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Lyapunov functions for $\mathcal{L}_2$ and input-to-state stability in a class of quantized control systems

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***Abstract*— $\mathcal{L}_2$ and input-to-state stability (ISS) properties of a class of linear quantized control systems are considered. The quantized control system differs slightly from the ones considered in the literature previously. A recently proposed hybrid modeling framework and corresponding Lyapunov analysis tools are used to calculate the finite gains of the closed loop system.**

I. INTRODUCTION

Quantized control systems (QCS) have attracted a lot of attention in the past two decades, see [1], [2], [3], [5], [7], [8], [12], [13], [14] due to certain novel applications of control in which control or measurement signals are transmitted via a communication channel with a severely restricted bandwidth. Such situations arise for instance in cases when the nature of the process restricts the communication bandwidth between the plant and the controller (e.g. automatic control of drilling in oil rigs) or in cases when for security reasons we want to control the process with a minimum amount of data transmitted between the plant and controller.

A particularly interesting question in this context is robustness of QCS to exogenous inputs, see for instance [6], [9], [10], [11]. The result in [10] is particularly interesting as it shows a fundamental limitation to robust stabilization of a class of QCS. Actually, results in [10] apply to a class of linear plants with so called finite-set feedback; a causal feedback map that takes measurements and produces a sequence of controls is termed finite-set if within each finite time interval its range is finite. Hence, bit-rate constrained QCS are a special case of systems with finite-set feedback. It was shown in [10] that finite gain l_p stabilization is impossible for linear systems with finite-set feedback; in other words, linear gains are impossible to achieve in this case. Moreover, it was shown in the same paper that a relaxed version of l_p stability is possible if one allows nonlinear gains that have infinite slope at the origin and infinity. Furthermore, it was shown that linear l_p gains necessitate logarithmic precision around the origin. Results in [10] are not constructive and they were proved by contradiction. Construction of such nonlinear gains for certain classes of QCS is studied, for instance, in more detail in [6], [9] but these constructions still rely on the existence of certain bounds on solutions of the system that were not explicitly computed; hence, the computed gains are still not explicit.

It is the purpose of this paper to explicitly construct $\mathcal{L}_2$ and input-to-state (ISS) finite gains for a class of QCS with exogenous disturbances. The class of systems we consider are not finite-set feedbacks as defined in [10] and we show that finite gains are possible in this case (see Remark 4). We consider a class of linear QCS with zooming protocols similar to [1], [7], [9] with a subtle difference. Indeed, in our case the zooming variable is restricted to be lower bounded by a strictly positive number ε; consequently, only practical stability of the closed loop systems can be proved but the ultimate bound for trajectories of the closed loop system can be reduced arbitrarily by reducing ε. Moreover, the model of QCS is written as a hybrid system following the framework presented in [4] and the switching strategy is subtly different from the ones found in the literature, such as [1], [7], [9].

Our main results provide explicit formulas for estimates of the finite ISS and $\mathcal{L}_2$ gains of a class of QCS systems. More importantly, our proofs are novel and they are based on the construction of appropriate Lyapunov functions for the closed loop system. Furthermore, our analysis applies in a unified manner to both zoom-in and zoom-out stages which was previously missing in the literature. The analysis uses the hybrid modeling framework and Lyapunov approach for hybrid systems presented in [4].

The paper is organized as follows. Some preliminary results are given in Section II. We introduce the model of the system and basic assumptions on the quantizer in Section III, which is followed by main results in Section IV. All proofs are provided in Section V and conclusions are given in the last section.

II. ANALYSIS PRELIMINARIES

Our approach is based on the recently proposed hybrid modeling framework in [4] and the corresponding Lyapunov techniques for hybrid systems. We summarize in this section certain results from [4] and prove several new results that are useful for our problem.

A. Lipschitz Lyapunov functions

Our analysis employs Lyapunov functions that are not smooth but are locally Lipschitz. Given a locally Lipschitz function $V : \mathcal{D} \to \mathbb{R}$ where $\mathcal{D} \subset \mathbb{R}^n$ is an open set, for each point $(x, w) \in \mathcal{D} \times \mathbb{R}^n$, $V^\circ(x; w)$ denotes the Clarke generalized derivative of V at x in the direction w. The utility of the Clarke generalized directional derivative

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is that it has a convenient calculus and, most importantly, the time derivative of $t \mapsto V(x(t))$ is upper bounded by $V^\circ(x(t);\dot{x}(t))$. If the set-valued mapping $x \mapsto F(x,w)$ is continuous, the function $x \mapsto \alpha(x,w)$ is continuous, and for almost all z in a neighborhood of x we have $\langle \nabla V(z), f\rangle \leq \alpha(z,w)$ for all $f \in F(z,w)$, then $V^\circ(x;f) \leq \alpha(x,w)$ for all $f \in F(x,w)$. If $V_3(x) = V_1(x) + V_2(x)$, then $V_3^\circ(x;w) \leq V_1^\circ(x;w) + V_2^\circ(x;w)$.

B. Hybrid signals

Given a measurable hybrid signal $\psi : \text{dom}\ \psi \to \mathbb{R}_{\geq 0}$ and $(t,j) \in \text{dom}\ \psi$, let $0 \leq t_0 \leq \cdots \leq t_{j+1}$ satisfy

$$\text{dom}\ \psi \cap ([0,t] \times \{0,\ldots,j\}) = \bigcup_{i=0}^{j} ([t_i, t_{i+1}] \times \{i\})$$

and define

$$\begin{aligned}
\left\|(\psi,t,j)\right\|_{c,1} &:= \sum_{i=0}^{j} \int_{t_i}^{t_{i+1}} \psi(s,i)ds \\
\left\|(\psi,t,j)\right\|_{d,1} &:= \sum_{i=0}^{j-1} \psi(t_{i+1},i) \\
\left\|(\psi,t,j)\right\|_{1} &:= \left\|(\psi,t,j)\right\|_{c,1} + \left\|(\psi,t,j)\right\|_{d,1} \ ,
\end{aligned}$$

as well as

$$\begin{aligned}
\left\|(\psi,t,j)\right\|_{c,\infty} &= \max_{i\in\{0,\ldots,j\}} \operatorname*{ess\,sup}_{s\in[t_i,t_{i+1}]} \psi(s,i) \\
\left\|(\psi,t,j)\right\|_{d,\infty} &= \max_{i\in\{0,\ldots,j-1\}} \psi(t_{i+1},i) \\
\left\|(\psi,t,j)\right\|_{\infty} &= \max\left\{\left\|(\psi,t,j)\right\|_{c,\infty}, \left\|(\psi,t,j)\right\|_{d,\infty}\right\} \ .
\end{aligned}$$

In addition, when dom ψ is unbounded, define

$$\begin{aligned}
\left\|\psi\right\|_{c,1} &:= \lim_{t+j\to\infty} \left\|(\psi,t,j)\right\|_{c,1} \\
\left\|\psi\right\|_{\infty} &:= \lim_{t+j\to\infty} \left\|(\psi,t,j)\right\|_{\infty} \ .
\end{aligned}$$

Moreover, again assuming dom ψ is unbounded, for each $(s,i) \in \text{dom}\ \psi$ define $\psi_{s,i} : \text{dom}\ \psi_{s,i} \to \mathbb{R}_{\geq 0}$ as $\psi_{s,i}(t,j) := \psi(s+t, i+j)$ for each (t,j) such that $t+j \geq 0$ and $(s+t, i+j) \in \text{dom}\ \psi$, and then define

$$\left\|\psi\right\|_a := \limsup_{s+i\to\infty} \left\|\psi_{s,i}\right\|_{\infty} \ .$$

We suppose we have

$$\dot{v}(t,j) \leq -\sigma v(t,j) + \gamma w(t,j) \ , \tag{1}$$

$$v(t,j+1) \leq v(t,j) \ , \tag{2}$$

where v and w are non-negative valued functions, $\gamma \geq 0$, and $\sigma > 0$ and where (1) holds for almost all t such that (t,j) is in the domain of the solution, and (2) holds for all (t,j) in the domain such that $(t,j+1)$ is also in the domain. $v(\cdot,j)$ is assumed to be locally absolutely continuous in t for each fixed j.

Lemma 1: For each (t,j) in the domain of the solution:

$$\left\|(v,t,j)\right\|_{\infty} \leq \max\left\{v(0,0), \frac{\gamma}{\sigma}\left\|(w,t,j)\right\|_{c,\infty}\right\} \tag{3a}$$

$$\left\|(v,t,j)\right\|_{c,1} \leq \frac{1}{\sigma}v(0,0) + \frac{\gamma}{\sigma}\left\|(w,t,j)\right\|_{c,1} \tag{3b}$$

$$\left\|(v,t,j)\right\|_{\infty} \leq v(0,0) + \gamma\left\|(w,t,j)\right\|_{c,1} \ . \tag{3c}$$

Proof. First we establish (3a). For the sake of later use, we replace (2) with $v(t,j+1) \leq \rho v(t,j)$ with $\rho \in (0,1]$. Fix (t,j) in the domain of the solution. Define $\overline{\omega} := \left\|(w,t,j)\right\|_{c,\infty}$. Then $\dot{v}(s,i) \leq -\sigma v(s,i) + \gamma\overline{\omega}$ and $v(s,i+1) \leq \rho v(s,i)$. Define $y(s,i) := v(s,i) - \frac{\gamma}{\sigma}\overline{\omega}$. Then

$$\begin{aligned}
\dot{y}(s,i) &= \dot{v}(s,i) \ \leq \ -\sigma v(s,i) + \gamma\overline{\omega} \\
&\leq -\sigma v(s,i) + \sigma\tfrac{\gamma}{\sigma}\overline{\omega} \ = \ -\sigma y(s,i)
\end{aligned}$$

and

$$\begin{aligned}
y(s,i+1) &= v(s,i+1) - \tfrac{\gamma}{\sigma}\overline{\omega} \ \leq \ \rho v(s,i) - \tfrac{\gamma}{\sigma}\overline{\omega} \\
&= \rho y(s,i) + \tfrac{(\rho-1)\gamma}{\sigma}\overline{\omega} \ \leq \ \rho y(s,i) \ .
\end{aligned}$$

It then follows that $y(s,i) \leq \exp(-\sigma s)\rho^i y(0,0)$ and hence

$$\begin{aligned}
v(s,i) &\leq \exp(-\sigma s)\rho^i v(0,0) + \frac{\gamma}{\sigma}\overline{\omega}\left(1 - \exp(-\sigma s)\rho^i\right) \\
&\leq \max\left\{v(0,0), \tfrac{\gamma}{\sigma}\overline{\omega}\right\} \ .
\end{aligned}$$

This establishes (3a). To establish (3b), we integrate and sum (1) and (2) to get

$$\sigma\left\|(v,t,j)\right\|_{c,1} \leq v(0,0) + \gamma\left\|(w,t,j)\right\|_{c,1}$$

which is (3b). For (3c), we take $\sigma = 0$ in (3b) and then integrate and sum (1) and (2). This gives (3c). ∎

The next lemma supposes that, in addition to (1) and (2), there exist $\rho \in (0,1)$ and $\overline{\beta} > \frac{\gamma}{\sigma}\left\|w\right\|_{c,\infty}$ such that, if both $(t,j+1)$ and (t,j) belong to the domain of the solution and $v(t,j) \in [\frac{\gamma}{\sigma}\left\|w\right\|_{c,a}, \overline{\beta}]$ then

$$v(t,j+1) \leq \rho v(t,j) \ . \tag{4}$$

Lemma 2: If $v(0,0) \leq \overline{\beta}$ then

$$\left\|v\right\|_a \leq \frac{\gamma}{\sigma}\left\|w\right\|_{c,a} \ . \tag{5}$$

Proof. The lemma follows a calculation similar to that for the proof of (3a) in the Lemma 1. In this case, we take $\overline{\omega} := ||w_{s,i}||_{\infty}$ and let $s+i \to \infty$. ∎

III. SYSTEM DESCRIPTION

We study a type of $\mathcal{L}_2$ stability and input-to-state stability for the linear control system

$$\dot{\zeta} = A\zeta + Bu + Ed \tag{6}$$

where $\zeta \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $d \in \mathbb{R}^k$, and an ideal stabilizing linear state feedback $u = K\zeta$ is implemented through a dynamic quantizer

$$u = K\mu q\left(\frac{\zeta}{\mu}\right) \tag{7}$$

where $\mu > 0$ is adjusted discretely online and $q : \mathbb{R}^n \to \mathbb{R}^n$ is a quantization function. We impose the following assumption on the matrix K:

Assumption 1: The matrix $A + BK$ is Hurwitz and $P = P^T > 0$, $\gamma > 0$, $\kappa > 0$ and $\nu > 0$ satisfy

$$\begin{bmatrix} (A+BK)^T P + P(A+BK) + \gamma^{-2} P & PBK & PE \\ (PBK)^T & -\nu^2 I & 0 \\ (PE)^T & 0 & -\kappa^2 I \end{bmatrix} < 0 \, . \tag{8}$$

Remark 1: If $(A+BK)^T P + P(A+BK) < 0$ then, for sufficiently large $\gamma > 0$ and $\kappa > 0$, we have

$$\begin{bmatrix} (A+BK)^T P + P(A+BK) + \gamma^{-2} P & PE \\ (PE)^T & -\kappa^2 I \end{bmatrix} < 0 \, . \tag{9}$$

In turn, if (9) is satisfied then there exists $\nu > 0$ sufficiently large so that (8) is satisfied. If (9) is satisfied then, with $V_0(\zeta) = \zeta^T P \zeta$, we have

$$\langle \nabla V_0(\zeta), (A+BK)\zeta + Ed \rangle \le -\gamma^{-2} V_0(\zeta) + \kappa^2 |d|^2 \tag{10}$$

from which we can derive the following stability properties:

$$\sup_{t \ge 0} V_0(\zeta(t)) \le \max\left\{ V_0(\zeta(0)), (\gamma\kappa)^2 \sup_{t\ge 0} |d(t)|^2 \right\} \tag{11a}$$

$$\limsup_{t\to\infty} V_0(\zeta(t)) \le (\gamma\kappa)^2 \limsup_{t\to\infty} |d(t)|^2 \tag{11b}$$

$$\int_0^\infty V_0(\zeta(t))dt \le \gamma^2 V_0(\zeta(0)) + (\gamma\kappa)^2 \int_0^\infty |d(t)|^2 dt \, . \tag{11c}$$

Our study of quantized control systems will investigate the degree to which these bounds are preserved under quantized feedback. ■

We impose the following assumption on the quantizer function q:

Assumption 2: There exist strictly positive real numbers Δ and M satisfying

$$\frac{M}{\Delta} > 2 + \left(2 + \frac{\gamma\nu}{\sqrt{\lambda_{min}(P)}} \right) \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}} \tag{12}$$

and characterizing the quantization function q as follows

$$\begin{array}{rclcrcl} |z| & \le & M & \implies & |q(z) - z| & \le & \Delta \\ |q(z)| & \le & M - \Delta & \implies & |z| & \le & M. \end{array} \tag{13}$$

Remark 2: Quantizers that satisfy (13) have been used in the literature before, see for instance [7], [1], [9]. Moreover, conditions similar to (12) were used in the same papers, see for instance equation (13) in [9]. The difference between our condition (12) and similar conditions in the literature comes mainly from the subtle differences between the classes of systems considered and the proof techniques used; the condition ensures that the precision of the quantizer that is quantified with Δ is sufficiently smaller than the saturation of the quantizer that is captured by M.

Discrete updates for μ precipitate a closed-loop hybrid dynamical system. We consider the situation where updates of μ are triggered by the size of $q(\zeta/\mu)$, using "zoom outs" for μ when the size of $q(\zeta/\mu)$ is large and using "zoom ins" for μ when the size of $q(\zeta/\mu)$ is small. Specifically, we consider the hybrid system

$$\begin{array}{rcll} \dot{\zeta} & = & A\zeta + BK\mu q\left(\frac{\zeta}{\mu}\right) + Ed & \\ & := & f(\zeta,\mu,d) & (\zeta,\mu) \in C \\ \mu^+ & = & \max\{\lambda_{in}\mu, \varepsilon\} & (\zeta,\mu) \in D_{in} \\ \mu^+ & = & \lambda_{out}\mu & (\zeta,\mu) \in D_{out} \end{array} \tag{14}$$

where

$$\begin{array}{rcl} C & = & \left\{ (\zeta,\mu) \in \mathbb{R}^n \times [\varepsilon,\infty) : \left| q\left(\frac{\zeta}{\mu}\right)\right| \in [\ell_{in}, \ell_{out}] \right\} \\ D_{in} & = & \left\{ (\zeta,\mu) \in \mathbb{R}^n \times [\varepsilon,\infty) : \left| q\left(\frac{\zeta}{\mu}\right)\right| < \ell_{in} \right\} \\ D_{out} & = & \left\{ (\zeta,\mu) \in \mathbb{R}^n \times [\varepsilon,\infty) : \left| q\left(\frac{\zeta}{\mu}\right)\right| > \ell_{out} \right\} . \end{array} \tag{15}$$

The parameter $\varepsilon \in (0,1)$ limits how small μ can become. The parameter $\lambda_{out} > 1$ forces μ to grow when $(\zeta,\mu) \in D_{out}$. The other positive parameters $\lambda_{in} < 1$, which shrinks μ when $(\zeta,\mu) \in D_{in}$, and ℓ_{in} and ℓ_{out}, which characterize the region C where μ remains constant, must satisfy the following assumption:

Assumption 3: The positive real numbers ε, λ_{out}, λ_{in}, ℓ_{in} and ℓ_{out} satisfy $\varepsilon \in (0,1)$ and

$$\lambda_{out} > 1 \tag{16a}$$

$$\lambda_{in} < 1 \tag{16b}$$

$$\ell_{in} > \Delta \left(1 + \frac{\gamma\nu}{\sqrt{\lambda_{min}(P)}} \right) \tag{16c}$$

$$\ell_{out} > \Delta + (\ell_{in} + \Delta)\lambda_{in}^{-1} \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}} \tag{16d}$$

$$\ell_{out} \le M - \Delta \, . \tag{16e}$$

The following claim can be verified by first picking λ_{in}, ℓ_{in}, and ℓ_{out} to satisfy (16b)-(16d), but with equalities rather than strict inequalities, and then noting that, by the condition (12) in Assumption 2, the condition (16e) is satisfied with a strict inequality. This means that λ_{in} can be decreased and ℓ_{in} and ℓ_{out} can be increased by small amounts in order to satisfy (16b)-(16d) without destroying (16e).

Lemma 3: Under condition (12) of Assumption 2, it is possible to choose the positive real numbers λ_{in}, ℓ_{in}, and ℓ_{out} so that the conditions (16b)-(16e) are satisfied.

Remark 3: Another way to view the constraints (16) is to pick $\ell_{in} = c_{in}\Delta$ and $\ell_{out} = c_{out}\Delta$ so that the conditions

(16) become

$$\lambda_{out} > 1 \tag{17a}$$

$$\lambda_{in} < 1 \tag{17b}$$

$$c_{in} > \left(1 + \frac{\gamma\nu}{\sqrt{\lambda_{min}(P)}}\right) \tag{17c}$$

$$c_{out} > 1 + (c_{in}+1)\lambda_{in}^{-1}\sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}} \tag{17d}$$

$$c_{out} \le \frac{M}{\Delta} - 1 \tag{17e}$$

which clearly can be satisfied for $\frac{M}{\Delta}$ sufficiently large. The condition (12) characterizes how large this ratio needs to be. ∎

Remark 4: We note that (14) differs slightly from QCS in the literature, such as [1], [7], [9], since the tuning variable μ is lower bounded by ε. Hence, we will only be able to show practical stability of the closed loop system. Note also that an arbitrarily large number of instantaneous zoom outs are possible for some initial states which requires an arbitrarily large bandwidth of the communication channel. Hence, (14) does not belong to the class of finite-set feedback systems considered in [10]. We will show that (14) has linear ISS and $\mathcal{L}_2$ gains.

Instead of analyzing the system (14)-(15), which involves the discontinuous function q, we find it just as easy and productive to analyze the system (14) with f, C, D_{in}, and D_{out} replaced, respectively, by certain $\widehat{F}$, $\widehat{C}$, $\widehat{D}_{in}$, and $\widehat{D}_{out}$ that satisfy the containments

$$C \subset \widehat{C}, \quad D_{in} \subset \widehat{D}_{in}, \quad D_{out} \subset \widehat{D}_{out} \tag{18}$$

$$f(\zeta,\mu,d) \in \widehat{F}(\zeta,\mu,d) \qquad \forall(\zeta,\mu)\in C. \tag{19}$$

These containments imply that every solution of (14)-(15) is a solution of

$$\begin{array}{rcll} \dot{\zeta} & \in & \widehat{F}(\zeta,\mu,d) & (\zeta,\mu)\in\widehat{C} \\ \mu^+ & = & \max\{\lambda_{in}\mu,\varepsilon\} & (\zeta,\mu)\in\widehat{D}_{in} \\ \mu^+ & = & \lambda_{out}\mu & (\zeta,\mu)\in\widehat{D}_{out}. \end{array} \tag{20}$$

Therefore, if we establish useful properties for the solutions of (20) then the solutions of (14)-(15) have those same properties. We choose

$$\begin{array}{rcl} \widehat{C} & := & \left\{(\zeta,\mu)\in\mathbb{R}^n\times[\varepsilon,\infty) : \left|\frac{\zeta}{\mu}\right| \in [\ell_{in}-\Delta, \ell_{out}+\Delta]\right\} \\ \widehat{D}_{in} & := & \left\{(\zeta,\mu)\in\mathbb{R}^n\times[\varepsilon,\infty) : \left|\frac{\zeta}{\mu}\right| \le \ell_{in}+\Delta\right\} \\ \widehat{D}_{out} & := & \left\{(\zeta,\mu)\in\mathbb{R}^n\times[\varepsilon,\infty) : \left|\frac{\zeta}{\mu}\right| \ge \ell_{out}-\Delta\right\} \end{array} \tag{21}$$

and, for all $(\zeta,\mu)\in\widehat{C}$ and all $d\in\mathbb{R}^n$,

$$\widehat{F}(\zeta,\mu,d) := \{f : f = (A+BK)\zeta + BKv + d, |v| \le \mu\Delta\}\ . \tag{22}$$

The next lemma establishes that these choices satisfy the desired containments.

Lemma 4: Under the conditions (13) in Assumption 2, the data $\widehat{C}$, $\widehat{D}_{in}$, $\widehat{D}_{out}$, and $\widehat{F}$ defined in (21)-(22) satisfy the containments (18)-(19).

Proof. The keys to establishing the set containments in (18) are the following bounds, which follow from the conditions (13) in Assumption 2:

$$\begin{array}{l} N \le M-\Delta, \\ |q(z)| \le N \end{array} \implies \begin{array}{rcl} |z| & \le & M, \\ |z| & \le & |q(z)| + |z-q(z)| \\ & \le & N+\Delta \end{array} \tag{23}$$

and

$$\begin{array}{l} N-\Delta \le M, \\ |z| < N-\Delta \end{array} \implies \begin{array}{rcl} |q(z)-z| & \le & \Delta, \\ |q(z)| & \le & |z| + |q(z)-z| \\ & < & N-\Delta+\Delta, \end{array} \tag{24}$$

the latter which can be written equivalently as

$$N-\Delta \le M, \quad |q(z)| \ge N \quad \implies \quad |z| \ge N-\Delta. \tag{25}$$

With the set containments established, the containment (19) follows from (16e) in Assumption 3 together with the first implication of (13) in Assumption 2. ∎

IV. Main results

In this section we present our main results (Theorems 1 and 2) where we construct Lyapunov functions for the closed loop system that can be used to show ISS and $\mathcal{L}_2$ stability. Corollaries that follow our main results provide conclusions that we can draw on the trajectories of the closed loop system from the constructed Lyapunov functions. The first main result establishes such conditions only for the state variable x whereas the second main result provides conditions for the state (x,μ).

Our main results are expressed in terms of the functions

$$V_0(\zeta) := \zeta^T P \zeta \tag{26a}$$

$$V_2(\mu) := \mu^2 - \varepsilon^2 \tag{26b}$$

$$V_4(\zeta) := \max\left\{0, V_0(\zeta) - \varepsilon^2\sigma^2\right\} \tag{26c}$$

where

$$\sigma := \sqrt{\lambda_{max}(P)}(\ell_{in}+\Delta)\lambda_{in}^{-1}\ . \tag{27}$$

We also use $g(\zeta,\mu)$ for the mapping that satisfies $\mu^+ = g(\zeta,\mu)$, and we define

$$\begin{array}{rcl} \underline{\theta} & := & \sqrt{\dfrac{\lambda_{min}(P)(\ell_{in}-\Delta)^2 - (\gamma\nu\Delta)^2}{\lambda_{min}(P)(\ell_{in}-\Delta)^2}}\ , \\ \overline{\theta} & := & \sqrt{\dfrac{\lambda_{min}(P)(\ell_{out}-\Delta)^2}{\lambda_{min}(P)(\ell_{out}-\Delta)^2-\sigma^2}}\ . \end{array} \tag{28}$$

It follows from (16c) that $\underline{\theta}$ is well defined and satisfies $\underline{\theta}\in(0,1)$. It follows from (16d) and (27) that $\overline{\theta}$ is well defined and satisfies $\overline{\theta}\in(1,\infty)$. Moreover, if we take $\ell_{in} = c_{in}\Delta$, $\ell_{out} = c_{out}\Delta$, where c_{in} and c_{out} satisfy (17), then $\underline{\theta}\to 1^-$ as c_{in} becomes arbitrarily large, and $\overline{\theta}\to 1^+$ if, in addition, the ratio c_{out}/c_{in} becomes arbitrarily large. It is possible to pick c_{in} and c_{out} large with a large ratio c_{out}/c_{in} when the ratio M/Δ is large.

Our first result is contained in the following theorem.

Theorem 1: Suppose Assumptions 1-3 hold. For each $\varepsilon \in (0,1)$, there exists a locally Lipschitz function $W_\varepsilon : \mathbb{R}^n \times [\varepsilon,\infty) \to \mathbb{R}_{\geq 0}$ such that:

1) for all $(\zeta,\mu) \in \mathbb{R}^n \times [\varepsilon,\infty)$,

$$V_4(\zeta) \leq W_\varepsilon(\zeta,\mu) \leq (1+\overline{\theta}^2)V_4(\zeta) + \sigma^2 V_2(\mu) \ , \quad (29)$$

2) for all $(\zeta,\mu) \in \widehat{C}$ and $f \in \widehat{F}(\zeta,\mu,d)$,

$$W_\varepsilon^\circ((\zeta,\mu);f) \leq -\gamma^{-2}\underline{\theta}^2 W_\varepsilon(\zeta,\mu) + \overline{\theta}^2\kappa^2|d|^2 \ , \quad (30)$$

3) and for all $(\zeta,\mu) \in \widehat{D}$ and

$$W_\varepsilon(\zeta, g(\zeta,\mu)) \leq W_\varepsilon(\zeta,\mu) \ . \quad (31)$$

The next result, which demonstrates finite $\mathcal{L}_\infty$ and $\mathcal{L}_2$ gains and how they are degraded from the ideal (no quantization) case described in Remark 1, follows from Theorem 1 together with Lemma 1.

Corollary 1: Under Assumptions 1-3, for each $\varepsilon \in (0,1)$, each complete solution (ζ,μ,d) of (20)-(22) satisfies

$$\left\|V_4(\zeta)\right\|_\infty \leq \max\left\{\sigma^2 V_2(\mu(0,0)) + (1+\overline{\theta}^2)V_4(\zeta(0,0)), (\gamma\kappa)^2\left(\frac{\overline{\theta}}{\underline{\theta}}\right)^2\left\|d^T d\right\|_{c,\infty}\right\} \quad (32a)$$

$$\left\|V_4(\zeta)\right\|_{c,1} \leq \left(\frac{\gamma}{\underline{\theta}}\right)^2\left(\sigma^2 V_2(\mu(0,0)) + (1+\overline{\theta}^2)V_4(\zeta(0,0))\right) + (\gamma\kappa)^2\left(\frac{\overline{\theta}}{\underline{\theta}}\right)^2\left\|d^T d\right\|_{c,1} \ . \quad (32b)$$

The $\mathcal{L}_2$ stability bound is expressed in terms of $\|\cdot\|_{c,1}$ since the quantities being integrated are quadratic in the disturbance and in the state. Note that the $\mathcal{L}_\infty$ and $\mathcal{L}_2$ gains with respect to d approach the ideal situation described in Remark 1 as $\overline{\theta}/\underline{\theta}$ tends toward one. Of course, it is a type of "practical" $\mathcal{L}_\infty$ and $\mathcal{L}_2$ stability due to the fact that $V_4(\zeta)$ is identically zero on a ball around the origin, the radius of which is proportional to $\varepsilon \in (0,1)$.

The results up to this point say nothing explicitly about the behavior of the state variable μ. The next results are aimed in this direction. Among other things, we establish that the compact set

$$\mathcal{A} := \left\{(\zeta,\mu) \in \mathbb{R}^n \times [\varepsilon,\infty) : \mu = \varepsilon,\ V_0(\zeta) \leq \varepsilon^2\sigma^2\right\} \quad (33)$$

is globally asymptotically stable for the system (20)-(22). The next theorem enables us to establish such a result. For technical reasons, we impose one additional condition on $\lambda_{out} > 1$. It is possible to establish global asymptotic stability without this technical condition, but it requires a more complicated Lyapunov function than the one we propose. Due to (16d), it is possible to satisfy the following assumption:

Assumption 4: The value of λ_{out} satisfies the condition

$$1 < \lambda_{out} < \lambda_{in}\frac{\ell_{out}-\Delta}{\ell_{in}+\Delta}\sqrt{\frac{\lambda_{min}(P)}{\lambda_{max}(P)}} \ .$$

Theorem 2: Suppose Assumptions 1-4 holds. There exist $\delta^* > 0$ and $\beta > 0$ and, for each $\delta \in (0,\delta^*]$ and each $\varepsilon \in (0,1)$, there exists a locally Lipschitz function $W_{\delta,\varepsilon} : \mathbb{R}^n \times [\varepsilon,\infty) \to \mathbb{R}_{\geq 0}$ such that:

1) for all $(\zeta,\mu) \in \mathbb{R}^n \times [\varepsilon,\infty)$,

$$\begin{aligned} \delta V_2(\mu) + V_4(\zeta) &\leq W_{\delta,\varepsilon}(\zeta,\mu) \\ W_{\delta,\varepsilon}(\zeta,\mu) &\leq (1+\beta\delta)\left((1+\overline{\theta}^2)V_4(\zeta) + \sigma^2 V_2(\mu)\right) \ , \end{aligned} \quad (34)$$

2) for all $(\zeta,\mu) \in \widehat{C}$ and $f \in \widehat{F}(\zeta,\mu,d)$,

$$\begin{aligned} W_{\delta,\varepsilon}^\circ((\zeta,\mu);f) \leq & -\gamma^{-2}\underline{\theta}^2(1-\beta\delta)W_{\delta,\varepsilon}(\zeta,\mu) \\ & +\overline{\theta}^2(1+\beta\delta)\kappa^2|d|^2 \ , \end{aligned} \quad (35)$$

3) for all $(\zeta,\mu) \in \widehat{D}$,

$$W_{\delta,\varepsilon}(\zeta, g(\zeta,\mu)) \leq W_{\delta,\varepsilon}(\zeta,\mu) \ , \quad (36)$$

and for all $(\zeta,\mu) \in \widehat{D} \setminus \mathcal{A}$,

$$W_{\delta,\varepsilon}(\zeta, g(\zeta,\mu)) < W_{\delta,\varepsilon}(\zeta,\mu) \ . \quad (37)$$

The next corollary, which addresses the existence of complete solutions, follows from Theorem 2 together with Lemma 1. It establishes that maximal solutions are complete if either the disturbance is bounded or has bounded energy.

Corollary 2: Under Assumptions 1-4, there exist $\delta > 0$, $\alpha > 0$ such that, for each $\varpi \in \{1,\infty\}$, each solution (ζ,μ,d) of (20)-(22), and each $(t,j) \in \text{dom}\ (\zeta,\mu,d)$,

$$\begin{aligned} \|((\delta V_2(\mu) + V_4(\zeta),t,j)\|_\infty \leq & \ \alpha\big(V_2(\mu(0,0)) \\ & +V_4(\zeta(0,0)) \\ & +\left\|d^T d,t,j)\right\|_{c,\varpi}\big) \ . \end{aligned}$$

In particular, with the definition $T = \sup\{t+j : (t,j) \in \text{dom}\ (\zeta,\mu,d)\}$, if $\lim_{t+j\to T}\left\|(d^T d,t,j)\right\|_{c,\varpi}$ is well defined and finite then (ζ,μ,d) is either complete or can be extended to a complete solution.

The next two corollaries also follow from Theorem 2, this time combined with Lemma 2. Implicitly, we also use the fact that if (37) and (34) hold, then for each pair (r,R) satisfying $0 < r < R < \infty$, there exists $\rho \in (0,1)$ such that, for all $(\zeta,\mu) \in \widehat{D}$ such that $W_{\delta,\varepsilon}(\zeta,\mu) \in [r,R]$,

$$W_{\delta,\varepsilon}(\zeta, g(\zeta,\mu)) \leq \rho W_{\delta,\varepsilon}(\zeta,\mu) \ . \quad (38)$$

The first of the two corollaries pertains to the case where $d \equiv 0$.

Corollary 3: Under Assumptions 1-4, for each $\varepsilon \in (0,1)$, the system (20)-(22) with $d \equiv 0$ has the compact set

$$\mathcal{A} := \left\{(\zeta,\mu) \in \mathbb{R}^n \times [\varepsilon,\infty) : \mu = \varepsilon,\ \zeta^T P\zeta \leq \varepsilon^2\sigma^2\right\}$$

globally asymptotically stable.

The second corollary addresses asymptotic gains when d is bounded. It is proved using Theorem 2 and Lemma 2 to get a linear gain that depends on $\delta > 0$ and then taking the limit of this gain as $\delta \to 0$.

Corollary 4: Under Assumptions 1-4, for each $\varepsilon \in (0,1)$, each complete solution of (20)-(22) satisfies

$$\left\|V_4(\zeta)\right\|_a \leq (\gamma\kappa)^2 \left(\frac{\overline{\theta}}{\underline{\theta}}\right)^2 \left\|d^T d\right\|_{c,a} \ . \tag{39}$$

With a little extra work, it is possible to construct a Lyapunov function that exhibits finite $\mathcal{L}_\infty$ and $\mathcal{L}_2$ gain and decreases exponentially at jumps. We omit these results for space reasons.

V. Sketch of Proof of Theorem 1

We only sketch the proof of Theorem 1 and omit other proofs due to space reasons.

1) A manipulation involving Lyapunov functions: In the proofs of all of the theorems, we exploit the following fact: Suppose there exists $\rho \in (0,1)$ and positive constants c_i, $i \in \{1,2,3,4\}$ such that

$$\begin{array}{rcl} V_a^\circ(x;f) & \leq & -c_1 V_a(x) + c_2|d|^2 \\ V_b^\circ(x;f) & \leq & -c_1 V_b(x) + c_3|d|^2 \end{array} \tag{40}$$

and

$$\begin{array}{rcl} V_a(g) & \leq & V_a(x) + c_4 V_b(x) \\ V_b(g) & \leq & \rho V_b(x) \ . \end{array} \tag{41}$$

Define $V_c(x) := V_a(x) + c_5 V_b(x)$. Then

$$\begin{array}{rcl} V_c^\circ(x;f) & \leq & -c_1 V_a(x) + c_2|d|^2 - c_1 c_5 V_b(x) + c_5 c_3 |d|^2 \\ & = & -c_1 V_c(x) + (c_2 + c_5 c_3)|d|^2 \end{array} \tag{42}$$

and

$$\begin{array}{rcl} V_c(g) & \leq & V_a(x) + c_4 V_b(x) + c_5 \rho V_b(x) \\ & = & V_c(x) + \big(c_4 + c_5(\rho - 1)\big) V_b(x) \ . \end{array} \tag{43}$$

In particular, if $c_5 \geq c_4/(1-\rho)$ then $V_c(g) \leq V_c(x)$.

2) System in transformed coordinates: We analyze the system (20)-(22) using the coordinates (z,μ) where $z := \zeta/\mu$. In these coordinates, the dynamics become

$$\begin{array}{ll} \dot z \in \widetilde{F}(z, d/\mu) & (z,\mu) \in \widetilde{C} \times [\varepsilon, \infty) \\ \left.\begin{array}{l} z^+ = g_1(z,\mu) \\ \mu^+ = g_2(z,\mu) \end{array}\right\} & (z,\mu) \in (\widetilde{D}_{in} \cup \widetilde{D}_{out}) \times [\varepsilon,\infty) \end{array} \tag{44}$$

where

$$\begin{array}{rcl} \widetilde{C} & = & \{z \in \mathbb{R}^n : |z| \in [\ell_{in} - \Delta, \ell_{out} + \Delta]\} \\ \widetilde{D}_{in} & = & \{z \in \mathbb{R}^n : |z| \leq \ell_{in} + \Delta\} \\ \widetilde{D}_{out} & = & \{z \in \mathbb{R}^n : |z| \geq \ell_{out} - \Delta\} \\ \widetilde{F}(z, d/\mu) & = & \{f : f = (A + BK)z + BKv + Ed/\mu, \\ & & |v| \leq \Delta\} \quad \forall (z,\mu,d) \in \widetilde{C} \times [\varepsilon,\infty) \times \mathbb{R}^k \\ g_1(z,\mu) & = & \begin{cases} z \cdot \frac{1}{\max\{\lambda_{in}, \varepsilon/\mu\}} & (z,\mu) \in \widetilde{D}_{in} \times [\varepsilon,\infty) \\ \lambda_{out}^{-1} z & (z,\mu) \in \widetilde{D}_{out} \times [\varepsilon,\infty) \end{cases} \\ g_2(z,\mu) & = & \begin{cases} \max\{\lambda_{in}\mu, \varepsilon\} & (z,\mu) \in \widetilde{D}_{in} \times [\varepsilon,\infty) \\ \lambda_{out}\mu & (z,\mu) \in \widetilde{D}_{out} \times [\varepsilon,\infty) \ . \end{cases} \end{array}$$

3) Lyapunov function:
We use a Lyapunov function of the form

$$W_\varepsilon(\zeta,\mu) := V_3(\zeta/\mu, \mu) + \overline{\theta}^2 \varepsilon^2 V_1(\zeta/\mu) \ , \tag{45}$$

where $V_1(z) := \max\{0, V_0(z) - \sigma^2\}$ and $V_3(z,\mu) := V_2(\mu)V_0(z)$. Then, it can be shown (the proofs are omitted due to space reasons) that for all $(\zeta,\mu) \in \widehat{C}$ and $f \in \widehat{F}(\zeta,\mu,d)$ we have that

$$\begin{array}{l} W_\varepsilon^\circ((\zeta,\mu);f) \\ \quad \leq -\gamma^{-2}\underline{\theta}^2 V_3(\zeta/\mu,\mu) + \dfrac{\mu^2 - \varepsilon^2}{\mu^2}\kappa^2|d|^2 \\ \quad \leq -\gamma^{-2}\underline{\theta}^2 W_\varepsilon(\zeta,\mu) + \overline{\theta}^2\kappa^2|d|^2 \ . \end{array} \tag{46}$$

Similarly, for all $(z,\mu) \in \widehat{D}$,

$$\begin{array}{rcl} W_\varepsilon(\zeta, g(\zeta,\mu)) & \leq & V_3(\zeta/\mu,\mu) + \overline{\theta}^2(1 - \lambda_{out}^{-2})\varepsilon^2 V_1(\zeta/\mu) \\ & & + \overline{\theta}^2 \varepsilon^2 \lambda_{out}^{-2} V_1(\zeta/\mu) \\ & = & W_\varepsilon(\zeta,\mu) \ . \end{array} \tag{47}$$

This establishes Theorem 1. ■

VI. Conclusions

We have presented a Lyapunov approach to analysis of $\mathcal{L}_2$ stability and ISS of a class of linear QCS. Estimates of linear gains are provided by constructing appropriate Lyapunov functions for the closed loop system. Our approach is novel and we treat the system in zoom in and zoom out modes in a unified manner using a hybrid model of the closed loop system and the hybrid modeling framework in [4].

References

[1] R. W. Brockett and D. Liberzon. Quantized feedback stabilization of linear systems. *IEEE Trans. Automat. Control*, 45:1279–1289, 2000.

[2] D. F. Delchamps. Stabilizing a linear system with quantized state feedback. *IEEE Trans. Automat. Control*, 35:916–924, 1990.

[3] N. Elia and S. K. Mitter. Stabilization of linear systems with limited information. *IEEE Trans. Automat. Control*, 46:1384–1400, 2001.

[4] R. Goebel, R. Sanfelice, and A.R. Teel. Hybrid dynamical systems. *IEEE Control Systems Magazine*, 29:28–93, 2009.

[5] H. Ishii and B. A. Francis. Stabilizing a linear system by switching control with dwell time. *IEEE Trans. Automat. Control*, 47:1962–1973, 2002.

[6] T. Kameneva and D. Nešić. On l_2 stabilization of linear systems with quantized control. *IEEE Trans. Automat. Control*, 53:399–405, 2008.

[7] D. Liberzon. Hybrid feedback stabilization of systems with quantized signals. *Automatica*, 39:1543–1554, 2003.

[8] D. Liberzon. On stabilization of linear systems with limited information. *IEEE Trans. Automat. Control*, 48:304–307, 2003.

[9] D. Liberzon and D. Nešić. Input-to-state stabilization of linear systems with quantized state measurements. *IEEE Trans. Automat. Control*, 52:767–781, 2005.

[10] N. Martins. Finite gain l_p stabilization requires analog control. *Systems Control Letters*, 55:949–954, 2006.

[11] G. N. Nair and R. J. Evans. Stabilizability of stochastic linear systems with finite feedback data rates. *SIAM J. Control Optim.*, 43:413–436, 2004.

[12] I. R. Petersen and A. V. Savkin. Multi-rate stabilization of multivariable discrete-time linear systems via a limited capacity communication channel. In *Proc. 40th IEEE Conf. on Decision and Control*, pages 304–309, 2001.

[13] S. Tatikonda and S. K. Mitter. Control under communication constraints. *IEEE Trans. Automat. Control*, 49:1056–1068, 2004.

[14] W. S. Wong and R. W. Brockett. Systems with finite communication bandwidth constraints II: stabilization with limited information feedback. *IEEE Trans. Automat. Control*, 44:1049–1053, 1999.