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Moment matching for linear port Hamiltonian systems

T. C. Ionescu, A. Astolfi

Abstract— **The problem of moment matching with preservation of port Hamiltonian structure is tackled. Based on the time-domain approach to linear moment matching, we characterize the (subset of) port Hamiltonian models from the set of parameterized models that match the moments of a given port Hamiltonian system, at a set of finite points. We also discuss the problem of finding port Hamiltonian reduced order models that match the Markov parameters of a given port Hamiltonian system.**

I. Introduction

Port Hamiltonian systems represent an important class of systems used in modeling, analysis and control of physical systems, see e.g. [1], [2]. Physical modeling often leads to systems of high dimension, usually difficult to analyze and simulate and unsuitable for control design.

In the problem of model reduction, moment matching techniques represent an efficient tool, see e.g. [3], [4], [5], [6], [7] for a complete overview for linear systems. With such techniques, the (reduced order) model is obtained by constructing a lower degree rational function that approximates a given transfer function (assumed rational). The low degree rational function matches the given transfer function at various points in the complex plane. Recently in [8], [9], [10], [11], Krylov methods have been applied to linear port Hamiltonian systems, resulting in reduced order models that match the Markov parameters of the given port Hamiltonian system. The procedure therein involves finding a change of coordinates such that the Hamiltonian becomes the square of the norm of the state vector, followed by the application of projection methods. The resulting model, that matches the moments of the given system, has coordinates such that the Hamiltonian is again the square of the norm of the state vector.

In this paper, we use the time-domain approach to moment matching from the recent works [12], [13], [14] and [15]. This approach yields a simple and direct parametrization of a complete family of reduced order models achieving moment matching at a set of finite interpolation points. These models depend on a set of free parameters, useful for enforcing properties such as, e.g., passivity, stability [16], relative degree, etc. We characterize the reduced order model that preserves the port Hamiltonian structure and matches the moments of the given port Hamiltonian system. In other words, from the family of models that achieve moment matching, we select the reduced order model that inherits the port Hamiltonian form, by picking a particular member. Furthermore, we obtain a family of parameterized port Hamiltonian systems that match the moments and inherit the structure of the given system. We also discuss the problem of Markov parameters matching, first for general linear systems and then for linear port Hamiltonian systems. Similar to the aforementioned linear moment matching problem, we obtain the family of linear port Hamiltonian models that achieve Markov parameters matching. Computationally, there is no need to calculate moments and any numerically efficient reduction algorithm can be used to determine the class of models that achieve moment matching, e.g., Krylov subspace methods (see also [13, Section II-C]). From this class, we compute the parameter that yields the reduced order model which preserves the port Hamiltonian structure.

The paper is organized as follows. In Section II, we give a brief overview of the definition of moments and moment matching for linear port Hamiltonian systems, as well as of presenting the family of parameterized reduced order models that achieve moment matching at a set of finite interpolation points. In Section III, we discuss the problem of moment matching with preservation of the port Hamiltonian structure, and characterize the port Hamiltonian reduced order models. Furthermore, we give a necessary and sufficient condition for a reduced order model that achieves moment matching to be a port Hamiltonian model. In Section IV, we tackle the problem of finding the reduced order models that match a set of Markov parameters of the given system, first for general linear systems and then for port Hamiltonian systems. The paper is completed by Conclusions.

II. Preliminaries

Let $J \in \mathbb{R}^{n\times n}$ be a skew symmetric matrix and $R \in \mathbb{R}^{n\times n}$, $Q \in \mathbb{R}^{n\times n}$ be two symmetric matrices. Consider the single-input, single-output, port Hamiltonian system

$$\Sigma : \begin{cases} \dot{x} = (J-R)Qx + Bu, \\ y = B^*Qx, \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $B \in \mathbb{R}^n$. The Hamiltonian is $\mathcal{H}(x) = \frac{1}{2}x^*Qx$.

Assumption 1. Q is invertible and $R \geq 0$.

The transfer function of system (1) is given by $K(s) = B^*Q(sI-(J-R)Q)^{-1}B$. Let $s_i \in \mathbb{C} - \sigma((J-R)Q)$. The moments of (1) at s_i are $\eta_0(s_i), \eta_1(s_i), \dots$, with $\eta_k(s_i) =$

T. C. Ionescu is with the Department of Electrical and Electronic Engineering, Imperial College London, SW7 2AZ, London, UK. E-mail: t.ionescu@imperial.ac.uk.

A. Astolfi is with the Department of Electrical and Electronic Engineering, Imperial College London, SW7 2AZ, London, UK and with Dipartimento di Informatica, Sistemi e Produzione, Università di Roma Tor Vergata, 00133 Roma, Italy. E-mail: a.astolfi@imperial.ac.uk.

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$\frac{(-1)^k}{k!}\,\frac{d^k K(s)}{ds^k}\Big|_{s=s_i}$. Let $L \in \mathbb{R}^{1\times\nu}$ and $S \in \mathbb{R}^{\nu\times\nu}$ be such that the pair (L, S) is observable.

Assumption 2. $\sigma(S) \cap \sigma((J-R)Q) = \emptyset$.

Let $\Pi \in \mathbb{R}^{n\times\nu}$ be the (unique) solution of the Sylvester equation

$$(J-R)Q\Pi + BL = \Pi S. \tag{2}$$

The moments $\eta_k(s_i)$, $k = 0, 1, 2, ...$ of (1) at $\{s_1, ..., s_\nu\} = \sigma(S)$, are in one-to-one relation with $B^*Q\Pi$ (see also [13, Lemma 3]). Let $G \in \mathbb{R}^\nu$.

Assumption 3. G is such that $\sigma(S) \cap \sigma(S - GL) = \emptyset$.

According to [13], a family of reduced order models of dimension ν, parameterized in G, that match the moments of (1) at $\sigma(S)$ is given by

$$\Sigma_G : \begin{cases} \dot{\xi} = (S - GL)\xi + Gu, \\ \psi = B^*Q\Pi\xi, \end{cases} \tag{3}$$

with $\xi(t) \in \mathbb{R}^\nu$.

A reduced order model which preserves the port Hamiltonian structure and matches the moments of (1) at $\{s_1, ..., s_\nu\}$ is obtained by means of Krylov projections as in [17].

Theorem 1. *[17] Consider system (1) and let $x = V\xi$, where $V \in \mathbb{R}^{n\times\nu}$ spans a Krylov subspace. Then, the reduced order port Hamiltonian system that matches the moments of (1) is*

$$\Sigma_V : \begin{cases} \dot{\xi} = (J_r - R_r)Q_r\xi + B_r u, \\ \psi = B_r^* Q_r \xi, \end{cases} \tag{4}$$

with

$$\begin{aligned} &J_r = V^*QJQV,\ R_r = V^*QRQV, \\ &Q_r = (V^*QV)^{-1},\ B_r = V^*QB. \end{aligned} \tag{5}$$

□

Let $\mathcal{Q} \in \mathbb{R}^{\nu\times\nu}$ and $\mathcal{R} \in \mathbb{R}^\nu$ be such that the pair $(\mathcal{Q}, \mathcal{R})$ is controllable.

Assumption 4. $\sigma(\mathcal{Q}) \cap \sigma((J-R)Q) = \emptyset$.

Let $\Upsilon \in \mathbb{R}^{\nu\times n}$ be the (unique) solution of the Sylvester equation

$$\mathcal{Q}\Upsilon = \Upsilon(J-R)Q + \mathcal{R}B^*Q. \tag{6}$$

The moments of (1) at $\{s_{\nu+1}, ..., s_{2\nu}\} = \sigma(\mathcal{Q})$ are in one-to-one relation with ΥB (see [18], [19]). Let $H \in \mathbb{R}^{1\times\nu}$.

Assumption 5. H is such that $\sigma(\mathcal{Q}) \cap \sigma(\mathcal{Q} - \mathcal{R}H) = \emptyset$.

According to [18] a family of reduced models of order ν, parameterized in H, that match the moments of (1) at $\sigma(\mathcal{Q})$ is given by

$$\Sigma_H : \begin{cases} \dot{\xi} = (\mathcal{Q} - \mathcal{R}H)\xi + \Upsilon Bu, \\ \psi = H\xi, \end{cases} \tag{7}$$

with $\xi(t) \in \mathbb{R}^\nu$.

III. Matching with preservation of the port Hamiltonian structure

In this section, given a port Hamiltonian system, we discuss the problem of finding a reduced order port Hamiltonian model that achieves moment matching at a set of finite interpolation points, i.e., we perform port Hamiltonian structure preservation moment matching. Throughout the rest of this section, we consider that Assumptions 1 to 5 hold.

Proposition 1. *Consider system (1). Let (L, S) be an observable pair and Π be the unique solution of (2). A port Hamiltonian reduced order model achieving moment matching at $\sigma(S)$ is given by*

$$\Sigma_\Pi : \begin{cases} \dot{\xi} = (\widetilde{J} - \widetilde{R})\widetilde{Q}\xi + \widetilde{B}u, \\ \psi = \widetilde{B}^*\widetilde{Q}\xi, \end{cases} \tag{8}$$

with $\xi(t) \in \mathbb{R}^\nu$ and

$$\begin{aligned} &\widetilde{J} = \Pi^*QJQ\Pi,\ \widetilde{R} = \Pi^*QRQ\Pi, \\ &\widetilde{Q} = (\Pi^*Q\Pi)^{-1},\ \widetilde{B} = \Pi^*QB. \end{aligned} \tag{9}$$

□

Remark 1. Let (8) be a reduced order model of (1). Then, according to [13], [19], model (8) matches the moments of (1) at $\sigma(S)$ if there exists an invertible matrix $P \in \mathbb{R}^{\nu\times\nu}$ such that

$$\begin{aligned} &(\widetilde{J} - \widetilde{R})\widetilde{Q}P + \widetilde{B}L = PS, \\ &B^*Q\Pi = \widetilde{B}^*\widetilde{Q}P. \end{aligned} \tag{10}$$

Conditions (10) hold for $P = \widetilde{Q}^{-1} = \Pi^*Q\Pi$. Furthermore, plugging P into (10) yields equation (2). □

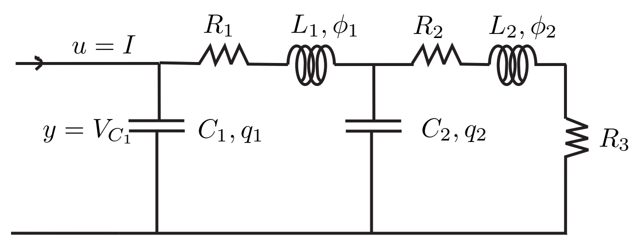

Fig. 1. Ladder network

Example 1. Consider the ladder network in Fig. 1, with C_1, C_2, L_1, L_2, R_1, R_2 the capacitances, inductances, and resistances of the corresponding capacitors, inductors, and resistors. The port Hamiltonian representation of this system is given by (1) with $x = [q_1\ \phi_1\ q_2\ \phi_2]^*$ and

$$J = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix},\ R = \mathrm{diag}\{0, R_1, 0, R_2 + R_3\},$$

$$Q = \mathrm{diag}\left\{\frac{1}{C_1}, \frac{1}{L_1}, \frac{1}{C_2}, \frac{1}{L_2}\right\},\ B = [1\ 0\ 0\ 0]^*. \tag{11}$$

Assume $C_1 = C_2$, $L_1 = L_2$, $R_1 = R_2 = R_3$

and $C_1 \neq 0$, $L_1 \neq 0$, $R_1 \neq 0$. The transfer function of the port Hamiltonian system (11) is $K(s) = \frac{L_1^2 C_1 s^3 + 3L_1 R_1 C_1 s^2 + \left(2L_1 + 2R_1^2 C_1\right)s + 3R_1}{C_1^2 L_1^2 s^4 + 3C_1^2 L_1 R_1 s^3 + \left(3C_1 L_1 + 2C_1^2 R_1^2\right)s^2 + 5R_1 C_1 s + 1}$. The first two moments of (11) at 0 are $\eta_0 = 3R_1$ and $\eta_1 = 2L_1 - 13R_1^2 C_1$. Let $L = [1\ 0]$ and $S = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Solving equation (2) yields

$$\Pi = \begin{bmatrix} 3R_1C_1 & C_1(L_1 - 13C_1R_1^2) \\ L_1 & -3R_1L_1C_1 \\ 2R_1C_1 & C_1(L_1 - 10R_1^2C_1) \\ L_1 & -5R_1L_1C_1 \end{bmatrix}.$$

A reduced order port Hamiltonian model that matches these two moments is given by (8) with

$$\begin{aligned} &\widetilde{J} = \begin{bmatrix} 0 & 2L_1 \\ -2L_1 & 0 \end{bmatrix}, \ \widetilde{R} = R_1 \begin{bmatrix} 3 & -13R_1C_1 \\ -13R_1C_1 & 59R_1^2C_1^2 \end{bmatrix}, \\ &\widetilde{Q} = \frac{1}{10L_1^3 - R_1^2C_1(11L_1^2 - 44R_1^2L_1 - 16R_1^4)} \\ &\cdot \begin{bmatrix} 5L_1^2 - R_1^2C_1(38L_1 - 269R_1^2) & 59R_1^3C_1 \\ 59R_1^3C_1 & 13R_1^2 + 2\frac{L_1}{C_1} \end{bmatrix}, \\ &\widetilde{B} = [3R_1 \ \ 2L_1 - 13R_1^2C_1]^*. \end{aligned} \tag{12}$$

The transfer function of the reduced order model is $K_\Pi(s) = \frac{a}{b}\frac{s + \frac{d}{a}}{s^2 + \frac{c}{b}s + \frac{e}{b}}$, with a, b, c given by

$$\begin{aligned} a &= R_1^2C_1(16C_1^2 + 28R_1^2L_1C_1 - 7L_1^2) + 8L_1^3, \\ b &= C_1(10L_1^3 - R_1^2C_1(11L_1^2 - 44R_1^2L_1 - 16R_1^4)), \\ c &= R_1C_1(40R_1^4C_1^2 + 15L_1^2 + 4R_1^2L_1C_1), \\ d &= 12R_1(L_1^2 + 2R_1^4C_1^2), \\ e &= 4(L_1^2 + 2R_1^4C_1^2). \end{aligned}$$

□

Remark 2. Let systems (8) and (4) be two reduced order port Hamiltonian models that match the moments of (1). Then there exists an invertible matrix T such that $\Pi T = V$ and $J_r = T^*\widetilde{J}T$, $R_r = T^*\widetilde{R}T$, $Q_r = T^{-*}\widetilde{Q}T^{-1}$ and $B_r = T^*\widetilde{B}$. □

Remark 3. Let $L = [l_1\ l_2\ \ldots\ l_\nu]$ be such that (L, S) is observable. Then the solution of the Sylvester equation (2) is given by a matrix $\Pi(L)$, yielding a family of reduced order port Hamiltonian models $\Sigma_{\Pi(L)}$ defined by (8) with $\widetilde{J}(L)$, $\widetilde{R}(L), \widetilde{Q}(L), \widetilde{B}(L)$, as in (9). Note that the input output behaviour is not affected by the choice of l_1, ..., l_ν, i.e., all models parameterized in L have the same transfer function. However, since the port Hamiltonian structure is a state-space property, the parameters l_i, $i = 1, ..., \nu$ can be used to enforce state-space/physical properties, e.g. the reduced order Hamiltonian defined by $\widetilde{Q}$, or the reduced order dissipation matrix $\widetilde{R}$, have a desired form. □

Example 2. Consider the ladder network described in Example 1 and let $C_1 = 1, C_2 = 2, L_1 = L_2 = 1$ and $R_1 = R_2 = R_3 = 1$. Furthermore, let $L = [l_1\ l_2]$, $l_1 \in \mathbb{R}$, $l_2 \in \mathbb{R}$ and $S = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. (L, S) is observable if and only if $l_1 \neq 0$. Note that $\Pi(l_1, l_2) = \begin{bmatrix} 3l_1 & l_1 & l_1 & l_1 \\ 3l_2 - l_1 & l_2 - 3l_1 & l_2 - \frac{7}{2}l_1 & l_2 - 4l_1 \end{bmatrix}^*$. The family of port Hamiltonian models, parameterized in l_1 and l_2 is given by

$$\begin{aligned} &\widetilde{J}(l_1, l_2) = \begin{bmatrix} 0 & 2l_1^2 \\ -2l_1^2 & 0 \end{bmatrix}, \\ &\widetilde{R}(l_1, l_2) = \begin{bmatrix} 3l_1^2 & 3l_2l_1 - 11l_1^2 \\ 3l_2l_1 - 11l_1^2 & 3l_2^2 - 22l_1l_2 + 41l_1^2 \end{bmatrix}, \\ &\widetilde{Q}(l_1, l_2) = \frac{1}{31l_1^4}\begin{bmatrix} 26l_2^2 - 164l_1l_2 + 261l_1^2 & 2l_1(41l_1 - 13l_2) \\ 2l_1(41l_1 - 13l_2) & 26l_1^2 \end{bmatrix}, \\ &\widetilde{B}(l_1, l_2) = [3l_1 \ \ 3l_2 - 9l_1]^*. \end{aligned} \tag{13}$$

For $l_2 = \frac{41}{13}l_1$, we obtain the subfamily of reduced order models with the following properties: they match the first two moments of (11) at 0, preserve the port Hamiltonian structure of the model and have diagonalized Hamiltonian. For $l_2 = \frac{11}{3}l_1$, we obtain a subfamily of port Hamiltonian reduced order models with diagonal dissipation matrix. All the parameterized models have the same input-output behaviour described by the transfer function $K_{\Pi(l_1,l_2)}(s) = \frac{9(3s+4)}{31s^2+45s+12}$. □

We now show that (8) is a subset of the family of reduced order models (3), obtained employing a particular choice of the parameter G.

Theorem 2. *Let (3) be a reduced order model of (1). Then (3) is equivalent*[1] *to a port Hamiltonian system (8), i.e. (3) preserves the port Hamiltonian structure of (1), if and only if* $G = (\Pi^*Q\Pi)^{-1}\Pi^*QB$. □

Remark 4. Theorem 2 offers a way to find a reduced order port Hamiltonian model, from a reduced order model that achieves matching of moments of the given (port Hamiltonian) system, by selecting the parameter G. Let (3) be a reduced order model and let P be such that $S^*P + PS \leq \Pi^*QBL + L^*B^*Q\Pi$. Then, according to [13, Theorem 4], there exists G such that the model is passive, i.e., $PG = \Pi^*QB$. If (3) is minimal then $0 < P_a \leq P \leq P_r$. From this set of matrices, $P = \Pi^*Q\Pi$ is the choice that gives the G which identifies the port Hamiltonian reduced order model that achieves moment matching and preserves the structure of the given system. □

Remark 5. The result in Theorem 2 is consistent with the result showing the equivalence between the family of reduced order models obtained by projection and the family of reduced order models obtained by time-domain moment matching, see [20]. In detail, let Σ_G and Σ_V be two reduced order models of (1). Then selecting $T = \Pi^*Q\Pi$, yields $\Pi^*Q\Pi(S - GL) = \widetilde{W}^*(J - R)Q\widetilde{V}\Pi^*Q\Pi$, $\Pi^*Q\Pi G = \widetilde{W}^*B$ and $B^*Q\Pi = B^*Q\widetilde{V}\Pi^*Q\Pi$, which shows that one port

[1] Two systems described by state-space equations are called equivalent if they have the same transfer functions, i.e., the same input-output behaviour.

Hamiltonian model can be obtained from the other via a coordinate transformation. $\square$

Proposition 2. *Consider system (1). Let $(\mathcal{Q}, R)$ be an observable pair and Υ be the unique solution of equation (6). A port Hamiltonian reduced order model achieving moment matching at $\sigma(\mathcal{Q})$ is given by*

$$\Sigma_\Upsilon : \begin{cases} \dot{\xi} = (\widetilde{J} - \widetilde{R})\widetilde{Q}\xi + \widetilde{B}u, \\ \psi = \widetilde{B}^*\widetilde{Q}\xi, \end{cases} \tag{14}$$

with $\xi(t) \in \mathbb{R}^\nu$ and

$$\begin{aligned} &\widetilde{J} = \Upsilon J \Upsilon^*, \ \widetilde{R} = \Upsilon R \Upsilon^*, \\ &\widetilde{Q} = (\Upsilon Q^{-1}\Upsilon^*)^{-1}, \ \widetilde{B} = \Upsilon B. \end{aligned} \tag{15}$$

$\square$

Remark 6. Let $\mathcal{R} = [r_1 \ r_2 \ \dots \ r_\nu]^*$ be such that $(\mathcal{Q}, \mathcal{R})$ is controllable. Then the solution of the Sylvester equation (6) is given by a matrix $\Upsilon(\mathcal{R})$, yielding a family of reduced order port Hamiltonian models $\Sigma_{\Upsilon(\mathcal{R})}$ defined by equation (14) with $\widetilde{J}(\mathcal{R}), \widetilde{R}(\mathcal{R}), \widetilde{Q}(\mathcal{R}), \widetilde{B}(\mathcal{R})$ as in (15). All models parameterized in $\mathcal{R}$ have the same transfer function. Since the port Hamiltonian structure is a state-space property, the parameters r_i, $i = 1, ..., \nu$ can be used to enforce state-space/physical properties. $\square$

Example 3. Consider the ladder network described in Example 2 and let $\mathcal{R} = [r_1 \ r_2]^*$, $r_1 \in \mathbb{R}$, $r_2 \in \mathbb{R}$. $(\mathcal{Q}, \mathcal{R})$ is controllable if and only if $r_1 \neq 0$. Solving (6), we obtain

$$\Upsilon(r_1, r_2) = \begin{bmatrix} 3r_1 & -r_1 & 2r_1 & -r_1 \\ 3r_2 - r_1 & 3r_1 - r_2 & 2r_2 - 7r_1 & 4r_1 - r_2 \end{bmatrix}.$$

The family of port Hamiltonian models, all with the transfer function $K_\Upsilon(s) = \frac{9(3s+2)}{32s^2+27s+6}$, parameterized in r_1 and r_2 is given by

$$\begin{aligned} &\widetilde{J}(r_1, r_2) = \begin{bmatrix} 0 & -2r_1^2 \\ 2r_1^2 & 0 \end{bmatrix}, \\ &\widetilde{R}(r_1, r_2) = \begin{bmatrix} 3r_1^2 & 3r_2r_1 - 11r_1^2 \\ 3r_2r_1 - 11r_1^2 & 3r_2^2 - 22r_1r_2 + 41r_1^2 \end{bmatrix}, \\ &\widetilde{Q}(r_1, r_2) = \tfrac{1}{32r_1^4}\begin{bmatrix} 204r_1^2 - 124r_1r_2 + 19r_2^2 & r_1(62r_1 - 19r_2) \\ r_1(62r_1 - 19r_2) & 19r_1^2 \end{bmatrix}, \\ &\widetilde{B}(r_1, r_2) = [3r_1 \ \ 3r_2 - 9r_1]^*. \end{aligned} \tag{16}$$

For $r_2 = \frac{62}{19}r_1$ we obtain a subfamily of port Hamiltonian reduced order models with diagonalized Hamiltonian. For $r_2 = \frac{11}{3}r_1$ we obtain a subfamily of port Hamiltonian reduced order models with diagonal dissipation matrix. $\square$

We now show that (14) is a member of the family of reduced order models (7), for a particular choice of the parameter H.

Lemma 1. *A family of models (7) contains a passive model if and only if there exists $P = P^* > 0 \in \mathbb{R}^{\nu \times \nu}$ such that $P\mathcal{Q}^* + \mathcal{Q}P \leq \mathcal{R}B^*\Upsilon^* - \Upsilon B\mathcal{R}^*$.* $\square$

The next result shows how to obtain a port Hamiltonian system from a model (7). It is the dual version of [8, Theorem 3].

Lemma 2. *Let (7) be a passive reduced order model of system (1) and let P be as in Lemma 1. Then there exist matrices $\widetilde{J} = \frac{1}{2}[P(\mathcal{Q} - \mathcal{R}H) - (\mathcal{Q} - \mathcal{R}H)^*P]$, $\widetilde{R} = -\frac{1}{2}[P(\mathcal{Q} - \mathcal{R}H) + (\mathcal{Q} - \mathcal{R}H)^*P]$, $\widetilde{Q} = P^{-1}$ and $H = (P^{-1}\Upsilon B)^*$ such that (7) is a port Hamiltonian model described by equations of the form (14).* $\square$

Theorem 3. *Let (3) be a reduced order model of system (1). Then (7) is equivalent to the port Hamiltonian system (14) if and only if $H = B^*\Upsilon^*(\Upsilon Q^{-1}\Upsilon^*)^{-1}$.* $\square$

IV. Markov parameters matching with preservation of the port Hamiltonian structure

A. Matching at $s = \infty$ - The general case

Consider a linear system described by the equations

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned} \tag{17}$$

with $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}$, $u(t) \in \mathbb{R}$. Let $K(s) = C(sI - A)^{-1}B$ be the transfer function. The first $\nu + 1$ Markov parameters are the coefficients of the series expansion of $K(s)$ around $s = \infty$, i.e. they are the first $\nu + 1$ moments of $K(s)$ at ∞, namely

$$\eta_0(\infty) = 0, \ \eta_k(\infty) = CA^{k-1}B, \ k = 1, ..., \nu. \tag{18}$$

Let $\tau \in \mathbb{C}$ and define the function $\widetilde{K}(\tau) = K\left(\frac{1}{\tau}\right)$. Note that $\widetilde{K}(\tau) = C(I - A\tau)^{-1}B\tau$ and $\frac{d^{k+1}\widetilde{K}(\tau)}{d\tau^{k+1}} = (k+1)!C[(I - A\tau)^{-k-1}A^kB + (I - A\tau)^{-k-2}A^{k+1}B\tau]$, yielding

$$\frac{1}{(k+1)!}\frac{d^{k+1}\widetilde{K}(\tau)}{d\tau^{k+1}} = C(I - A\tau)^{-k-2}A^kB.$$

The moments of $\widetilde{K}(\tau)$ at $\tau = \tau^* \in \mathbb{C}$ are given by

$$\widetilde{\eta}_k(\tau^*) = \frac{1}{(k+1)!}\left.\frac{d^{k+1}\widetilde{K}(\tau)}{d\tau^{k+1}}\right|_{\tau=\tau^*} \tag{19}$$

and the moments $\eta_0(\infty), ..., \eta_\nu(\infty)$ are given by

$$\eta_k(\infty) = \frac{1}{(k+1)!}\left.\frac{d^{k+1}\widetilde{K}(\tau)}{d\tau^{k+1}}\right|_{\tau=0} = \widetilde{\eta}_k(0).$$

We now consider the following matching problem. Given the function $\widetilde{K}(\tau)$ and the point $\tau^* \in \mathbb{C}$ find $\hat{K}(\tau)$ such that the first $\nu + 1$ moments at τ^* match, i.e. $\widetilde{\eta}_k(\tau) = \frac{d^{k+1}\widetilde{K}}{d\tau^{k+1}}(\tau^*) = \frac{d^{k+1}\bar{K}}{d\tau^{k+1}}(\tau^*) = \bar{\eta}_k\left(\frac{1}{\tau^*}\right)$, for all $k = 0, ..., \nu$. In particular, we are interested in the case $\tau^* = 0$, which recovers the Markov parameter matching problem.

Proposition 3. *Consider the system (17) and $\tau^* \in \mathbb{C}$. Let*

$$L = [1 \ 0 \ 0 \ \dots \ 0] \in \mathbb{R}^{1 \times (\nu+1)},$$

$$S = \begin{bmatrix} \tau^* & 1 & 0 & \dots & 0 \\ 0 & \tau^* & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \tau^* & 1 \\ 0 & \dots & \dots & 0 & \tau^* \end{bmatrix} \in \mathbb{R}^{(\nu+1)\times(\nu+1)}.$$

1) *Let* $\Pi \in \mathbb{R}^{n\times(\nu+1)}$ *be the solution of the generalized Sylvester equation*

$$A\Pi S + BL = \Pi. \tag{20}$$

Then the moments of $\widetilde{K}(\tau) = K(1/\tau)$ *at* $\sigma(S)$ *are in one-to-one relation with*

$$[\widetilde{\eta}_0(\tau^*)\ \widetilde{\eta}_1(\tau^*)\ \ldots\ \widetilde{\eta}_\nu(\tau^*)] = C\Pi S. \tag{21}$$

2) *Let* $\bar{\Pi}$ *be the solution of the generalized Sylvester equation*

$$A\bar{\Pi}S + BLS = \bar{\Pi}. \tag{22}$$

Then the moments of $\widetilde{K}(\tau) = K(1/\tau)$ *at* $\sigma(S)$ *are in one-to-one relation with*

$$[\bar{\eta}_0(\tau^*)\ \bar{\eta}_1(\tau^*)\ \ldots\ \bar{\eta}_\nu(\tau^*)] = C\bar{\Pi}. \tag{23}$$

□

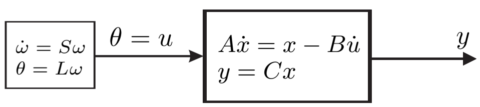

Fig. 2. Interconnection of the signal generator and the system (25), with transfer function $\widetilde{K}(s)$.

Remark 7. Consider the signal generator

$$\dot{\omega} = S\omega,\ \theta = L\omega, \tag{24}$$

with (L, S) observable, interconnected to the system

$$\begin{aligned} A\dot{x} &= x - B\dot{u} \\ y &= Cx, \end{aligned} \tag{25}$$

through $u = \theta$. Note that the transfer function of (25) is $\widetilde{K}(\tau)$. The moments of $\widetilde{K}(\tau)$ at $\sigma(S)$ are in a one-to-one relation with the steady-state response of the interconnection between the signal generator and (25). Similarly, consider the interconnection, through $u = \theta$, of the signal generator with the system

$$\begin{aligned} A\dot{x} &= x - Bu, \\ y &= C\dot{x}, \end{aligned} \tag{26}$$

having the transfer function $\widetilde{K}(\tau)$. The moments of $\widetilde{K}(\tau)$ at $\sigma(S)$ are in a one to one relation with the steady-state response of the interconnection between the signal generator and (25). □

Assumption 6. $\lambda\mu \neq 1$, for any $\lambda \in \sigma(A)$ and $\mu \in \sigma(S)$.

By Assumption 6, Π is the unique solution of (20) and $\bar{\Pi}$ is the unique solution of (22). Consider a pair (L, S) and the system

$$\Sigma_{\mathbf{\Pi}} : \begin{cases} \dot{\xi} = F\xi + Gu, \\ \psi = H\xi, \end{cases} \tag{27}$$

with $\xi(t) \in \mathbb{R}^\nu$ and $\mathbf{\Pi} = \Pi$, or $\mathbf{\Pi} = \bar{\Pi}$. Let $\widetilde{K}_{\mathbf{\Pi}}(\tau)$ be the transfer function of system (27). Then, if $\mathbf{\Pi} = \Pi$ and $\lambda\mu \neq 1$, for any $\lambda \in \sigma(F)$ and $\mu \in \sigma(S)$, (27) is a reduced order model that matches the first ν moments of $\widetilde{K}(\tau)$ at τ^* if there exists an invertible matrix $P \in \mathbb{R}^{\nu\times\nu}$ such that

$$C\Pi S = HPS,\ FPS + GL = P. \tag{28}$$

If $\tau^* = 0$, then the Markov parameters of $\widetilde{K}_\Pi(\tau)$ match the first ν Markov parameters of $K(s)$. Furthermore, for $\mathbf{\Pi} = \bar{\Pi}$, the system (27) is a reduced order model that matches the first ν moments of $\widetilde{K}(\tau)$ at τ^* if there exists an invertible matrix $\bar{P} \in \mathbb{R}^{\nu\times\nu}$ such that

$$C\bar{\Pi} = H\bar{P},\ F\bar{P}S + GLS = \bar{P}. \tag{29}$$

Remark 8. Assume S is invertible, i.e. $\tau^* \neq 0$. Let $P = I$. A reduced order model that matches the moments of $\widetilde{K}(\tau)$ at τ^* is given by equations (27) with $F = (I - GL)S^{-1}$ and $H = C\Pi$. Furthermore, if $\bar{P} = I$, according to (29) another reduced order model that matches the moments of $\widetilde{K}(\tau)$ at τ^* is given by equations (27) with $F = S^{-1} - GL$ and $H = C\bar{\Pi}$. □

B. Matching at $s = \infty$ - The port Hamiltonian case

Consider the port Hamiltonian system (1) with the transfer function $K(s) = B^*Q(sI - (J - R)Q)^{-1}B$. Suppose Assumption 6 holds. The moments of $\widetilde{K}(\tau) = K(1/\tau)$ at $\tau = \tau^*$ are in a one-to-one relation with $B^*Q\Pi S$, where Π is the solution of the generalized Sylvester equation

$$(J - R)Q\Pi S + BL = \Pi. \tag{30}$$

In addition, let $\bar{\Pi}$ be the unique solution of the generalized Sylvester equation

$$(J - R)Q\bar{\Pi}S + BLS = \bar{\Pi}. \tag{31}$$

The moments of $\widetilde{K}(\tau) = K(1/\tau)$ at $\tau = \tau^*$ are in a one-to-one relation with $B^*Q\bar{\Pi}$. The first ν Markov parameters of (1) are the moments of $\widetilde{K}(\tau)$ for $\tau^* = 0$. Assume there exists an invertible matrix P such that a reduced order model described by equations (27) exists and the relations (28) are satisfied. Furthermore, assume there exists an invertible matrix $\bar{P}$ such that conditions (29) are satisfied.

Proposition 4. *Consider system(1). Let* (L, S) *be an observable pair. Assume* $\lambda\mu \neq 1$, *for any* $\lambda \in \sigma((\widetilde{J} - \widetilde{R})\widetilde{Q})$ *and* $\mu \in \sigma(S)$. *Consider the port Hamiltonian system*

$$\Sigma^{\mathrm{pH}}_{\mathbf{\Pi}} : \begin{cases} \dot{\xi} = (\widetilde{J} - \widetilde{R})\widetilde{Q}\xi + \widetilde{B}u, \\ \psi = \widetilde{B}^*\widetilde{Q}\xi, \end{cases} \tag{32}$$

with $\xi(t) \in \mathbb{R}^\nu$ *and*

$$\begin{aligned} &\widetilde{J} = \mathbf{\Pi}^*QJQ\mathbf{\Pi},\ \widetilde{R} = \mathbf{\Pi}^*QRQ\mathbf{\Pi}, \\ &\widetilde{Q} = (\mathbf{\Pi}^*Q\mathbf{\Pi})^{-1},\ \widetilde{B} = \mathbf{\Pi}^*QB, \end{aligned} \tag{33}$$

and $\mathbf{\Pi} = \Pi$, *or* $\mathbf{\Pi} = \bar{\Pi}$, *where* Π *is the unique solution of equation (30) and* $\bar{\Pi}$ *is the unique solution of equation (31). Then (32) is a reduced order model matching the moments of* $\widetilde{K}(\tau) = K(1/s)$ *at* $\sigma(S)$, *where* $K(s)$ *is the transfer function of (1).* □

Theorem 4. *Let system (27) be a reduced order model of system (1). Then equations (27) are equivalent to a port*

Hamiltonian system (32) if and only if $G = \Pi^* Q B$ and $H = G^*(\Pi^* Q \Pi)^{-1}$. □

Remark 9. Theorem 4 holds also for $\Pi = \bar{\Pi}$, where $\bar{\Pi}$ is the (unique) solution of (31). □

Remark 10. If $\tau^* = 0$, the model Σ_Π matches the first ν Markov parameters of (1) and preserves the port Hamiltonian structure of the given system. This result is along the lines of [9], with the difference that we do not need to compute any additional coordinates transformation such that in the new coordinates the Hamiltonian is the square of the norm of the state vector. □

Example 4. Consider the ladder network in Example 1, with $C_1 = 1, C_2 = 2, L_1 = L_2 = 1$ and $R_1 = R_2 = R_3 = 1$. Let $L = [l_1\ l_2\ l_3]^*$, $l_1 \in \mathbb{R}$, $l_2 \in \mathbb{R}$, $l_3 \in \mathbb{R}$ and $S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Solving equation (30) yields

$$\Pi(l_1, l_2, l_3) = \begin{bmatrix} l_1 & 0 & 0 & 0 \\ l_2 & l_1 & 0 & 0 \\ l_3 - l_1 & l_2 - l_1 & l_1 & 0 \end{bmatrix}^*.$$

The family of port Hamiltonian models, parameterized in $l_1,\ l_2,\ l_3$ is given by

$$\widetilde{J}(l_1, l_2, l_3) = \begin{bmatrix} 0 & -l_1^2 & l_1(l_1 - l_2) \\ l_1^2 & 0 & -3l_1^2 + l_1 l_3 - l_2^2 + l_1 l_3 \\ l_1(l_1 - l_2) & -3l_1^2 + l_1 l_3 - l_2^2 + l_1 l_3 & 0 \end{bmatrix}, \tag{34}$$

$$\begin{aligned} \widetilde{R}(l_1, l_2, l_3) &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & l_1^2 & l_1(l_1 - l_2) \\ 0 & l_1(l_1 - l_2) & (l_1 - l_2)^2 \end{bmatrix}, \\ \widetilde{Q}(l_1, l_2, l_3) &= \\ \frac{1}{2l_1^6} &\begin{bmatrix} q_{11}(l_1, l_2, l_3) & q_{12}(l_1, l_2, l_3) & q_{13}(l_1, l_2, l_3) \\ q_{12}(l_1, l_2, l_3) & q_{22}(l_1, l_2, l_3) & q_{23}(l_1, l_2, l_3) \\ q_{13}(l_1, l_2, l_3) & q_{23}(l_1, l_2, l_3) & q_{33}(l_1, l_2, l_3) \end{bmatrix}, \\ \widetilde{B}(l_1, l_2, l_3) &= [l_1\ l_2\ l_3 - l_1]^*, \end{aligned} \tag{35}$$

with

$$\begin{aligned} q_{11}(l_1, l_2, l_3) &= 5l_1^2 l_2^2 - 2l_1 l_3 l_2^2 + l_2^4 - 2l_2^3 l_1 + 3l_1^4 - 2l_1^3 l_3 \\ &\quad + l_3^2 l_1^2 - 2l_1^3 l_2 + 2l_2 l_3 l_1^2, \\ q_{12}(l_1, l_2, l_3) &= l_1^4 - 4l_1^3 l_2 + l_1^2 l_2 l_3 - l_2^3 l_1 + 2l_1^2 l_2^2 - l_1^3 l_2, \\ q_{13}(l_1, l_2, l_3) &= l_1^2(l_2^2 - l_1 l_2 + l_1^2 - l_1 l_3), \\ q_{22}(l_1, l_2, l_3) &= l_1^2(l_2^2 - 2l_1 l_2 + 3l_1^2), \\ q_{23}(l_1, l_2, l_3) &= l_1^3(l_1 - l_1 l_2), \\ q_{33}(l_1, l_2, l_3) &= l_1^4. \end{aligned} \tag{36}$$

The input-output behaviour of the family of models (34) is given by the transfer function $\widetilde{K}(s) = \frac{s^2+s+2}{s(s^2+s+3)}$. □

V. Conclusions

In this paper, within the family of reduced order models that achieve moment matching, we have characterized the port Hamiltonian models. First, we have characterized the equations of the port Hamiltonian reduced order models, based on the definition of the set of the moments to be matched. Then, we have shown how to find a port Hamiltonian reduced order model from a parameterized family of reduced order models that achieve moment matching. We have also discussed the problem of Markov parameters matching and given a characterization of the port Hamiltonian models, within the family of models that achieve Markov parameters matching.

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