbrüggen and S. J. Glaser, *Optimal control of coupled spin dynamics: design of NMR pulse sequences by gradient ascent algorithms*, J. of Magnetic Reson., **172**, 2, 296 (2005)

[13] D. Sugny, C. Kontz and H. R. Jauslin, *Time-optimal control of a two-level dissipative quantum system*, Phys. Rev. A, **76**, 023419 (2007).

[14] C. Westbrook and C. Roth, *MRI in practice* (3rd Edition), Blackwell Publishing Ltd. (2005)