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Finite Frequency Range Control Law Synthesis for Differential Linear Repetitive Processes

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***Abstract*— This paper develops a new set of necessary and sufficient conditions for the stability of differential linear repetitive processes, based on application of the Kalman-Yakubovich-Popov lemma. These new conditions reduce the problem of determining the stability of an example to checking for the existence of a solution of a set of linear matrix inequalities. A relatively easy extension to enable stabilizing control law design, with additional perfromance specifications if required, is established. The inclusion of extra design specifications is developed for the case of regional constraints on the eigenvalues of state matrix and a finite frequency range design. Finally, a possible application in iterative learning control is briefly discussed.**

I. Introduction

Repetitive processes make a series of sweeps, termed passes, through a set of dynamics defined over a finite duration known as the pass length. On each pass an output, termed the pass profile, is produced which as a forcing function on, and hence contributes to, the dynamics of the next pass profile.

Let $\alpha < \infty$ denote the pass length and denote the, vector or scalar valued, pass profile by $y_k(t)$, $0 \le t \le \alpha$. Then in a repetitive process the pass profile $y_k(t)$ acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile $y_{k+1}(t)$, $0 \le t \le \alpha$, $k \ge 0$. This inter-pass interaction is the source of the unique control problem where the sequence of pass profiles $\{y_k\}$ generated can contain oscillations that increase in amplitude in the pass-to-pass (k) direction.

Repetitive processes have their origins in the coal mining and metal rolling industries where references to the original papers are given in [1]. In coal mining, the cutting machine rests on the previous pass profile, the height of the stone/coal interface above some datum line, during the production of the current one and the basic geometry confirms that this industrial application is a repetitive process in the sense defined above. The stability problem for this repetitive process is caused, in the main, by the machine's weight and can result in undulations of a level that require productive work is no longer possible without their removal.

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Applications exist where adopting a repetitive process setting for analysis has distinct advantages over alternatives. Examples include classes of iterative learning control schemes [1], [2] and iterative algorithms for solving nonlinear dynamic optimal control problems based on the maximum principle [3]. In the latter example, use of the repetitive process setting provides the basis for the development of highly reliable and efficient solution algorithms that have advantages over alternatives in some cases of practical interest.

Recently iterative learning control algorithms designed in using a repetitive process setting have been experimentally tested [4]. Also there has been work on the use of this setting for the analysis of OL-Nash games with a gas pipeline application [5].

This paper considers differential linear repetitive processes where the dynamics along the pass are governed by a matrix linear differential equation and the pass-to-pass dynamics by a discrete linear matrix equation. It is also possible to consider cases where the dynamics along the pass are also governed by a discrete linear matrix equation to form a discrete linear repetitive process.

Recognizing the unique control problem, the stability theory [1] for linear repetitive processes is of the bounded-input bounded-output (BIBO) type and is based on an abstract model in a Banach space setting that includes a large range of examples as special cases. In terms of their dynamics it is the pass-to-pass coupling, noting again their unique feature, which is critical and in physical terms a bounded initial pass profile ($k = 0$) is required to produce a bounded sequence of pass profiles. Two practically relevant forms of stability are possible, termed asymptotic and along the pass, respectively, where the former demands this property over the finite and fixed pass length α for a given example and the latter for all possible pass lengths. Also asymptotic stability is a necessary condition for stability along the pass.

Application of the stability along the pass stability theory to the processes considered in this paper produces three conditions that can be tested by direct application of standard, or 1D, linear systems stability tests. Two of these tests require that the eigenvalues of the matrices which describe the previous pass profile contribution to the current pass profile and the current pass state vector contribution to the along the pass dynamics lie in the open unit circle and open left-half of the complex plane, respectively. The third test requires the computation of the eigenvalues of the transfer-function matrix representation of the contribution of the previous pass profile dynamics to current one for $s =$

$j\omega$, $\omega \geq 0$, where s denotes the Laplace transform variable. Assuming that the first two conditions hold, stability along the pass requires that the loci generated by the eigenvalues of this transfer-function matrix lie in the open unit circle in the complex plane.

The last condition for stability along the pass of the three discussed above can be computationally intensive and is, in general, not suitable for the synthesis of control laws for stability and performance. Furthermore, there is no link between existing results and practical requirements for the control of repetitive processes that are often described by multiple Frequency Domain Inequalities (FDIs) in semi-finite frequency ranges. To overcome these problems, this paper makes extensive use of the Kalman-Yakubovich-Popov (KYP) lemma to establish the equivalence between FDIs for a transfer-function matrix and Linear Matrix Inequalities (LMIs) defined in terms of its state-space realization, as in [6] for 1D linear systems.

The new results in this paper start with the development of LMI based tests for stability along the pass. This analysis leads on to control law design algorithms procedure that can include multiple design specifications, whereas the vast majority of currently known designs ensure stability but cannot impose many useful additional performance specifications [1]. These design algorithms are based on sufficient but not necessary conditions and a possible application to the design of ILC schemes is briefly discussed.

Throughout this paper, the null and identity matrices with appropriate dimensions are denoted by 0 and I, respectively. Moreover, $\mathrm{sym}(X)$ is used to denote $X+X^T$ and $X^\perp$ denotes the orthogonal complement. The notation $X \succeq Y$ (respectively $X \succ Y$) means that the matrix $X-Y$ is symmetric and positive semi-definite (symmetric and positive definite, respectively). The symbol $(\star)$ denotes entries in symmetric matrices, $r(\cdot)$ denotes the spectral radius and $\mathbb{C}_-$ the open left-half of the complex plane. Use will also be made of the following results whose proofs can be found in [7], [8], [9].

Lemma 1: For linear time-invariant systems with transfer-function matrix $G(s)$ and frequency response matrix

$$G(j\omega)=\mathcal{C}(j\omega I-\mathcal{A})^{-1}\mathcal{B}+\mathcal{D},$$

the following inequalities are equivalent

(i) the frequency domain inequality

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^T \Pi \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \prec 0, \quad \forall \omega_l \leq \omega \leq \omega_h,$$

where Π is a given real symmetric matrix

(ii) the LMI

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ I & 0 \end{bmatrix}^T \Xi \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ I & 0 \end{bmatrix} + \begin{bmatrix} \mathcal{C} & \mathcal{D} \\ 0 & I \end{bmatrix}^T \Pi \begin{bmatrix} \mathcal{C} & \mathcal{D} \\ 0 & I \end{bmatrix} \prec 0, \quad (1)$$

where $Q \succ 0$, P is a symmetric matrix and the matrix Ξ is specified as follows

$$\Xi = \begin{bmatrix} -Q & P+j\omega_c Q \\ P+j\omega_c Q & -\omega_l\omega_h Q \end{bmatrix},$$

where $\omega_c = (\omega_l+\omega_h)/2$ for a finite frequency range or

$$\Xi = \begin{bmatrix} -Q & P \\ P & 0 \end{bmatrix},$$

for an entire frequency range, that is, $\omega_l = 0$, $\omega_h = \infty$.

Lemma 2: Given a symmetric matrix $\Psi \in \mathbb{R}^{m\times m}$ and two matrices Υ, Σ of column dimension m, there exists a matrix W such that the following LMI holds

$$\Psi + \Upsilon^T W^T \Sigma + \Sigma^T W \Upsilon \prec 0$$

if and only if the following two inequalities with respect to W are satisfied

$$\begin{aligned} \Upsilon^\perp \Psi \Upsilon^{\perp T} &\prec 0, \\ \Sigma^\perp \Psi \Sigma^{\perp T} &\prec 0. \end{aligned} \quad (2)$$

II. Differential Linear Repetitive Processes

Following [1], the state-space model of a differential linear repetitive process has the following form over $0 \leq t \leq \alpha$, $k \geq 0$

$$\begin{aligned} \dot{x}_{k+1}(t) &= Ax_{k+1}(t) + B_0 y_k(t) + Bu_{k+1}(t) \\ y_{k+1}(t) &= Cx_{k+1}(t) + D_0 y_k(t) + Du_{k+1}(t) \end{aligned} \quad (3)$$

where $\alpha < +\infty$ denotes the pass length, and on pass k $x_k(t) \in \mathbb{R}^n$ is the state vector, $y_k(t) \in \mathbb{R}^m$ is the pass profile (output) vector and $u_k(t) \in \mathbb{R}^r$ is the input vector.

To complete the process description, it is necessary to specify the boundary conditions, that is, the state initial vector on each pass and the initial pass profile, that is, on pass 0. For the purposes of this paper, no loss of generality arises from assuming that $x_{k+1}(0) = 0$, $k \geq 0$, and the initial pass profile $y_0(t)$ consists of entries that are known functions of t over $[0, \alpha]$.

A. Stability Theory

As discussed in the previous section, the stability theory [1] for linear repetitive processes is based on an abstract model in a Banach space setting which includes a wide range of such processes as special cases, including those described by the state-space model and boundary conditions considered in this paper. In terms of the process dynamics it is the pass-to-pass coupling, noting again their unique feature, which is critical and has the form $y_{k+1} = L_\alpha y_k$, where $y_k \in E_\alpha$, E_α is a Banach space with norm $||\cdot||$ and L_α is a bounded linear operator mapping E_α into itself. In the special case of processes described by (3) with the given boundary conditions L_α is a convolution operator.

Asymptotic stability demands that a bounded initial profile produces a bounded sequence of pass profiles over the finite and fixed pass length. It can be shown [1] that this property is equivalent to the existence of real scalars $M_\alpha > 0$ and $\lambda_\alpha \in (0,1)$ such that $||L_\alpha^k|| \leq M_\alpha \lambda_\alpha^k$, $k \geq 0$, where $||\cdot||$ also denotes the induced norm. The necessary and sufficient condition for this property is $r(L_\alpha) < 1$ and for processes described by (3) asymptotic stability holds if and only if $r(D_0) < 1$, that is, all eigenvalues of D_0 must lie in the open unit circle in the complex plane.

Suppose that (3) is asymptotically stable and the input sequence applied $\{u_{k+1}\}_k$ converges strongly as $k \to \infty$, that is, in the sense of the norm on the underlying function space, to u_∞. Then the strong limit $y_\infty := \lim_{k\to\infty} y_k$ is termed the limit profile corresponding to this input sequence and is described by a 1D linear systems state-space model with state matrix $A_{lp} := A + B_0(I - D_0)^{-1}C$. Hence under asymptotic stability the process dynamics can, after a sufficiently large number of passes have elapsed, be replaced by those of a 1D linear systems state-space model.

Asymptotic stability does not guarantee that the limit profile is stable. A simple counter-example is the case when $A = -1,\ B = 1,\ B_0 = 1+\beta,\ C = 1,\ D = 0,\ D_0 = 0$, where β is a real scalar. This example is asymptotically stable with resulting limit profile state matrix $A_{lp} = \beta$ and hence is unstable for $\beta \geq 0$.

To prevent cases such as the above example from arising, stability along the pass demands the BIBO property for all possible values of the pass length (mathematically this can be analyzed by letting $\alpha \to \infty$). Also it can be shown that stability along the pass requires the existence of finite real scalars $M_\infty > 0$ and $\lambda_\infty \in (0, 1)$, which are independent of α, such that $||L_\alpha^k|| \leq M_\infty \lambda_\infty^k,\ k \geq 0$.

Several sets of necessary and sufficient conditions for stability along the pass of differential linear repetitive processes described by (3) are known [1], such as the following.

Theorem 1: A differential linear repetitive process of the form (3) is stable along the pass if and only if

i) $r(D_0) < 1$,
ii) all eigenvalues of the matrix A lie in $\mathbb{C}_-$, and
iii) all eigenvalues of $G(s) = C(sI - A)^{-1}B_0 + D_0$ $s = j\omega, \forall \omega \geq 0$, have modulus strictly less than unity

The first two conditions of this theorem are the stability condition for 1D discrete and differential linear systems, respectively, and the third has a Nyquist based interpretation. In the single-input single-output (SISO) case condition iii) of Theorem 1 requires that the Nyquist plot generated by $G(s)$, lies inside the unit circle in the complex plane for all $s = j\omega,\ \forall\,\omega$. In physical terms this condition requires that each frequency component of the initial pass profile is attenuated from pass-to-pass. For the remainder of this paper attention is restricted to SISO examples as this case arises most often in application areas such as ILC and also extension to the multivariable case is straightforward.

Most applications will require control law design to ensure stability along the pass and additional performance objectives, where most of the currently available methods do not allow these performance specifications to be imposed as design constraints. This paper develops algorithms with this property for stability along the pass in the presence of design specifications on the locations of the eigenvalues of the state matrix of the controlled process.

The starting point for analysis is condition iii) of Theorem 1 expressed for SISO examples in the form

$$|G(j\omega)| < 1, \forall \omega \geq 0, \tag{4}$$

or, equivalently,

$$\begin{bmatrix} G(j\omega) \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & -\gamma^2 \end{bmatrix} \begin{bmatrix} G(j\omega) \\ 1 \end{bmatrix} \prec 0, \quad \forall \omega \geq 0, \tag{5}$$

with $\gamma = 1$. Also choosing the matrix Π as

$$\Pi = \begin{bmatrix} 1 & 0 \\ 0 & -\gamma^2 \end{bmatrix} \tag{6}$$

and making use of Lemma 1 for the entire frequency range, (5) is equivalent to

$$\begin{bmatrix} A & B_0 \\ I & 0 \end{bmatrix}^T \Xi \begin{bmatrix} A & B_0 \\ I & 0 \end{bmatrix} + \begin{bmatrix} C & D_0 \\ 0 & 1 \end{bmatrix}^T \Pi \begin{bmatrix} C & D_0 \\ 0 & 1 \end{bmatrix} \prec 0, \tag{7}$$

where $Q \succ 0$, P is a symmetric matrix, and the matrix Ξ is given by

$$\Xi = \begin{bmatrix} -Q & P \\ P & 0 \end{bmatrix}.$$

To be useful in control law design, (7) must be transformed to an equivalent representation with no product terms involving P, Q, and process state-space model matrices. The following result establishes the required transformation.

Theorem 2: A SISO differential linear repetitive process described by (3) is stable along the pass if and only if there exist matrices W, $Q \succ 0$, $R \succ 0$, $S \succ 0$, and a symmetric matrix P such that the following LMIs are feasible

$$D_0^T R D_0 - R \prec 0, \tag{8}$$

$$A^T S + SA \prec 0, \tag{9}$$

$$\begin{bmatrix} -Q & P + W^T & 0 & 0 \\ P + W & -A^T W^T - WA & -WB_0 & C^T \\ 0 & -B_0^T W^T & -1 & D_0^T \\ 0 & C & D_0 & -1 \end{bmatrix} \prec 0. \tag{10}$$

Proof: Two first LMIs follow immediately from Lyapunov stability theory for 1D discrete and differential linear systems applied to conditions i) and ii) in Theorem 1, respectively. To establish the LMI (10), (7) can be rewritten as

$$\begin{bmatrix} A^T & I & 0 \\ B_0^T & 0 & 1 \end{bmatrix} \begin{bmatrix} -Q & P & 0 \\ P & C^T C & C^T D_0 \\ 0 & D_0^T C & D_0^T D_0 - 1 \end{bmatrix} \begin{bmatrix} A & B_0 \\ I & 0 \\ 0 & 1 \end{bmatrix} \prec 0, \tag{11}$$

which is of form of the first inequality in (2) of Lemma 2 with

$$\Upsilon^\perp = \begin{bmatrix} A^T & I & 0 \\ B_0^T & 0 & 1 \end{bmatrix}, \Psi = \begin{bmatrix} -Q & P & 0 \\ P & C^T C & C^T D_0 \\ 0 & D_0^T C & D_0^T D_0 - 1 \end{bmatrix},$$

and hence

$$\Upsilon^T = \begin{bmatrix} -I & A & B_0 \end{bmatrix}.$$

Consequently for a matrix Σ that satisfies the second inequality of (2), application of Lemma 2 gives that (11) is feasible if

$$\begin{bmatrix} -Q & P & 0 \\ P & C^T C & C^T D_0 \\ 0 & D_0^T C & D_0^T D_0 - 1 \end{bmatrix} - \mathrm{sym}(\Upsilon W^T \Sigma^T) \prec 0, \tag{12}$$

which is satisfied for $\Sigma = [0\ I\ 0]^T$.

Suppose that there exist matrices $Q \succ 0$, W, and symmetric P such that the LMIs (10) are feasible. Then on applying the Schur's complement formula it follows that (12) is equivalent to the LMI of (10) and the proof is complete. ∎

Remark 1: In the SISO case, the LMI (8) can be replaced by the simple scalar inequality $|D_0| < 1$.

III. Control Law Design

The control law considered in this paper is of the form

$$u_{k+1} = \begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} x_{k+1}(t) \\ y_k(t) \end{bmatrix}, \tag{13}$$

where K_1 and K_2 are compatibly dimensioned matrices. This control law is formed as a weighted sum of current pass state feedback and feedforward of the previous pass profile, see [1] for further background on this form of control action.

Application of the control law (13) to (3) gives the controlled process state-space model

$$\begin{bmatrix} \dot{x}_{k+1}(t) \\ y_{k+1}(t) \end{bmatrix} = \begin{bmatrix} A+BK_1 & B_0+BK_2 \\ C+DK_1 & D_0+DK_2 \end{bmatrix} \begin{bmatrix} x_{k+1}(t) \\ y_k(t) \end{bmatrix}. \tag{14}$$

The existence of stabilizing K_1 and K_2 can be characterized in LMI terms as follows.

Theorem 3: Suppose that a control law of the form (13) is applied to a SISO differential linear repetitive process described by (3). Then the resulting controlled process is stable along the pass if there exist matrices Y, X_1, X_2 $Q \succ 0$, $S \succ 0$, a symmetric matrix P, and scalars p, q such that the following LMIs are feasible

$$\begin{bmatrix} -1 & D_0 + DX_2 \\ X_2^T D^T + D_0^T & -1 \end{bmatrix} \prec 0, \tag{15}$$

$$\begin{bmatrix} -pY-pY^T & S+pAY+pBX_1-qY^T \\ (\star) & \mathrm{sym}(qAY+qBX_1) \end{bmatrix} \prec 0, \tag{16}$$

$$\begin{bmatrix} -\widehat{Q} & \widehat{P}+Y^T & 0 & 0 \\ \widehat{P}+Y & -\mathrm{sym}(AY+BX_1) & -B_0-BX_2 & YC^T+X_1^T D^T \\ 0 & -B_0^T - X_2^T B^T & -1 & D_0^T+X_2^T D^T \\ 0 & CY+DX_1 & D_0+DX_2 & -1 \end{bmatrix} \prec 0, \tag{17}$$

where the scalars p and q are chosen to satisfy

$$qp+pq < 0. \tag{18}$$

If the LMIs (15)–(17) are feasible, stabilizing control law matrices K_1 and K_2 can be calculated using $K_1 = X_1 Y^{-1}$, $K_2 = X_2$.

Proof: Application of the result of Theorem 2 to the controlled process state-space model shows that stability along the pass holds if and only if

$$(D_0+DK_2)^T R(D_0+DK_2) - R \prec 0, \tag{19}$$

$$(A+BK_1)^T S + S(A+BK_1) \prec 0, \tag{20}$$

$$\begin{bmatrix} -Q & P+W^T & 0 & 0 \\ P+W & -(A+BK_1)^T W^T - W(A+BK_1) & -W(B_0+BK_2) & (C+DK_1)^T \\ 0 & -(B_0+BK_2)^T W^T & -1 & (D_0+DK_2)^T \\ 0 & C+DK_1 & D_0+DK_2 & -1 \end{bmatrix} \prec 0. \tag{21}$$

Also in the SISO case $D_0 + DK_2$ and $R \succ 0$ in (19) are scalars. Hence the LMI

$$(D_0 + DK_2)^T R(D_0 + DK_2) - R \prec 0,$$

for $R \succ 0$ is equivalent to

$$(D_0+DK_2)^2 - 1 \prec 0,$$

and application of the Schur's complement formula to this last inequality gives (15). Next, it follows immediately from (20) that

$$\begin{bmatrix} (A+BK_1)^T & I \end{bmatrix} \begin{bmatrix} 0 & S \\ S & 0 \end{bmatrix} \begin{bmatrix} A+BK_1 \\ I \end{bmatrix} \prec 0, \tag{22}$$

and, since for arbitrary chosen real numbers p and q satisfying (18) an annihilator of $[-qI\ pI]^T$ is $[pI\ qI]$,

$$\begin{bmatrix} pI & qI \end{bmatrix} \begin{bmatrix} 0 & S \\ S & 0 \end{bmatrix} \begin{bmatrix} pI \\ qI \end{bmatrix} \prec 0.$$

Introducing $\widetilde{A} = A+BK_1$, and noting that an annihilator of $[-I\ \ \widetilde{A}]^T$ is $[\widetilde{A}^T\ \ I]$, application of Lemma 2 gives that (22) is equivalent to

$$\begin{bmatrix} 0 & S \\ S & 0 \end{bmatrix} \prec \mathrm{sym}\left(\begin{bmatrix} -I \\ \widetilde{A}^T \end{bmatrix} Y \begin{bmatrix} -qI & pI \end{bmatrix} \right), \tag{23}$$

which is just (16). The LMI (17) is directly obtained by pre- and post-multiplying (21) by $diag(W^{-1}, W^{-1}, 1, 1)$ and setting $Y = W^{-1}$, $\widehat{P} = W^{-1}PW^{-1}$, $\widehat{Q} = W^{-1}QW^{-1}$, $X_1 = K_1 W^{-1}$, $X_2 = K_2$, and the proof is complete. ∎

The reason why this last result is sufficient but not necessary, unlike Theorem 2 for stability along the pass in the uncontrolled case, is due to the introduction of the same matrix variable Y in the LMIs (16) and (17).

IV. Control Law Design with Finite Frequency Range Attenuation

By analogy with the 1D linear systems case, enforcing the frequency attenuation as required by condition iii) of Theorem 1 over the complete frequency range is either unobtainable or very restrictive. Hence the subject of this section is control law design where the attenuation is only required over a finite frequency range $\omega_l \leq \omega \leq \omega_h$, where the lower and upper frequency values are selected based on knowledge of the particular example considered. Moveover, the eigenvalues of the state matrix (A and $A+BK_1$ in the uncontrolled and controlled cases, respectively) govern the dynamics produced along any pass and it will be required in some applications to place these in particular locations in the open left-half of the complex plane to meet performance specifications. This section therefore considers control law design in the SISO case where the control law is required to a) ensure that $r(D_0 + DK_2) < 1$, b) place the eigenvalues of $\widetilde{A} = A + BK_1$, inside a pre-specified region of the open left-half of the complex plane and c) ensure that $|G_c(j\omega)| < 1$ over a pre-specified finite frequency range where

$$G_c(s) = (C + DK_1)(sI - A - BK_1)^{-1}(B_0 + BK_2) + D_0 + DK_2.$$

The region of interest for the eigenvalues of $A + BK_1$ is the interior of the circle of radius r with center at c given by

$$\mathcal{C}(c, r) := \{x + jy \in \mathbb{C} : |x + jy - c| < r\}. \tag{24}$$

To guarantee that the interior of this circle is located in open left-half complex plane requires $c < 0$ and $|c| > r$. Also by choosing

$$\Phi = \begin{bmatrix} 1 & -c \\ -c & |c|^2 - r^2 \end{bmatrix},$$

and using results in, for example, [7], the eigenvalues of $\widetilde{A} = A + BK_1$ are located inside the disc $\mathcal{C}(c, r)$ if there exists a matrix Y such that

$$\Phi \otimes S \prec \mathrm{sym}\left(\begin{bmatrix} -I \\ \widetilde{A}^T \end{bmatrix} Y \begin{bmatrix} -qI & pI \end{bmatrix}\right),$$

where scalars p and q are real numbers satisfying

$$p^2 - 2cpq + q^2(c^2 - r^2) < 0 \tag{25}$$

and $\otimes$ denotes the matrix Kronecker product.

The following result enables the control law to be designed to satisfy the design constraints given above.

Theorem 4: Suppose that a control law of the form (13) is applied to a SISO differential linear repetitive process described by (3). Then the resulting controlled process is stable along the pass over the finite frequency range $\omega_l \leq \omega \leq \omega_h$ with $\omega_c = (\omega_l + \omega_h)/2$ and eigenvalues of the state matrix $\widetilde{A} = A + BK_1$ located inside the sector $\mathcal{C}(c, r)$ of the open left-half of the complex plane defined by (24) if there exist matrices Y, X_1, X_2, $Q \succ 0$, $S \succ 0$, a symmetric matrix P, and scalars p, q such that the following LMIs are feasible

$$\begin{bmatrix} -1 & D_0 + DX_2 \\ X_2^T D^T + D_0^T & -1 \end{bmatrix} \prec 0, \tag{26}$$

$$\begin{bmatrix} S - pY - pY^T & -cS + pAY + pBX_1 - qY^T \\ (\star) & \mathrm{sym}(qAY + qBX_1) + (c^2 - r^2)S \end{bmatrix} \prec 0, \tag{27}$$

$$\begin{bmatrix} -\widehat{Q} & \widehat{P} + Y^T + j\omega_c Q & 0 & 0 \\ \widehat{P} + Y - j\omega_c Q & -\omega_l\omega_h Q - \mathrm{sym}(AY + BX_1) & -B_0 - BX_2 & YC^T + X_1^T D^T \\ 0 & -B_0^T - X_2^T B^T & -1 & D_0^T + X_2^T D^T \\ 0 & CY + DX_1 & D_0 + DX_2 & -1 \end{bmatrix} \prec 0, \tag{28}$$

where the scalars p and q are chosen to satisfy (25). If LMIs (26)–(28) are feasible, control law matrices K_1 and K_2 can be calculated using $K_1 = X_1 Y^{-1}$, $K_2 = X_2$.

Proof: The LMIs in this result are obtained by performing similar transformations to those in the proofs of Theorem 3 and 2, except for application the finite frequency results of Lemma 1 and the regional eigenvalue location constraints. ■

A. Numerical example

Consider the case of (3) when

$$A = \begin{bmatrix} -0.6 & 1.0 \\ 0.1 & -0.4 \end{bmatrix}, \; B_0 = \begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix}, B = \begin{bmatrix} 0.2 \\ 1.4 \end{bmatrix}$$
$$C = \begin{bmatrix} -1.0 & 0.1 \end{bmatrix}, \; D_0 = 1.1, \; D = 1.4$$

and the frequency range over which attenuation is to be achieved given as $0.0001 \leq \omega \leq 20$ [rad/sec]. Setting $p = 10$, $q = -1$ and solving the set of LMIs (26)-(27) for the regional constraint $C(-10, 4)$ on the eigenvalues matrix $\widetilde{A} = A + BK_1$ gives

$$K_1 = \begin{bmatrix} -46.5749 & -5.6923 \end{bmatrix}, \; K_2 = -0.8589.$$

The Nyquist plots for the uncontrolled and controlled processes are shown in Fig. 1 and confirm that for the chosen frequency range the Nyquist plot of the controlled process is inside the unit circle. Furthermore, the eigenvalues of $\widetilde{A} =$ $A + BK_1$ are $\{-12.2424, -6.0418\}$ and obviously lie inside the specified region C(-10,4).

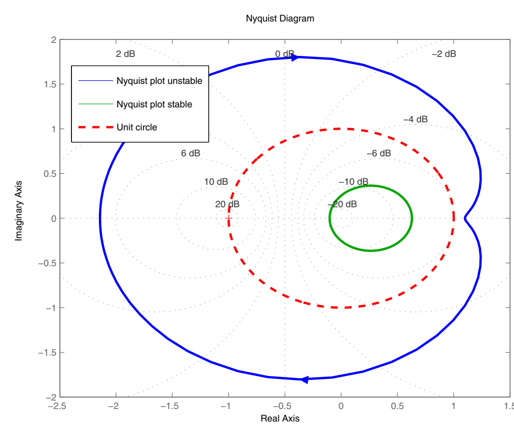

Fig. 1. Nyquist plots for the uncontrolled and controlled processes.

V. Application to ILC

Iterative learning control (ILC) is a technique for controlling systems operating in a repetitive (or pass-to-pass) mode with the requirement that a reference trajectory $y_{ref}(t)$ defined over a finite interval $0 \leq t \leq \alpha$ is followed to a high precision [10], [11]. Examples of such systems include robotic manipulators that are required to repeat a given task and chemical batch processes.

Consider an ILC application where there is a need to regulate along the pass behavior in addition to forcing pass-to-pass error convergence. Then for discrete dynamics one way to proceed is to design a feedback control loop for the plant and then enforce pass-to-pass error convergence based on the lifted model. The lifted model is a 1D linear systems model of the dynamics which is static in the pass number. This option is not available for differential dynamics and by adopting a repetitive process setting for analysis it is possible to consider control law for along the pass performance and pass-to-pass error convergence simultaneously.

Introduce for analysis purposes the following vector defined in terms of the difference between the current and previous pass state vector

$$\eta_{k+1}(t) = \int_0^t \left(x_{k+1}(\tau) - x_k(\tau)\right) d\tau, \tag{29}$$

and let $e_k(t) = y_{ref}(t) - y_k(t)$ denote the current pass error, where $y_{ref}(t)$ is the pre-specified reference vector for the ILC problem. Also let $\Delta u_{k+1}(t)$ be the change in the control signal between two successive passes. Then it is possible to proceed as in, for example, [4] and use an ILC law which requires the current trial state vector $x_k(t)$ of the plant using $\Delta u_{k+1}(t) = K_1 \dot{\eta}_{k+1}(t) + K_2 \dot{e}_k(t)$. The controlled system dynamics can be written in the form

$$\begin{aligned} \dot{\eta}_{k+1}(t) =& (A+BK_1)\eta_{k+1}(t) + (BK_2)e_k(t), \\ e_{k+1}(t) =& -C(A+BK_1)\eta_{k+1}(t) + (I-CBK_2)e_k(t). \end{aligned} \tag{30}$$

The following result now gives an ILC design procedure.

Theorem 5: The ILC scheme (30) is stable along the pass over the finite frequency range $\omega_l \le \omega \le \omega_h$ with $\omega_c = (\omega_l + \omega_h)/2$ and eigenvalues of the state matrix $\widetilde{A} = A + BK_1$ located inside the sector $C(c, r)$ of the open left-half of the complex plane defined by (24) if there exist matrices Y, X_1, X_2, $Q \succ 0$, $S \succ 0$, a symmetric matrix P and scalars p and q such that the following LMIs are feasible

$$\begin{bmatrix} -1 & 1 - CBX_2 \\ 1 - X_2^T B^T C^T & -1 \end{bmatrix} \prec 0, \tag{31}$$

$$\begin{bmatrix} S - pY - pY^T & -cS + pAY + pBX_1 - qY^T \\ (\star) & \text{sym}(qAY + qBX_1) + (c^2 - r^2)S \end{bmatrix} \prec 0, \tag{32}$$

$$\begin{bmatrix} -\widehat{Q} & \widehat{P} + Y^T + j\omega_c Q & 0 & 0 \\ \widehat{P} + Y - j\omega_c Q & -\omega_l\omega_h Q - \text{sym}(AY + BX_1) & -B_0 - BX_2 & -YA^TC^T - X_1^T B^T C^T \\ 0 & -B_0^T - X_2^T B^T & -1 & 1 - X_2^T B^T C^T \\ 0 & -CAY - CBX_1 & 1 - CBX_2 & -1 \end{bmatrix} \prec 0, \tag{33}$$

where scalars p and q are chosen to satisfy (25). If LMIs (31)–(33) are feasible, the ILC control law matrices K_1 and K_2 can be calculated using $K_1 = X_1 Y^{-1}$, $K_2 = X_2$.

VI. Conclusions

This paper has developed new LMI based conditions for stability of differential linear repetitive processes, leading to new control law design algorithms. These new algorithms allow control law design in the presence of practically motivated design specifications such as regional constraints on the location of the eigenvalues of the state matrix of the controlled process and finite frequency ranges.

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