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# Model Reference Adaptive Control For Nonminimum-Phase Systems Using A Surrogate Tracking Error

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*Abstract*— **This paper presents a direct model reference adaptive controller for single-input, single-output, linear, continuous-time systems that are possibly nonminimum phase, provided that the nonminimum-phase zeros are known. The adaptive controller uses a surrogate tracking error, which approximates the true tracking error. The recursive-least-squares-based adaptive law is developed by minimizing an integral quadratic cost of the surrogate tracking error.**

## I. Introduction

The model reference adaptive control (MRAC) architecture has been widely studied and is a common architecture employed in adaptive control [1]–[5]. The goal of MRAC is to force an unknown or uncertain system to behave like a known reference model. However, MRAC is generally restricted to minimum-phase systems.

For nonminimum-phase systems, indirect adaptive control techniques, such as adaptive pole placement [3]–[6], can be used. Adaptive pole placement differs from MRAC in that it does not employ a reference model. However, adaptive pole placement can be used in conjunction with the internal model principle to address the adaptive tracking problem (see for example [4, Chap. 7.2] or [5, Chap. 6.3]). Nevertheless, there are drawbacks to indirect adaptive control. In particular, indirect techniques typically assume that the estimates of the numerator and denominator polynomials of the plant are coprime for all time, and if this assumption is not satisfied, then singularities can occur when computing the controller parameters from the estimates of the plant parameters [5], [6]. There are approaches that address this drawback, including techniques where the parameter estimates are projected into a known convex set [1], [7] as well as techniques where the parameter estimates are perturbed [8], [9]. The modifications proposed in [1], [7]–[9] typically require additional assumptions, particularly, that the plant parameters exist within a known set. For a summary of these approaches, see [5, Chap. 6.6.2].

In contrast to the indirect adaptive pole placement approach, this paper presents a direct MRAC algorithm that is effective for systems that are either minimum phase or nonminimum phase, provided that the nonminimum-phase zeros are known. The direct continuous-time MRAC algorithm presented in this paper shares certain features with retrospective cost adaptive control (RCAC), which is a direct discrete-time adaptive control technique for systems that are possibly nonminimum-phase [10]–[13].

A key feature of RCAC is the use of a retrospective performance measure, in which the performance measurement is modified based on the difference between the actual past control inputs and the recomputed past control inputs, assuming that the current controller had been used in the past. The present paper adopts a related technique for continuous-time systems. In particular, we define a surrogate tracking error, which approximates the true tracking error. Next, a recursive-least-squares-based adaptive law is developed by minimizing an integral quadratic cost of the surrogate tracking error. Finally, this paper analyzes aspects of the surrogate tracking error MRAC algorithm.

## II. Problem Formulation

Let $\mathbf{p} = \frac{\mathrm{d}}{\mathrm{dt}}$ denotes the differential operator, and consider the continuous-time linear system

$$\alpha(\mathbf{p})y(t) = \beta_d\beta(\mathbf{p})u(t) + \gamma(\mathbf{p})w(t), \tag{1}$$

where $t \geq 0$; $y(t) \in \mathbb{R}$ is the output; $u(t) \in \mathbb{R}$ is the control; $w(t) \in \mathbb{R}^{l_w}$ is the exogenous disturbance; $\beta_d \in \mathbb{R}$; $\alpha(\mathbf{p})$ is a monic polynomial with degree $n > 0$; $\gamma(\mathbf{p})$ is a polynomial with degree at most $n$; $\beta(\mathbf{p})$ is a monic polynomial with degree $n - d$, where $d > 0$ is the relative degree; and the initial condition is $y_0 = [\ y^{(n-1)}(0) \quad \cdots \quad y(0)\ ]$.

Next, consider the reference model

$$\alpha_{\mathrm{m}}(\mathbf{p})y_{\mathrm{m}}(t) = \beta_{\mathrm{m}}(\mathbf{p})r(t), \tag{2}$$

where $t \geq 0$; $y_{\mathrm{m}}(t) \in \mathbb{R}$ is the reference model output; $r(t) \in \mathbb{R}$ is the reference model command; $\alpha_{\mathrm{m}}(\mathbf{p})$ is a monic polynomial with degree $n_{\mathrm{m}} > 0$; $\beta_{\mathrm{m}}(\mathbf{p})$ is a polynomial with degree $n_{\mathrm{m}} - d_{\mathrm{m}}$, where $d_{\mathrm{m}} > 0$ is the relative degree of (2); $\alpha_{\mathrm{m}}(\mathbf{p})$ and $\beta_{\mathrm{m}}(\mathbf{p})$ are coprime; $\alpha_{\mathrm{m}}(\mathbf{p})$ is asymptotically stable; and $r(t)$ is bounded and piecewise continuous.

Next, define the tracking error $z(t) \triangleq y(t) - y_{\mathrm{m}}(t)$. The goal is to drive $z(t)$ to zero in the presence of $w(t)$. We make the following assumptions regarding the system (1):

(A1) $\alpha(\mathbf{p})$ and $\beta(\mathbf{p})$ are coprime.
(A2) $d$ is known.
(A3) $\beta_d$ is known.
(A4) If $\lambda \in \mathbb{C}$, Re $\lambda \geq 0$, and $\beta(\lambda) = 0$, then $\lambda$ and its multiplicity are known.
(A5) There exists a known integer $\bar{n}$ such that $n \leq \bar{n}$.

We make the following assumptions regarding $w(t)$:

(A6) For all $t \geq 0$, the exogenous disturbance $w(t)$ is bounded and satisfies $\alpha_w(\mathbf{p})w(t) = 0$, where $\alpha_w(\mathbf{p})$ is a nonzero monic polynomial that has distinct roots, all of which lie on the imaginary axis, and none of which coincide with the roots of $\beta(\mathbf{p})$.

(A7) There exists a known integer $\bar{n}_w$ such that $n_w \triangleq \deg \alpha_w(\mathbf{p}) \le \bar{n}_w$.

We make the following assumptions regarding (2):

(A8) If $\lambda \in \mathbb{C}$, Re $\lambda \ge 0$, and $\beta(\lambda) = 0$, then $\beta_{\rm m}(\lambda) = 0$ and the multiplicity of $\lambda$ with respect to $\beta_{\rm m}(\mathbf{p})$ equals the multiplicity of $\lambda$ with respect to $\beta(\mathbf{p})$.

(A9) $d_{\rm m} \ge d$.

The disturbance $w(t)$ is not assumed to be measured, and its spectrum $\alpha_w(\mathbf{p})$ is not assumed to be known. It follows from assumption (A6) that $w(t)$ is sinusoidal.

Next, consider the factorization of $\beta(\mathbf{p})$ given by

$$\beta(\mathbf{p}) = \beta_d \beta_{\rm u}(\mathbf{p}) \beta_{\rm s}(\mathbf{p}), \tag{3}$$

where $\beta_{\rm u}(\mathbf{p})$ and $\beta_{\rm s}(\mathbf{p})$ are monic polynomials; and if $\lambda \in \mathbb{C}$, Re $\lambda \ge 0$, and $\beta(\lambda) = 0$, then $\beta_{\rm u}(\lambda) = 0$ and $\beta_{\rm s}(\lambda) \ne 0$. Let $n_{\rm u} \ge 0$ be the degree of $\beta_{\rm u}(\mathbf{p})$, and thus $n_{\rm s} \triangleq n - n_{\rm u} - d$ is the degree of $\beta_{\rm s}(\mathbf{p})$. Assumption (A4) implies that the nonminimum-phase zeros of (1) from the control to the output (i.e., the roots of $\beta(\mathbf{p})$ that lie in the closed-right-half plane) are known, which is equivalent to the assumption that $\beta_{\rm u}(\mathbf{p})$ and $n_{\rm u}$ are known.

Assumption (A8) implies that the nonminimum-phase zeros of (1) from the control to the output are zeros of the reference model. Assumption (A8) is a model matching condition, which arises from the fact that nonminimum-phase zeros cannot be moved through feedback or pole-zero cancellation. However, the reference model may have additional zeros. Thus, $\beta_{\rm m}(\mathbf{p})$ has the factorization $\beta_{\rm m}(\mathbf{p}) = \beta_{\rm u}(\mathbf{p})\beta_{\rm c}(\mathbf{p})$, where $\beta_{\rm c}(\mathbf{p})$ is a polynomial with degree $n_{\rm m} - d_{\rm m} - n_{\rm u}$.

## III. Surrogate Tracking Error MRAC

This section introduces the surrogate tracking error and presents an adaptive controller, which uses the surrogate tracking error. Let

$$n_{\rm c} \ge \max(2\bar{n} + 2\bar{n}_w - n_{\rm u} - d, n_{\rm m} - n_{\rm u} - d), \tag{4}$$

where assumptions (A2), (A4), (A5), and (A7) imply that the lower bound on $n_{\rm c}$ given by (4) is known. Next, let $a_{\rm f}(s)$ be an asymptotically stable monic polynomial with degree $n_{\rm c}$, and let $c_{\rm f}(s)$ be an asymptotically stable monic polynomial with degree $n_{\rm c}+n_{\rm u}+d-n_{\rm m}$. For all $i = 1, 2, \ldots, n_{\rm c}$, define the filters

$$G_{a_{\rm f},i}(s) \triangleq \frac{s^{n_{\rm c}-i}}{a_{\rm f}(s)}, \tag{5}$$

and let $\bar{y}_i(t)$ and $\bar{u}_i(t)$ be the signals obtained by passing $y(t)$ and $u(t)$, respectively, through the filter $G_{a_{\rm f},i}(s)$. Next, define the filter

$$G_{c_{\rm f}}(s) \triangleq \frac{\beta_{\rm c}(s) c_{\rm f}(s)}{a_{\rm f}(s)}, \tag{6}$$

and let $\bar{r}(t)$ be the signal obtained by passing $r(t)$ through the filter $G_{c_{\rm f}}(s)$.

Now, consider the controller

$$u(t) = \sum_{i=1}^{n_{\rm c}} L_i(t)\bar{y}_i(t) + \sum_{i=1}^{n_{\rm c}} M_i(t)\bar{u}_i(t) + N(t)\bar{r}(t), \tag{7}$$

where, for all $i = 1, \ldots, n_{\rm c}$, $L_i : [0,\infty) \to \mathbb{R}$ and $M_i : [0,\infty) \to \mathbb{R}$, and $N : [0,\infty) \to \mathbb{R}$ are given by the adaptive law (14) and (15) presented below. The controller (7) can be expressed as

$$u(t) = \phi^{\rm T}(t)\theta(t), \tag{8}$$

where

$$\theta(t) \triangleq \begin{bmatrix} L_1(t) & \cdots & L_{n_{\rm c}}(t) & M_1(t) & \cdots & M_{n_{\rm c}}(t) & N(t) \end{bmatrix}^{\rm T},$$
$$\phi(t) \triangleq \begin{bmatrix} \bar{y}_1(t) & \cdots & \bar{y}_{n_{\rm c}}(t) & \bar{u}_1(t) & \cdots & \bar{u}_{n_{\rm c}}(t) & \bar{r}(t) \end{bmatrix}^{\rm T}.$$

Next, let $b_{\rm f}(s)$ be an asymptotically stable monic polynomial with degree $n_{\rm c} + n_{\rm u} + d$, and define the filters

$$G_{b_{\rm f},1}(s) \triangleq \frac{\alpha_{\rm m}(s) c_{\rm f}(s)}{b_{\rm f}(s)}, \quad G_{b_{\rm f},2}(s) \triangleq \frac{\beta_d \beta_{\rm u}(s) a_{\rm f}(s)}{b_{\rm f}(s)}. \tag{9}$$

Let $z_{\rm f}(t)$ be the filtered tracking error, which is obtained by passing $z(t)$ through the filter $G_{b_{\rm f},1}(s)$; let $u_{\rm f}(t)$ be the filtered control, which is obtained by passing $u(t)$ through the filter $G_{b_{\rm f},2}(s)$; and let $\Phi(t) \in \mathbb{R}^{2n_{\rm c}+1}$ be the filtered regressor, which is obtained by passing each element of $\phi(t)$ through the filter $G_{b_{\rm f},2}(s)$.

Now, let $\hat{\theta} \in \mathbb{R}^{2n_{\rm c}+1}$ be an optimization variable used to develop the adaptive controller update equations, and define the surrogate tracking error

$$\hat{z}(\hat{\theta}, t) \triangleq z_{\rm f}(t) + \Phi^{\rm T}(t)\hat{\theta} - u_{\rm f}(t). \tag{10}$$

Furthermore, for all $t \ge 0$, define the surrogate tracking error measure

$$z_{\rm s}(t) \triangleq \hat{z}(\theta(t), t) = z_{\rm f}(t) + \Phi^{\rm T}(t)\theta(t) - u_{\rm f}(t). \tag{11}$$

Note that if, for all $t \ge 0$, $\theta(t) = C$, where $C \in \mathbb{R}^{2n_{\rm c}+1}$, then $\Phi^{\rm T}(t)\theta(t) = u_{\rm f}(t)$ and $z_{\rm s}(t) = z_{\rm f}(t)$. Thus, the surrogate tracking error measure $z_{\rm s}(t)$ can be interrupted as a modification to filter tracking error $z_{\rm f}(t)$ based on the difference between the actual filtered control $u_{\rm f}(t)$ and the recomputed filtered control $\Phi^{\rm T}(t)\theta(t)$.

To develop the adaptive control law, define the cost function

$$J(\hat{\theta}, t) \triangleq \int_0^t \eta(\tau)\hat{z}^2(\hat{\theta}, \tau) d\tau + \left[\hat{\theta} - \theta(0)\right]^{\rm T} R \left[\hat{\theta} - \theta(0)\right], \tag{12}$$

where $R \in \mathbb{R}^{(2n_{\rm c}+1)\times(2n_{\rm c}+1)}$ is positive definite, $\theta(0) \in \mathbb{R}^{2n_{\rm c}+1}$, and $\eta(t) \triangleq \frac{1}{1+\eta_1 \Phi^{\rm T}(t)\Phi(t)}$, where $\eta_1 \in [0,\infty)$. If $\eta_1 > 0$, then $\eta(t)$ normalizes the first term of the cost function (12). However, if $\eta_1 = 0$, then $\eta(t) = 1$ and the cost function (12) is unnormalized. A normalized cost results in a normalized adaptive law whereas an unnormalized cost results in an unnormalized adaptive law.

**Theorem 1.** *For all $t \ge 0$, the unique global minimizer of the cost function (12) is given by*

$$\theta(t) = \left[R + \int_0^t \eta(\tau)\Phi(\tau)\Phi^{\rm T}(\tau)d\tau\right]^{-1} \times \left[R\theta(0) - \int_0^t \eta(\tau)\left(z_{\rm f}(\tau) - u_{\rm f}(\tau)\right)\Phi(\tau)d\tau\right]. \tag{13}$$

*Furthermore, (13) satisfies*

$$\dot{\theta}(t) = -\eta(t)z_{\rm s}(t)P(t)\Phi(t), \quad (14)$$

*where*

$$\dot{P}(t) = -\eta(t)P(t)\Phi(t)\Phi^{\rm T}(t)P(t), \quad (15)$$

*and* $P(0) = R^{-1}$.

The batch solution for $\theta(t)$, given by (13), is not desirable for implementation because it requires the online calculation of a matrix inverse of rank $2n_{\rm c}+1$. Thus, we implement the recursive solution (14) and (15). In particular, the surrogate tracking error model reference adaptive controller is given by (8), (14), and (15), where $z_{\rm s}$ is given by (11).

*Proof of Theorem 1.* It follows from (10) and (12) that $J(\hat{\theta},t) = \hat{\theta}^{\rm T}\Gamma_1(t)\hat{\theta} + \Gamma_2(t)\hat{\theta} + \Gamma_3(t)$, where $\Gamma_1(t) \triangleq R + \int_0^t \eta(\tau)\Phi(\tau)\Phi^{\rm T}(\tau)d\tau$, $\Gamma_2(t) \triangleq -2\theta^{\rm T}(0)R + 2\int_0^t \eta(\tau)\left(z_{\rm f}(\tau) - u_{\rm f}(\tau)\right)\Phi^{\rm T}(\tau)d\tau$, and $\Gamma_3(t) \triangleq \theta^{\rm T}(0)R\theta(0) + \int_0^t \eta(\tau)\left(z_{\rm f}(\tau) - u_{\rm f}(\tau)\right)^2 d\tau$. Since, for all $t \geq 0$, $\Gamma_1(t)$ is nonsingular, it follows that the cost function (12) has the unique global minimizer $\theta(t) \triangleq -\frac{1}{2}\Gamma_1^{-1}(t)\Gamma_2^{\rm T}(t) = \bar{P}(t)X(t)$, where

$$\bar{P}(t) \triangleq \Gamma_1^{-1}(t) = \left[R + \int_0^t \eta(\tau)\Phi(\tau)\Phi^{\rm T}(\tau)d\tau\right]^{-1}, \quad (16)$$

$$X(t) \triangleq R\theta(0) - \int_0^t \eta(\tau)\left(z_{\rm f}(\tau) - u_{\rm f}(\tau)\right)\Phi(\tau)d\tau, \quad (17)$$

thus verifying (13).

Next, note that $\frac{d}{dt}\left[\bar{P}(t)\bar{P}^{-1}(t)\right] = \dot{\bar{P}}(t)\bar{P}^{-1}(t) + \bar{P}(t)\frac{d}{dt}\left[\bar{P}^{-1}(t)\right] = 0$, and thus it follows from (16) that

$$\begin{aligned}\dot{\bar{P}}(t) &= -\bar{P}(t)\frac{d}{dt}\left[\bar{P}^{-1}(t)\right]\bar{P}(t)\\ &= -\eta(t)\bar{P}(t)\Phi(t)\Phi^{\rm T}(t)\bar{P}(t).\end{aligned}$$

Since $P(t)$ and $\bar{P}(t)$ satisfy the same differential equation and have the same initial condition, it follows that $P(t) = \bar{P}(t)$, which verifies (15).

Next, differentiating (13) with respect to $t$, and using (11), (15), and (17) yields

$$\begin{aligned}\dot{\theta}(t) &= \dot{P}(t)X(t) + P(t)\dot{X}(t)\\ &= -\eta(t)P(t)\Phi(t)\Phi^{\rm T}(t)P(t)X(t) + P(t)\dot{X}(t)\\ &= -\eta(t)P(t)\Phi(t)\left(\Phi^{\rm T}(t)\theta(t) + z_{\rm f}(t) - u_{\rm f}(t)\right)\\ &= -\eta(t)z_{\rm s}(t)P(t)\Phi(t),\end{aligned}$$

which verifies (14). □

The surrogate tracking error MRAC architecture is shown in Figure 1. Designing the surrogate tracking error MRAC includes choosing three asymptotically stable polynomials, namely, $a_{\rm f}(s)$, $b_{\rm f}(s)$, and $c_{\rm f}(s)$, which are used to construct the filters (5), (6), and (9). While the polynomials $a_{\rm f}(s)$, $b_{\rm f}(s)$, and $c_{\rm f}(s)$ can be selected by the user to meet design criteria, we now present one option, which simplifies the filters $G_{b_{\rm f},1}(s)$, $G_{b_{\rm f},2}(s)$, and $G_{c_{\rm f}}(s)$. Specifically, let $\bar{a}_{\rm f}(s)$ be an asymptotically stable monic polynomial with degree $n_{\rm m} - n_{\rm u} - d$, and let $c_{\rm f}(s)$ be an asymptotically stable monic polynomial with degree $n_{\rm c} + n_{\rm u} + d - n_{\rm m}$. Next, let $a_{\rm f}(s) = \bar{a}_{\rm f}(s)c_{\rm f}$ and $b_{\rm f}(s) = \alpha_{\rm m}(s)c_{\rm f}(s)$. In this case, it follows from (6) and (9) that

$$G_{b_{\rm f},1}(s) = 1,\quad G_{b_{\rm f},2}(s) = \frac{\beta_d\beta_{\rm u}(s)\bar{a}_{\rm f}(s)}{\alpha_{\rm m}(s)},\quad G_{c_{\rm f}}(s) \triangleq \frac{\beta_{\rm c}(s)}{\bar{a}_{\rm f}(s)},$$

and $z_{\rm f}(t) = z(t)$, that is, the filtered tracking error equals the tracking error.

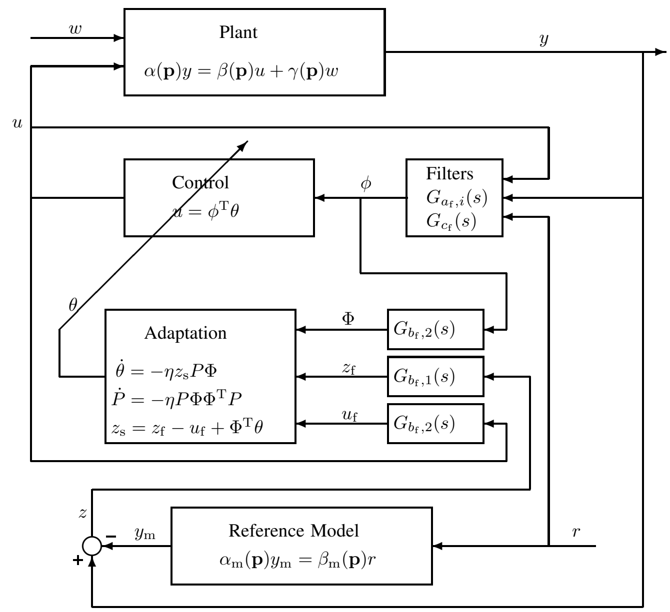


Fig. 1. Schematic diagram of the surrogate tracking error MRAC.

## IV. Numerical Examples

For all numerical examples presented in this section, the adaptive controller is normalized with $\eta_1 = 0.01$, and the adaptive controller is initialized to zero (i.e., $\theta(0) = 0$). For all examples $a_{\rm f}(s) = (s+3)^{n_{\rm c}}$, $c_{\rm f}(s) = (s+3)^{n_{\rm c}+n_{\rm u}+d-n_{\rm m}}$, and $b_{\rm f}(s) = \alpha_{\rm m}(s)c_{\rm f}(s)$. Therefore, it follows that the filters (9) and (6) are $G_{b_{\rm f},1}(s) = 1$, $G_{b_{\rm f},2}(s) = \frac{\beta_d\beta_{\rm u}(s)(s+3)^{n_{\rm m}-d-n_{\rm u}}}{\alpha_{\rm m}(s)}$, and $G_{c_{\rm f}}(s) = \frac{\beta_{\rm c}(s)}{(s+3)^{n_{\rm m}-d-n_{\rm u}}}$.

### *A. Asymptotically stable, nonminimum-phase system*

Consider the asymptotically stable, nonminimum-phase system

$$(\mathbf{p}+5)(\mathbf{p}^2+10\mathbf{p}+50)y = 3(\mathbf{p}^2-4)u + (\mathbf{p}^2-9)w,$$

where $y_0 = [1 \quad -3 \quad 1]$. For this problem, it follows that $n = 3$, $n_{\rm u} = 1$, $d = 1$, $\beta_d = 3$, and $\beta_{\rm u}(\mathbf{p}) = \mathbf{p} - 2$. Next, consider the reference model (2), where $\alpha_{\rm m}(\mathbf{p}) = (\mathbf{p}+10)^3$, $\beta_{\rm m}(\mathbf{p}) = (\mathbf{p}+8)\beta_{\rm u}(\mathbf{p})[\alpha_{\rm m}(0)/8/\beta_{\rm u}(0)]$, and $r(t) = \sin(7\pi t) + 2\sin(2\pi t)$.

First, we let $w(t) = 0$, and consider the tracking problem without disturbance. The surrogate tracking error MRAC (8), (14), and (15) is implemented in feedback with $n_{\rm c} = 8$, which satisfies (4), and $P(0) = 10^{12}I_{17}$. Figure 2 shows the time history of $y$, $y_{\rm m}$, and $z$. The system is simulated in open-loop for 2 seconds to demonstrate the open-loop response, then the adaptive controller is turned on and the tracking error $z$ tends to zero.

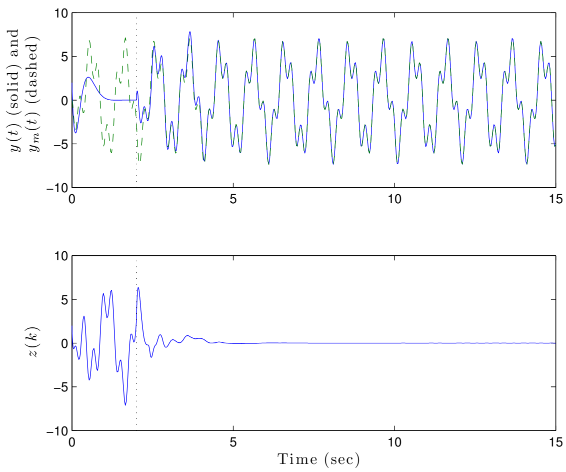


Fig. 2. *Command following for an asymptotically stable, nonminimum-phase system:* The surrogate tracking error MRAC (8), (14), and (15) is implemented in feedback with $n_{\rm c} = 8$, $\theta(0) = 0$, and $P(0) = 10^{12} I_{17}$. The controller drives $z$ to zero. Thus, the controller forces $y$ to follow $y_{\rm m}$ asymptotically.

Next, we consider the same example but with a disturbance; specifically, let $w(t) = 10\sin(3\pi t)$. Note that the disturbance spectrum is unknown and the disturbance is unmeasured. The controller parameters are the same as above. The system is simulated in open-loop for 2 seconds, then the adaptive controller is turned on and the tracking error $z$ tends to zero. Figure 3 shows that $y$ follows $y_{\rm m}$ asymptotically, while rejecting the disturbance $w$.

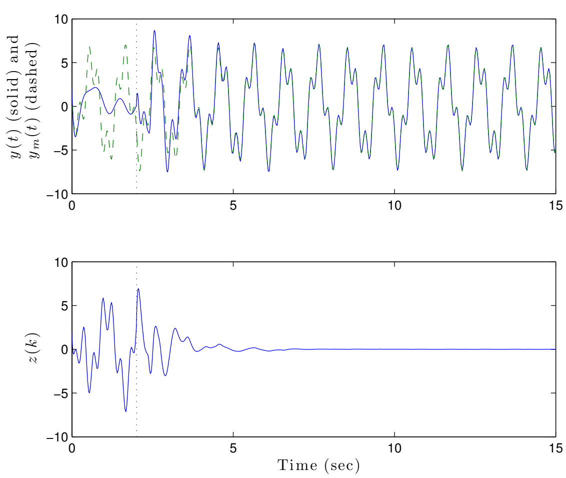


Fig. 3. *Command following and disturbance rejection for an asymptotically stable, nonminimum-phase system:* The surrogate tracking error MRAC (8), (14), and (15) is implemented in feedback with $n_{\rm c} = 8$, $\theta(0) = 0$, and $P(0) = 10^{12} I_{17}$. The controller drives $z$ to zero. Thus, the controller forces $y$ to follow $y_{\rm m}$ asymptotically, while rejecting the disturbance $w$.

### *B. Unstable, nonminimum-phase system*

Consider the unstable, nonminimum-phase system

$$(\mathbf{p}-1)(\mathbf{p}^2+4\mathbf{p}+29)y = -2(\mathbf{p}-0.5)u, \qquad (18)$$

where $y_0 = [-1 \quad -0.1 \quad 0]$. The system (18) has an unstable pole at 1 and a nonminimum-phase zero at 0.5. The system (18) is not strongly stabilizable (i.e., an unstable linear controller is required to stabilize the system) [14]. For this problem, $n = 3$, $n_{\rm u} = 1$, $d = 2$, $\beta_d = -2$, and $\beta_{\rm u}(\mathbf{p}) = \mathbf{p} - 0.5$. Next, consider the reference model (2), where $\alpha_{\rm m}(\mathbf{p}) = (\mathbf{p}+7)^4$, $\beta_{\rm m}(\mathbf{p}) = (\mathbf{p}+5)\beta_{\rm u}(\mathbf{p})[\alpha_{\rm m}(0)/5/\beta_{\rm u}(0)]$, and $r(t) = 2\sin(6\pi t) + \sin(2\pi t)$. The surrogate tracking error MRAC (8), (14), and (15) is implemented in feedback with $n_{\rm c} = 4$ and $P(0) = 10^{14} I_9$. Figure 4 shows that the tracking error $z$ tends to zero. Furthermore, the controller parameters $\theta$ converges numerically. In fact, $\theta$ converges to a value, where the fixed-time controller has an unstable pole (at approximately 5.6), which is required to stabilize (18).

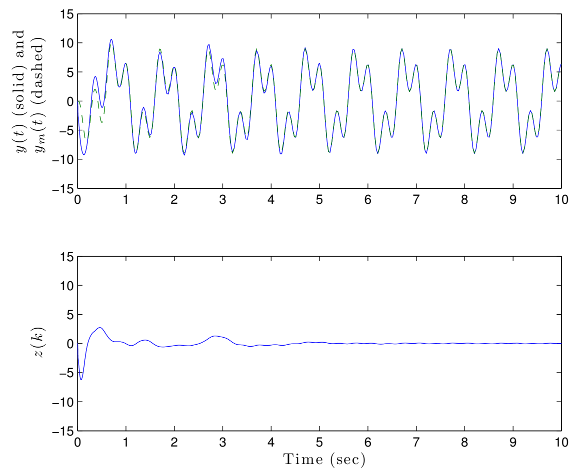


Fig. 4. *Command following for an unstable, nonminimum-phase system:* The surrogate tracking error MRAC (8), (14), and (15) is implemented in feedback with $n_{\rm c} = 4$, $\theta(0) = 0$, and $P(0) = 10^{14} I_9$. The controller drives $z$ to zero.

## V. Ideal Fixed-Gain Controller

The remainder of this paper is dedicated to analyzing the stability properties of the surrogate tracking error MRAC algorithm. In this section, we prove the existence of an ideal fixed-gain controller, which is used in the next section to analyze the closed-loop adaptive system. An ideal fixed-gain controller, whose structure is illustrated in Figure 5, includes four parts, specifically, a precompensator, which cancels the stable zeros $\beta_{\rm s}(\mathbf{p})$; an internal model of the disturbance dynamics $\alpha_w(\mathbf{p})$; feedback controller whose input is $y$; and a feedforward controller whose input is $r$.

For all $t \geq 0$, consider the system (1) with $u(t) = u_*(t)$, where $u_*(t)$ is the signal generated by the ideal fixed-gain controller. More precisely, for all $t \geq 0$, consider the system

$$\alpha(\mathbf{p})y_*(t) = \beta(\mathbf{p})u_*(t) + \gamma(\mathbf{p})w(t), \qquad (19)$$

where $y_{*,0} \triangleq [y_*^{(n-1)}(0) \quad \cdots \quad y_*(0)]$ is the initial condition, and $u_*(t)$ is given by the ideal fixed-gain controller

$$u_*(t) = \sum_{i=1}^{n_{\rm c}} L_{*,i}(t)\bar{y}_{*,i}(t) + \sum_{i=1}^{n_{\rm c}} M_{*,i}(t)\bar{u}_{*,i}(t) + N_*\bar{r}(t) + \varepsilon(t), \qquad (20)$$

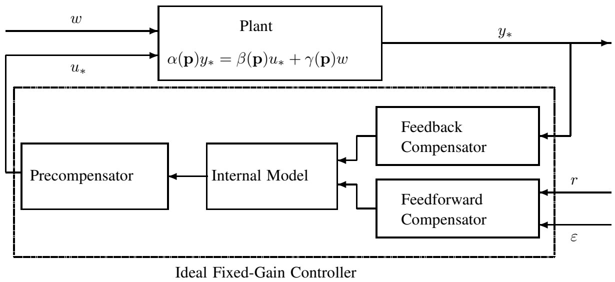


Fig. 5. Schematic diagram of the closed-loop system with the ideal fixed-gain controller.

where $L_{*,1},\dots,L_{*,n_\mathrm{c}} \in \mathbb{R}$; $M_{*,1},\dots,M_{*,n_\mathrm{c}} \in \mathbb{R}$; $N_* \in \mathbb{R}$; for all $t \geq 0$, $\varepsilon(t)$ is an arbitrary signal; and for $i = 1,\dots,n_\mathrm{c}$, $\bar{y}_{*,i}(t)$ and $\bar{u}_{*,i}(t)$ are the signals obtained by passing $y_*(t)$ and $u_*(t)$, respectively, through the filter $G_{a_\mathrm{f},i}(s)$. The ideal controller (20) can be written as

$$u_*(t) = \phi_*^\mathrm{T}(t)\theta_* + \varepsilon(t), \tag{21}$$

where

$$\theta_* \triangleq \begin{bmatrix} L_{*,1} & \cdots & L_{*,n_\mathrm{c}} & M_{*,1} & \cdots & M_{*,n_\mathrm{c}} & N_* \end{bmatrix},$$
$$\phi_*(t) \triangleq \begin{bmatrix} \bar{y}_{*,1}(t) & \cdots & \bar{y}_{*,n_\mathrm{c}}(t) & \bar{u}_{*,1}(t) & \cdots & \bar{u}_{*,n_\mathrm{c}}(t) & \bar{r}(t) \end{bmatrix}.$$

**Theorem 2.** *Let $n_\mathrm{c} \geq \max(2n - n_\mathrm{u} - d, n_\mathrm{m} - n_\mathrm{u} - d)$. Then there exists an ideal fixed-gain controller (20), such that, for all initial conditions $y_{*,0}$ and all $t \geq 0$,*

$$\alpha_\mathrm{m}(\mathbf{p})c_\mathrm{f}(\mathbf{p})y_*(t) = \beta_\mathrm{m}(\mathbf{p})c_\mathrm{f}(\mathbf{p})r(t) + \beta_d\beta_\mathrm{u}(\mathbf{p})a_\mathrm{f}(\mathbf{q})\varepsilon(t). \tag{22}$$

*Proof.* We construct the ideal fixed-gain controller (20), which is depicted in Figure 5, and show that it satisfies (22). First, it follows from (5), (6), and (20) that the ideal control (20) satisfies

$$M_*(\mathbf{p})u_*(t) = L_*(\mathbf{p})y_*(t) + N_*\beta_\mathrm{c}(\mathbf{p})c_\mathrm{f}(\mathbf{p})r(t) + a_\mathrm{f}(\mathbf{p})\varepsilon(t), \tag{23}$$

where $L_*(\mathbf{p}) \triangleq L_{*,1}\mathbf{p}^{n_\mathrm{c}-1} + \cdots + L_{*,n_\mathrm{c}-1}\mathbf{p} + L_{*,n_\mathrm{c}}$, and $M_*(\mathbf{p}) \triangleq a_\mathrm{f}(\mathbf{p}) - (M_{*,1}\mathbf{p}^{n_\mathrm{c}-1} + \cdots + M_{*,n_\mathrm{c}-1}\mathbf{p} + M_{*,n_\mathrm{c}})$. Since $a_\mathrm{f}(\mathbf{q})$ is a monic polynomial with degree $n_\mathrm{c}$, it follows that that choice of $M_{*,1},\dots,M_{*,n_\mathrm{c}} \in \mathbb{R}$ uniquely determines $M_*(\mathbf{p})$ and admits all possible monic polynomials with degree $n_\mathrm{c}$. Therefore, it suffices to show that there exists $L_*(\mathbf{p})$, $M_*(\mathbf{p})$, and $N_*$, such that (22) is satisfied.

Next, let

$$M_*(\mathbf{p}) = \bar{M}_*(\mathbf{p})\alpha_w(\mathbf{p})\beta_\mathrm{s}(\mathbf{p}), \tag{24}$$

where $\bar{M}_*(\mathbf{p})$ is a monic polynomial with degree $n_1 \triangleq n_\mathrm{c} - n_w - n_\mathrm{s}$. Now, it suffices to show that there exists $N_*$, $L_*(\mathbf{p})$, and $\bar{M}_*(\mathbf{p})$, such that (22) is satisfied.

Next, it follows from (3) and (19) that

$$\alpha(\mathbf{p})y_*(t) = \beta_d\beta_\mathrm{u}(\mathbf{p})\beta_\mathrm{s}(\mathbf{p})u_*(t) + \gamma(\mathbf{p})w(t). \tag{25}$$

Multiplying (25) by $\bar{M}_*(\mathbf{p})\alpha_w(\mathbf{p})$ and using (24) yields

$$\bar{M}_*(\mathbf{p})\alpha_w(\mathbf{p})\alpha(\mathbf{p})y_*(t) = \bar{M}_*(\mathbf{p})\gamma(\mathbf{p})\alpha_w(\mathbf{p})w(t) + \beta_d\beta_\mathrm{u}(\mathbf{p})M_*(\mathbf{p})u_*(t). \tag{26}$$

Since (A6) implies that $\bar{M}_*(\mathbf{p})\gamma(\mathbf{p})\alpha_w(\mathbf{p})w(t) = 0$, it follows from (23) and (26) that

$$[\bar{M}_*(\mathbf{p})\alpha_w(\mathbf{p})\alpha(\mathbf{p}) - \beta_d\beta_\mathrm{u}(\mathbf{p})L_*(\mathbf{p})]y_*(t) = \beta_d N_*\beta_\mathrm{u}(\mathbf{p})\beta_\mathrm{c}(\mathbf{p})c_\mathrm{f}(\mathbf{p})r(t) + \beta_d\beta_\mathrm{u}(\mathbf{p})a_\mathrm{f}(\mathbf{p})\varepsilon(t).$$

Since $\beta_\mathrm{m}(\mathbf{p}) = \beta_\mathrm{u}(\mathbf{p})\beta_\mathrm{c}(\mathbf{p})$, letting $N_* = 1/\beta_d$ implies

$$[\bar{M}_*(\mathbf{p})\alpha_w(\mathbf{p})\alpha(\mathbf{p}) - \beta_d\beta_\mathrm{u}(\mathbf{p})L_*(\mathbf{p})]y_*(t) = \beta_\mathrm{m}(\mathbf{p})c_\mathrm{f}(\mathbf{p})r(t) + \beta_d\beta_\mathrm{u}(\mathbf{p})a_\mathrm{f}(\mathbf{p})\varepsilon(t). \tag{27}$$

Next, we show that there exist polynomials $L_*(\mathbf{p})$ and $\bar{M}_*(\mathbf{p})$ such that $\bar{M}_*(\mathbf{p})\alpha_w(\mathbf{p})\alpha(\mathbf{p}) - \beta_d\beta_\mathrm{u}(\mathbf{p})L_*(\mathbf{p}) = \alpha_\mathrm{m}(\mathbf{p})c_\mathrm{f}(\mathbf{p})$. First, note that deg $\bar{M}_*(\mathbf{p})\alpha_w(\mathbf{p})\alpha(\mathbf{p}) = n_1 + n_w + n = n_\mathrm{c} + n_\mathrm{u} + d =$ deg $\alpha_\mathrm{m}(\mathbf{p})$ + deg $c_\mathrm{f}(\mathbf{p})$. Next, if $n_w + n_\mathrm{s} > 0$, then let $L_{*,1},\dots,L_{*,n_w+n_\mathrm{s}} = 0$, which implies that $L_*(\mathbf{p})$ is a polynomial with degree $n_\mathrm{c} - n_w - n_\mathrm{s} - 1 = n_1 - 1$. Thus, deg $\bar{M}_*(\mathbf{p}) = n_1$, deg $L_*(\mathbf{p}) = n_1 - 1$, and deg $\alpha_w(\mathbf{p})\alpha(\mathbf{p}) = n_w + n \leq n_\mathrm{c} - n - n_w + n_\mathrm{u} + d = n_1$. Since, in addition, assumptions (A1) and (A6) imply that $\alpha_w(\mathbf{p})\alpha(\mathbf{p})$ and $\beta_\mathrm{u}(\mathbf{p})$ are coprime, it follows from the Diophantine equation that the roots of $\bar{M}_*(\mathbf{p})\alpha_w(\mathbf{p})\alpha(\mathbf{p}) - \beta_\mathrm{u}(\mathbf{p})L_*(\mathbf{p})$ can be assigned arbitrarily by choice of $L_*(\mathbf{p})$ and $\bar{M}_*(\mathbf{p})$. Therefore, there exist polynomials $L_*(\mathbf{p})$ and $\bar{M}_*(\mathbf{p})$ such that $\bar{M}_*(\mathbf{p})\alpha_w(\mathbf{p})\alpha(\mathbf{p}) - \beta_d\beta_\mathrm{u}(\mathbf{p})L_*(\mathbf{p}) = \alpha_\mathrm{m}(\mathbf{p})c_\mathrm{f}(\mathbf{p})$. Thus, for all $t \geq 0$, (27) becomes (22). □

## VI. Preliminary Stability Analysis

In this section, we analyze the surrogate tracking error MRAC (8), (14), and (15). First, let $\theta_* \in \mathbb{R}^{2n_\mathrm{c}+1}$ be the ideal fixed-gain controller given by Theorem 2, and define the estimation error $\tilde{\theta}(t) \triangleq \theta(t) - \theta_*$. The following result relates $z_\mathrm{f}(t)$ to $u_\mathrm{f}(t)$, $\Phi(t)$, and $\theta_*$.

**Lemma 1.** *Consider the open-loop system (1) with the feedback (8). Then, for all initial conditions $y_0$, all sequences $\theta(t)$, and, all $t \geq 0$,*

$$z_\mathrm{f}(t) = u_\mathrm{f}(t) - \Phi^\mathrm{T}(t)\theta_*. \tag{28}$$

*Proof.* Adding and subtracting $\phi^\mathrm{T}(t)\theta_*$ to $u(t)$ yields $u(t) = \sum_{i=1}^{n_\mathrm{c}} L_{*,i}(t)\bar{y}_i(t) + \sum_{i=1}^{n_\mathrm{c}} M_{*,i}(t)\bar{u}_i(t) + N_*(t)\bar{r}(t) + u(t) - \phi^\mathrm{T}(t)\theta_*$, which has the same form as (20) with with $\varepsilon(t) = u(t) - \phi^\mathrm{T}(t)\theta_*$. Thus, it follows from Theorem 2 (with $\varepsilon(t) = u(t) - \phi^\mathrm{T}(t)\theta_*$) that, for all $t \geq 0$, $\alpha_\mathrm{m}(\mathbf{p})c_\mathrm{f}(\mathbf{p})y(t) = \beta_\mathrm{m}(\mathbf{p})c_\mathrm{f}(\mathbf{p})r(t) + \beta_d\beta_\mathrm{u}(\mathbf{p})a_\mathrm{f}(\mathbf{p})\left[u(t) - \phi^\mathrm{T}(t)\theta_*\right]$. Subtracting $\alpha_\mathrm{m}(\mathbf{p})c_\mathrm{f}(\mathbf{p})y_\mathrm{m}(t)$ from both sides and using (2) yields $\alpha_\mathrm{m}(\mathbf{p})c_\mathrm{f}(\mathbf{p})z(t) = \beta_d\beta_\mathrm{u}(\mathbf{p})a_\mathrm{f}(\mathbf{p})u(t) - \beta_d\beta_\mathrm{u}(\mathbf{p})a_\mathrm{f}(\mathbf{p})\phi^\mathrm{T}(t)\theta_*$, and applying the filter $\frac{1}{b_\mathrm{f}(s)}$ to each term yields (28). □

Lemma 1 relates $z_\mathrm{f}(t)$ to $\theta_*$ but does not relate $z_\mathrm{f}(t)$ to $\tilde{\theta}(t)$. However, the next result follows from (11) and (28), and shows that $z_\mathrm{s}(t)$ is a linear regression in $\tilde{\theta}(t)$.

**Lemma 2.** *Consider the open-loop system (1) with the feedback (8). Then, for all initial conditions $y_0$, all sequences $\theta(t)$, and, all $t \geq 0$,*

$$z_\mathrm{s}(t) = \Phi^\mathrm{T}(t)\tilde{\theta}(t). \tag{29}$$

The following theorem is the main result of the paper and provides properties of the surrogate tracking error MRAC algorithm (8), (14), and (15).

**Theorem 3.** *Consider the open-loop system (1) satisfying assumptions (A1)-(A9), and the surrogate tracking error model reference adaptive controller (8), (14), and (15), where $n_{\rm c}$ satisfies (4). Then, for all initial conditions $y_0$, $\theta(0)$, and $P(0)$ the following properties hold:*

*(i)* $\theta(t)$ *and* $P(t)$ *are bounded.*

*(ii)* $\lim_{t\to\infty} P(t)$ *exists.*

*(iii)* $\lim_{t\to\infty} \theta(t)$ *exists.*

*(iv)* $\int_0^\infty \eta(t) z_{\rm s}^2(t) dt$ *exists.*

*Proof.* Since $\frac{d}{dt}\left[P(t)P^{-1}(t)\right] = \dot{P}(t)P^{-1}(t) + P(t)\frac{d}{dt}\left[P^{-1}(t)\right] = 0$, it follows from (15) that

$$\frac{d}{dt}\left[P^{-1}(t)\right] = -P^{-1}(t)\dot{P}(t)P^{-1}(t) = \eta(t)\Phi(t)\Phi^{\rm T}(t). \tag{30}$$

Next, define the positive-definite Lyapunov-like function $V(\tilde{\theta}(t), P(t)) \stackrel{\triangle}{=} \tilde{\theta}^{\rm T}(t)P^{-1}(t)\tilde{\theta}(t)$. Evaluating the derivative of $V(\tilde{\theta}(t), P(t))$ along the trajectories of (14) and (30) yields

$$\begin{aligned}\dot{V}(\tilde{\theta}(t), P(t)) &= 2\tilde{\theta}^{\rm T}(t)P^{-1}(t)\dot{\theta}(t) + \tilde{\theta}^{\rm T}(t)\frac{d}{dt}\left[P^{-1}(t)\right]\tilde{\theta}(t)\\ &= -\eta(t)\tilde{\theta}^{\rm T}(t)\Phi(t)\left[2z_{\rm s}(t) - \Phi^{\rm T}(t)\tilde{\theta}(t)\right].\end{aligned}$$

Next, it follows from Lemma 2 that

$$\dot{V}(\tilde{\theta}(t), P(t)) = -\eta(t) z_{\rm s}^2(t). \tag{31}$$

Since $V$ is a positive-definite radially unbounded function of $\tilde{\theta}(t)$ and $\dot{V}$ is non-positive, it follows that $\tilde{\theta}(t)$ is bounded and thus $\theta(t)$ is bounded.

To show that $P(t)$ is bounded, it follows from (30) that $P^{-1}(t) = P^{-1}(0) + \int_0^t \eta(\tau)\Phi(\tau)\Phi^{\rm T}(\tau)d\tau$. Since $P(0)$ is positive definite, it follows that, for all $t \ge 0$ and all $\delta \ge 0$, $0 < P^{-1}(t) \le P^{-1}(t+\delta)$. Consequently, for all $t \ge 0$ and all $\delta \ge 0$, $0 < P(t+\delta) \le P(t)$. Therefore, $0 < P(t) \le P(0)$, which implies that $P(t)$ is bounded and verifies *(i)*.

Next, we show *(iv)*. Since $V$ is positive definite and $\dot{V}$ is non-positive, it follows from (31) that

$$\begin{aligned}0 \le \int_0^\infty \eta(t) z_{\rm s}^2(t) dt &= -\int_0^\infty \dot{V}(\tilde{\theta}(t), P(t)) dt\\ &= V(\tilde{\theta}(0), P(0)) - \lim_{t\to\infty} V(\tilde{\theta}(t), P(t))\\ &\le V(\tilde{\theta}(0), P(0)),\end{aligned}$$

which verifies *(iv)*.

To show *(ii)*, let $q \in \mathbb{R}^{2n_{\rm c}+1}$ and define $f_q(t) \stackrel{\triangle}{=} q^{\rm T}P(t)q$. Since $P(t)$ is nonincreasing and positive definite, it follows that $f_q(t)$ is nonincreasing and bounded from below. Thus, $\lim_{t\to\infty} f_q(t)$ exists. Next, for $i = 1, \ldots, 2n_{\rm c}+1$, let $q = e_i$, where $e_i$ is the $i$th column of the $(2n_{\rm c}+1) \times (2n_{\rm c}+1)$ identity matrix. Thus, each entry along the diagonal of $P(t)$ converges. Next, for $i = 2, \ldots, 2n_{\rm c}+1$, let $q = e_1 + e_i$, and it follows that each entry in the first column (and row) of $P(t)$ converges. Next, for $i = 3, \ldots, 2n_{\rm c}+1$, let $q = e_1 + e_2 + e_i$, and it follows that each entry in the second column (and row) of $P(t)$ converges. Continuing this process yields that each entry of $P(t)$ converges. Thus, $P(\infty) \stackrel{\triangle}{=} \lim_{t\to\infty} P(t)$ exists, which confirms *(ii)*.

To show *(iii)*, it follows from (14), (29), and (30) that $\dot{\theta}(t) = -\eta(t)P(t)\Phi(t)\Phi^{\rm T}(t)\tilde{\theta}(t) = -P(t)\frac{d}{dt}[P^{-1}(t)]\tilde{\theta}(t)$, which implies that $\frac{d}{dt}[P^{-1}(t)\tilde{\theta}(t)] = P^{-1}(t)\dot{\theta}(t) + \frac{d}{dt}[P^{-1}(t)]\tilde{\theta}(t) = 0$. Integrating from 0 to $t$ yields $P^{-1}(t)\tilde{\theta}(t) = P^{-1}(0)\tilde{\theta}(0)$, which implies $\tilde{\theta}(t) = P(t)P^{-1}(0)\tilde{\theta}(0)$. Therefore, $\lim_{t\to\infty}\tilde{\theta}(t) = \lim_{t\to\infty} P(t)P^{-1}(0)\tilde{\theta}(0) = P(\infty)P^{-1}(0)\tilde{\theta}(0)$. Finally, since $\lim_{t\to\infty}\tilde{\theta}(t)$ exists and $\theta(t) = \tilde{\theta}(t) + \theta_*$, it follows that $\lim_{t\to\infty}\theta(t)$ exists, which confirms *(iii)* □

Theorem 3 provides preliminary stability properties for the surrogate tracking error MRAC algorithm (8), (14), and (15). Although, numerical simulations demonstrate that the tracking error $z(t)$ tends to zero, a proof of this results remains open. However, techniques related to those used in [12], [13] may provide the tools required to prove that the tracking error tends to zero.

## VII. Conclusions

This paper presented a direct MRAC algorithm for continuous-time systems that are possibly nonminimum phase. The adaptive controller requires knowledge of the first nonzero Markov parameter and the nonminimum-phase zeros of the transfer function from the control to the output. The present paper provided the construction and preliminary stability analysis of the surrogate tracking error MRAC algorithm.

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