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Convergent Series Observer Design for A Class of Nonlinear Systems

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Abstract— **This paper deals with convergence analysis for power series solutions to a partial differential equation for nonlinear observer design with linear observer error dynamics. This power series solution is used to design the gain matrix for a Luenberger-like observer for nonlinear systems. The conditions are identified to guarantee the convergence of the series in l_2. The linearized model of the original system is assumed to be anti-stable at the origin for the convenience of presentation. The convergent conditions can provide a guideline for nonlinear observer design with a truncated series for the observer gain.**

I. Introduction

Results for observer design of nonlinear systems started to appear in 1970 [1], [2], and observer design for nonlinear systems continue to attract significant attention of control research with many results appeared in literature (for example, see [3], [4], [5], [6], [7] etc). One significant result was on observer design for nonlinear systems with linear observer error dynamics by introducing output injection [8]. A more general formulation of observers with linear observer error dynamics is reported in [9] based on Lyapunov's auxiliary theorem in which a Luenberger-like observer is presented with the observer gain to be determined by a nonlinear function of the system state. This nonlinear function is a solution to a partial differential equation. This result has attracted significant attend in nonlinear observer design, as evidenced by some of the recent results [10], [11], [12], [13], [14], [15].

The partial differential equation in [9] depends on the dynamics of the original system and the chosen dynamics of observer errors. Although conditions have been identified for the existence of a solution, a general solution to the partial differential equation is difficult to obtain in general. When a closed form solution is not available, series solutions can be considered, and an iterative method of obtaining high order polynomials has been introduced [9]. This series can be truncated to a certain order and the truncated series can then be used for computing observer gain in the nonlinear observer design. In this paper, we consider the convergence issue of the power series to the partial differential equation for a class of nonlinear systems with nonlinear functions in polynomials, and identify conditions for the series to converge in l_2. The conditions depend on the higher order terms in relative to the first order one, and on the radius of the state variables with respect to the origin. In establishing the convergence result, we formulate the individual terms in the series as states of a discrete-time system through proper matrix manipulations, and obtain the convergence result based the dynamic of the discrete time system. An easy assumption is made for the linearized model at the origin to be anti-stable for the convenience of presenting the basic concepts. With the result of the convergence of the series, users would be more confident in selecting the order of approximation and identifying the domain for the observer errors to converge. A example is included to demonstrate the notations used in the paper and to reveal some issues in the convergence of the power series in the observer design.

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II. Problem Formulation

Consider a nonlinear system

$$\dot{x} = f(x), \quad (1)$$

$$y = h(x) \quad (2)$$

where $x \in \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}^m$ is the output, $f : \mathbb{R}^n \to \mathbb{R}^n$, and $h : \mathbb{R}^n \to \mathbb{R}^m$ are continuous nonlinear functions with $f(0) = 0$, and $h(0) = 0$, and $n > m$.

As shown in [9], an observer can be designed if there exists a nonlinear function $p : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\frac{\partial p(x)}{\partial x} f(x) = Ap(x) + Bh(x) \quad (3)$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are constant matrices with A Hurwitz and $\{A, B\}$ controllable. Let $\xi = p(x)$, and its dynamics are then described by

$$\begin{aligned} \dot{\xi} &= A\xi + By, \\ y &= h(p^{-1}(\xi)). \end{aligned}$$

from which an observer is designed as

$$\begin{aligned} \dot{\hat{\xi}} &= A\hat{\xi} + By, \\ \hat{x} &= p^{-1}(\hat{\xi}) \end{aligned} \quad (4)$$

It is easy to see that the observer error dynamics is linear as

$$\dot{\tilde{\xi}} = A\tilde{\xi}.$$

The observer can also be implemented in the original state as

$$\dot{\hat{x}} = f(\hat{x}) + \left(\frac{\partial p}{\partial \hat{x}}(\hat{x})\right)^{-1} B(y - h(\hat{x})). \quad (5)$$

which is in the same structure as the standard Luengerger observer for linear systems by viewing $\left(\frac{\partial p}{\partial \hat{x}}(\hat{x})\right)^{-1} B$ as the observer gain.

The key step in designing this type of nonlinear observers is to solve the nonlinear function $p(x)$ in (3). Sufficient conditions for the existence of solution of (3) have been given in [9]. In case that a close-form solution is difficult to obtain, solutions in power series can then be obtained. In this paper, we analyze the convergence of the series solutions to the partial differential equation (3) for nonlinear observer design.

For the convergence analysis, we assume that functions $f(x)$ and $h(x)$ are polynomials of x with finite orders, and we introduce an assumption on the eigenvalues of $\frac{\partial f}{\partial x}(0)$ and A.

Assumption 1. All the eigenvalues of $\frac{\partial f}{\partial x}(0)$ are positive real numbers and distinct, and all the eigenvalues of A are negative real numbers and distinct.

Remark 1: Even though functions $f(x)$ and $h(x)$ are polynomials are with finite orders, the power series solution to (3) is not of finite order in general. When $f(x)$ and $h(x)$ are other smooth nonlinear functions, they can be approximated by polynomials.

Assumption 1 guarantees the existence of a solution to the partial differential equation (3), following the result shown in [9].

Based on Assumption 1, we can state, with loss of generality, that both $\frac{\partial f}{\partial x}(0)$ and A are diagonal.

Remark 2: When the eigenvalues of $\frac{\partial f}{\partial x}(0)$ and A are distinct, we can introduce suitable state transforms such that the transformed system and transformed A satisfy Assumption 1 with diagonal $\frac{\partial f}{\partial x}(0)$ and A. Let $\frac{\partial f}{\partial x}(0) = T_1^{-1}\bar{F}_1 T_1$ where $\bar{F}_1$ is a diagonal matrix. Let $\bar{x} = T_1 x$, and we have

$$\begin{aligned} \dot{\bar{x}} &= \bar{f}(\bar{x}) \\ y &= \bar{h}(\bar{x}) \end{aligned}$$

where $\bar{f}(\cdot) = T_1 f(T_1^{-1}(\cdot))$, and $\bar{h}(\cdot) = h(T_1^{-1}(\cdot))$. It is easy to verify that $\frac{\partial \bar{f}}{\partial x}(0) = \bar{F}_1$ that is diagonal. For A, we can choose a controllable pair $\{A, B\}$ with A diagonal. If A is not diagonal, but with distinct eigenvalues, a transform can be introduced. If $A = T_2^{-1}\bar{A}T_2$ with $\bar{A}$ diagonal, multiplying both sides of (3) by T_2, we have

$$T_2 \frac{\partial p(x)}{\partial w} f(x) = \bar{A} T_2 p(x) + T_2 B y$$

which gives

$$\frac{\partial (T_2 p(x))}{\partial x} f(x) = \bar{A}(T_2 p(x)) + T_2 B y.$$

This means that the convergence analysis can be carried out with the diagonal $\bar{A}$ together with $Tp(x)$.

III. Solutions in Power Series

Series solutions for (3) are discussed in [9]. In this section, we introduce an set of iterative matrix equations, by taking similar notations for series solutions for nonlinear output regulation equations in [16] for the convenience of convergence analysis.

Let us denote a solution $p(x)$ by

$$p(x) = \sum_{i=1}^{\infty} p_i(x) \tag{6}$$

where $p_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$ denotes all the polynomial terms with the order i, ie, all the terms of $x_1^{a_1} w_2^{a_2} \ldots x_n^{a_n}$ with $\sum_{k=1}^{n} a_k = i$ and $a_k \in \mathbb{N}$. Similarly, we denote

$$f(x) = \sum_{i=1}^{n_f} f_i(x), \tag{7}$$

$$h(x) = \sum_{i=1}^{n_h} h_i(x) \tag{8}$$

with the notations $f_i = 0$ for $i > n_f$, and $h_i = 0$ for $i > n_h$. Substituting the above expressions into (3), we have

$$\sum_{i=1}^{\infty} \frac{\partial p_i(x)}{\partial x} \sum_{i=1}^{n_f} f_i(x) = A \sum_{i=1}^{\infty} p(x) + B \sum_{i=1}^{n_h} h_i(x). \tag{9}$$

Comparing the order of the polynomial terms in above equation, we have

$$\begin{aligned} &\frac{\partial p_1(x)}{\partial x} f_1(x) = A p_1(x) + B h_1(x), \\ &\frac{\partial p_2(x)}{\partial x} f_1(x) + \frac{\partial p_1(x)}{\partial w} f_2(x) = A p_2(x) + B h_2(x) \\ &\ldots \\ &\sum_{j=1}^{i} \frac{\partial p_{i-j+1}(x)}{\partial w} f_j(x) = A p_i(x) + B h_i(x), \\ &\ldots \end{aligned} \tag{10}$$

Re-arranging them, we have

$$\frac{\partial p_1(x)}{\partial x} f_1(x) - A p_1(x) = B h_1(x), \tag{11}$$

$$\begin{aligned} \frac{\partial p_2(x)}{\partial x} f_1(x) - A p_2(x) &= -\frac{\partial p_1(x)}{\partial x} f_2(x) + B h_2(x) \\ &\ldots \\ \frac{\partial p_i(x)}{\partial w} f_1(x) - A p_i(x) &= \\ -\sum_{j=2}^{i} &\frac{\partial p_{i-j+1}(x)}{\partial x} f_j(x) + B h_i(x), \\ &\ldots \end{aligned} \tag{12}$$

An iterative solution can start from (11) for $p_1(x)$ and then follows (12) for $p_i(x)$ for $i = 1, 2, 3$ etc.

For the first order term, we have $p_1(x) = P_1 x$ and $f_1(x) = F_1 x$ with $P_1, F_1 \in \mathbb{R}^{n \times n}$ being constant matrices, and $h_1(x) = H_1 x$ with $H_1 \in \mathbb{R}^{n \times m}$ being a constant matrix. Hence the equation (11) can be written as

$$P_1 F_1 x - A P_1 x = B H_1 x$$

which leads to a matrix equation

$$P_1 F_1 - A P_1 = B H_1.$$

The solution of this equation is guaranteed by the condition specified in Assumption 1.

For the high order terms, we need to introduce a few notations. Let us introduce a base function for polynomials such that we can write

$$p_i(x) = P_i v_i(x).$$

where $v_i(x)$ contains all the unique polynomial terms of order i.

Adopting some notations from [16], we define

$$v^{(1)} = x = [x_1, \dots x_n]^T,$$

and

$$v^{(i)} = v^{(i-1)} \otimes x, \text{ for } i > 1,$$

where $\otimes$ denotes Kronecker product. For example, with $n = 2$, we have

$$\begin{aligned} v^{(3)} &= [x_1^3, x_1^2x_2, x_1^2x_2, x_1x_2^2, x_1^2x_2, x_1x_2^2, x_1x_2^2, x_2^3]^T, \\ v_3 &= [x_1^3, x_1^2x_2, x_1x_2^2, x_2^3]^T. \end{aligned}$$

Hence, v_i can be formed by taking the unique elements from $v^{(i)}$, and it can be denoted by

$$v_i = M_i v^{(i)}.$$

Note that M_i has more columns than rows for $i > 1$, and therefore it is not invertible. However, since the unique entries in $v^{(i)}$ are contained in v_i, we can write

$$v^{(i)} = N_i v_i.$$

It can be seen that $M_iN_i = I$, from their definitions.

With the notations v_i, we can write $f_i(x) := F_iv_i(x)$ and $h_i(x) := H_iv_i(x)$ with F_i and H_i are constant matrices with proper dimensions.

Furthermore, we denote matrix notations Σ_i and $Q_{i,j}$ for partial differential operations by

$$\frac{\partial v_i(x)}{\partial x} F_1 x := \Sigma_i v_i(x), \tag{13}$$

$$\frac{\partial v_{i-j}(x)}{\partial w} f_{j+1}(x) := -Q_{i,j} v_i(x) \tag{14}$$

for $j = 1, \dots, n_f - 1$. From Lemma 4.6 of [16], we have the explicit expressions of Σ_i and Q_i as

$$\Sigma_i := M_i \bar{\Sigma}_i N_i, \tag{15}$$

$$Q_{i,j} := M_{i-j} \bar{Q}_{i,j} N_i \tag{16}$$

where

$$\bar{\Sigma}_i = \left[\sum_{k=1}^{i} I_n^{(k-1)} \otimes F_1 \otimes I_n^{(i-k)} \right], \tag{17}$$

$$\bar{Q}_{i,j} = -\left[\sum_{k=1}^{i-j} I_n^{(k-1)} \otimes F_j \otimes I_n^{(i-j-k)} \right], \tag{18}$$

and the superscript (k) denotes k times Kronecker products, ie, $I_n^{(3)} = I_n \otimes I_n \otimes I_n$.

With the notations introduced, we can re-write the equation (12) as

$$P_i \Sigma_i v_i(x) - AP_i v_i(x) = \sum_{j=1}^{n_f-1} P_{i-j} Q_{i,j} v_i(x) + BH_i v_i(x),$$

and therefore

$$P_i \Sigma_i - AP_i = \sum_{j=1}^{n_f-1} P_{i-j} Q_{i,j} + BH_i. \tag{19}$$

for $i \in \mathbb{N}$. If we have solutions of P_i form (19), we will then have a solution for the regulation equation (3). Based on Assumption 1, the solution of P_i is guaranteed.

Theorem 1: There exists a unique solution for the matrix equation (19) if Assumption 1 is satisfied.

Proof. The existence of unique solution can be established based on the results shown [9], [16].

IV. Convergence Analysis of Power Series Solutions

In this section, we will analyze the convergence issue of the power series solutions obtained form the iterative polynomial method. As shown in the previous section that P_i can be obtained from (19) iteratively for $i = 1, \dots, N$, with N denoting any big positive integer. Hence we can write

$$p(x) = \sum_{i=1}^{\infty} P_i v_i(x).$$

We introduce a number of notations for the convergence analysis. We use σ_1 to denote the largest singular value and σ_n for the smallest singular values, regardless the actual dimension of the matrix, and similarly, λ_1 and λ_n for the eigenvalues of a matrix with largest and smallest modules. The notation $\text{vec}(A)$ is to denote the vector formed by stacking all the vectors of A. We have used $\|\cdot\|$ to denote 2-norm for a vector or its induced norm for a matrix, and $\|\cdot\|_\infty$ for the infinity norm for a matrix $A \in R^{n\times m}$ as

$$\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{m} |a_{i,j}|.$$

We also define $\mu_n \in R^n$ with all the elements equal 1, ie, $\mu_3 = [1,\ 1,\ 1]^T$, and for a matrix $A = \{a_{i,j}\} \in R^{n\times m}$ we denote $A^+ = \{|a_{i,j}|\}$. Furthermore, for $A, B \in R^{n\times m}$, we define $A \le B$ if $a_{i,j} \le b_{i,j}$ for $i = 1, \dots n$ and $j = 1, \dots m$.

For the convergence of this series, we have the following theorem.

Theorem 2: The series $\{p_i(x)\}$ for $i \in \mathbb{N}$ converges in l_2 for $|x_i| \le d_x$, $i = 1, \dots, n$, if

$$|\lambda_1(S)| < 1, \tag{20}$$

where $S \in \mathbb{R}^{(n_f-1)\times(n_f-1)}$ and

$$S = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ddots & 1 \\ s_{n_f-1} & s_{n_f-1} & \dots & s_1 \end{bmatrix} \tag{21}$$

with

$$s_j = \frac{\|F_{j+1}\|_\infty}{\sigma_n(F_1)} d_x^j$$

for $j = 1, \dots, n_f - 1$.

We need a few technical results for the proof of this theorem.

In the following lemma, we list a number of results on matrices and singular values that are needed later on.

Lemma 3: The following facts hold:
3.1 For $A \in R^{n\times m}$, $\sigma_1(I \otimes A) = \sigma_1(A \otimes I) = \sigma_1(A)$ and $\sigma_n(I \otimes A) = \sigma_n(A \otimes I) = \sigma_n(A)$.
3.2 For $A \in R^{n\times m}$, $\sigma_n(A) = |\lambda_n(A)|$ if A is a normal matrix.
3.3 For A, X and B in proper dimensions such that AXB exists, $\text{vec}(AXB) = (B^T \otimes A)\text{vec}(X)$.
3.4 For $A \in R^{n\times m}$, $\|A \otimes I\|_\infty = \|I \otimes A\|_\infty = \|A\|_\infty$.
3.5 $A \in R^{n\times m}$, $A^+\mu_m \leq \|A\|_\infty \mu_n$.
3.6 For A and B in proper dimensions such that AB exists, $(AB)^+ \leq A^+B^+$, and if A is diagonal, $(AB)^+ \leq \sigma_1(A)B^+$.

Proof. The results 3.1 to 3.3 can be directly found or derived from the results in the reference [17]. The result shown in 3.1 follows Theorem 4.2.15, p.246 of [17]. 3.2 follows a result shown on p.162 of [17]. 3.4 follows from Lemma 4.3.1, p.254 of [17]. 3.4 follows from direct evaluation of Kronecker products. The results in 3.5 and 3.6 follows from direct evaluation of matrix products.

Lemma 4: For the matrices M_i and N_i, the following facts hold:
4.1 $N_iM_i\bar{v}^{(i)} = \bar{v}^{(i)}$ with $\bar{v}^{(i)} := \mu_{n^i}$.
4.2 Let D be a diagonal matrix with proper dimensions such that M_iDN_i, can be evaluated, and D invertible. The matrix M_iDN_i is diagonal too, and $(M_iDN_i)^{-1} = M_iD^{-1}N_i$.

Proof. From the definitions, we have $v^{(i)} = N_iv_i = N_iM_iv^i$. Therefore, 4.1 can be obtained by setting $x_j = 1$ for $j = 1, \ldots, n$.

From the definition, M_i has one and only one no-zero entry with value 1 in each row. N_i, too, has one and only one no-zero entry with value 1 in each row, but it will have at least one 1 in the entry of each column. The number of "1"s depends on the combinations of x_j to make up the polynomial entry. From the definitions, we have $M_iN_i = I$, ie, the nonzero entry in the jth row of M_i matches with a nonzero entry of 1 in the jth column. Hence with a diagonal matrix D, M_iDN_i is a diagonal matrix with only the possible entries from the diagonal elements of D. If we denote $D = \text{diag}\{d_j\}$ for $j \in J := \{1, 2, \ldots n^i\}$, we have $M_iDN_i = \text{diag}\{d_{j'}\}$ for $j' \in J' \subset J$. Similarly, it can be shown that $M_iD^{-1}N_i = \text{diag}\{1/d_{j'}\}$ for $j' \in J'$. Therefore, we conclude $(M_iDN_i)^{-1} = M_iD^{-1}N_i$.

We are ready to prove Theorem 2.

Proof of Theorem 2. We first establish a relationship between P_i and P_j with $j = i-1, \ldots, i-n_f+1$. Let us assume that $i > n_h$ which implies $H_i = 0$. From (19), we then have

$$P_i\Sigma_i - AP_i = \sum_{j=1}^{n_f-1} P_{i-j}Q_{i,j}. \tag{22}$$

Using the notation of vec$(\cdot)$, and the result 3.3 in Lemma 3, we have

$$\text{vec}(P_i) = \Omega_i^{-1}\sum_{j=1}^{n_f-1}[Q_{i,j}^T \otimes I]\text{vec}(P_{i-j}) \tag{23}$$

where

$$\Omega_i = [\Sigma_i^T \otimes I - I \otimes A].$$

From (17) and 4.2 of Lemma 4, we know that Σ_i is diagonal and furthermore that Ω_i is diagonal, as A is diagonal. From 3.2 of Lemma 3, we have

$$\sigma_n(\Omega_i) = |\lambda_n(\Omega_i)| \geq i\sigma_n(F_1) + \sigma_n(A). \tag{24}$$

Notice that Ω_i is a diagonal matrix with all its elements positive, ie, $\Omega_i^+ = \Omega_i$.

From (23), using the result 3.6, we have

$$\begin{aligned}
\text{vec}(P_i^+) &\leq (\Omega_i^{-1})^+ \sum_{j=1}^{n_f-1}[(Q_{i,j}^T)^+ \otimes I]\text{vec}(P_{i-j}^+) \\
&\leq \sigma_1(\Omega_i^{-1}) \sum_{j=1}^{n_f-1}[(Q_{i,j}^T)^+ \otimes I]\text{vec}(P_{i-j}^+) \\
&= \sigma_n^{-1}(\Omega_i) \sum_{j=1}^{n_f-1}[(Q_{i,j}^T)^+ \otimes I]\text{vec}(P_{i-j}^+)
\end{aligned}$$

which implies, with the result 3.3 in Lemma 3, that

$$\begin{aligned}
P_i^+ &\leq \sigma_n^{-1}(\Omega_i) \sum_{j=1}^{n_f-1} P_{i-i}^+ Q_{i,j}^+ \\
&\leq \sigma_n^{-1}(\Omega_i) \sum_{j=1}^{n_f-1} P_{i-1}^+ M_{i-j}\bar{Q}_{i,j}^+ N_i
\end{aligned} \tag{25}$$

Multiplying both sides of (25) by $M_i\bar{v}^{(i)}$ and using the results 4.1, 3.5 and 3.6, we have,

$$\begin{aligned}
& P_i^+ M_i \bar{v}^{(i)} \\
\leq\; & \sigma_n^{-1}(\Omega_i) \sum_{j=1}^{n_f-1} P_{i-j}^+ M_{i-j}\bar{Q}_{i,j}^+ N_i M_i \bar{v}^{(i)} \\
=\; & \sigma_n^{-1}(\Omega_i) \sum_{j=1}^{n_f-1} P_{i-j}^+ M_{i-j}\bar{Q}_{i,j}^+ \bar{v}^{(i)} \\
\leq\; & \sigma_n^{-1}(\Omega_i) \sum_{j=1}^{n_f-1} P_{i-j}^+ M_{i-j}\|\bar{Q}_{i,j}^+\|_\infty \bar{v}^{(i-j)}
\end{aligned}$$

From (18), it can be obtained that

$$\begin{aligned}
\|\bar{Q}_{i,j}^+\|_\infty &= \|\sum_{k=1}^{i-j} I_n^{(i-1)} \otimes F_{j+1} \otimes I_n^{(i-1-k)}\|_\infty \\
&\leq \sum_{k=1}^{i-j} \|I_n^{(i-1)} \otimes F_{j+1} \otimes I_n^{(i-1-k)}\|_\infty \\
&\leq \sum_{k=1}^{i-j} \|F_{j+1} \otimes I_n^{(i-1-k)}\|_\infty \\
&\leq (i-j)\|F_{j+1}\|_\infty.
\end{aligned}$$

Therefore we have

$$\begin{aligned} & P_i^+ M_i \bar{v}^{(i)} \\ \leq & \sum_{j=1}^{n_f-1} \frac{(i-j)\|F_{j+1}\|_\infty}{i\sigma_n(F_1)+\sigma_n(A)} P_{i-j}^+ M_{i-j}\bar{v}^{(i-j)} \\ < & \sum_{j=1}^{n_f-1} \frac{\|F_{j+1}\|_\infty}{\sigma_n(F_1)} P_{i-j}^+ M_{i-j}\bar{v}^{(i-j)}. \end{aligned} \tag{26}$$

With the result shown in (26), we can establish a bound of $p_i(x)$ as

$$\begin{aligned} & \|p_i(x)\| \\ \leq & \|P_i^+ M_i (v^{(i)})^+\| \\ \leq & \|P_i^+ M_i \bar{v}^{(i)}\| d_x^i \\ < & \sum_{j=1}^{n_f-1} \frac{\|F_{j+1}\|_\infty}{\sigma_n(F_1)} \|P_{i-j}^+ M_{i-j}\bar{v}^{(i-j)}\| d_x^i \\ \leq & \sum_{j=1}^{n_f-1} \frac{\|F_{j+1}\|_\infty}{\sigma_n(F_1)} \|P_{i-j}^+ M_{i-j}\bar{v}^{(i-j)} d_x^{i-j}\| d_x^j \end{aligned}$$

Define

$$\bar{p}_i = \|P_i^+ M_{i-1}\bar{v}^{(i)}\| d_x^i.$$

We then have that

$$\bar{p}_i < \sum_{j=1}^{n_f-1} \frac{\|F_{j+1}\|_\infty}{\sigma_n(F_1)} d_x^j \bar{p}_{i-j} \tag{27}$$

Let us define a discrete-time system with state variable $z \in \mathbb{R}^{n_f-1}$ with

$$z(i) = [\bar{p}_{i-n_f+1}, \ldots, \bar{p}_{i-1}, \bar{p}_i]^T$$

which implies that

$$z(i+1) = [\bar{p}_{i-n_f+2}, \ldots, \bar{p}_i, \bar{p}_{i+1}]^T.$$

From (27), we have

$$\bar{p}_{i+1} < \sum_{j=1}^{n_f-1} \frac{\|F_{j+1}\|_\infty}{\sigma_n(F_1)} d_x^j \bar{p}_{i-j+1}$$

and therefore we obtain that

$$z(i+1) \leq S z(i). \tag{28}$$

From (28), there exist positive definite matrices $U, W \in \mathbb{R}^{(n_f-1)\times(n_f-1)}$ such that

$$S^T U S - U = -W. \tag{29}$$

Let

$$J_k = \sum_{i=k}^{\infty} z^T(i) W z(i)$$

for an integer $k \geq \max\{n_f, n_h\}$. From (28) and (29), we have

$$\begin{aligned} J_k &= \sum_{i=k}^{\infty} z^T(i)(U - S^T U S) z(i) \\ &\leq \sum_{i=k}^{\infty} z^T(k)(S^T)^{i-k}(U - S^T U S) S^{i-k} z(k) \\ &\leq \sum_{i=k}^{\infty} z^T(k)[(S^T)^{i-k} U S^{i-k} \\ &\qquad - (S^T)^{i-k+1} U S^{i-k+1}] z(k) \\ &= z^T(k)(S^T)^{i-k} U S^{i-k} z(k). \end{aligned}$$

With $\|h_i(x)\| < \bar{h}_i(x)$, we have

$$\sum_{i=k}^{\infty} \|h_i(x)\|^2 < \frac{1}{w_{n_f-1,n_f-1}} J_k$$

where w_{n_f-1,n_f-1} is the last diagonal element of W, and therefore we can conclude $h_i(x)$ converges in l_2.

V. An Example

Consider a nonlinear system

$$\begin{aligned} \dot{x}_1 &= 2.1x_1 + x_2 + 0.3x_1^2 - 0.5x_1^3, \\ \dot{x}_2 &= -x_1, \\ y &= x_1. \end{aligned} \tag{30}$$

For this system, we have

$$F_1 = \begin{bmatrix} 2.1 & 1 \\ -1 & 0 \end{bmatrix}, \; F_2 = \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$F_3 = \begin{bmatrix} -0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

With

$$A = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

we obtain that

$$\begin{aligned} P_1 &= \begin{bmatrix} 0.1352 & 0.3104 \\ 0.3791 & 1.0665 \end{bmatrix}, \\ P_2 &= \begin{bmatrix} -0.0081 & 0.0032 & -0.0042 \\ -0.0212 & 0.0083 & -0.0093 \end{bmatrix}. \end{aligned}$$

Note that even though the nonlinear functions in the dynamical system (30) are polynomials up to third order, the solution $p(x)$ to the equation (3) is not a polynomial with finite order.

Observers in the form of (5) are implemented with $p(x) = P_1 v_1$ for the first order approximation of $p(x)$ and with $p = P_1 v_1 + P_2 v_2$ for the second order approximation. The simulation results are shown in Figures 1 and 2. The second order approximation shows improvements over the first order approximation. A third order approximation was also studied, but no improvements were observed, and this might be due to the fact the x was outside the convergence region for computation of $p(x)$. Further study on this issue will be carried out in future.

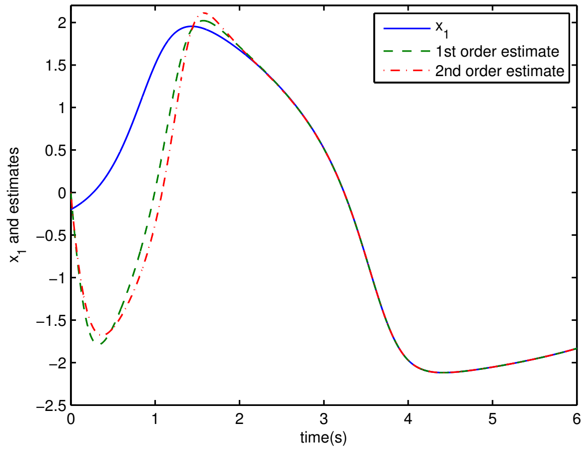

Fig. 1. The unmeasured state variable and its estimates from observers

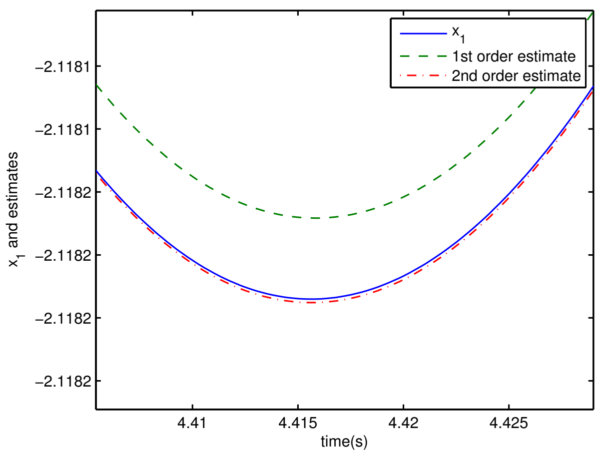

Fig. 2. An enlargement of Figure 1

VI. Conclusion

In this paper, we have identified a set of conditions for the convergence of a series solution to the observer gain design for nonlinear dynamic systems. This result is obtained by analyzing the relationship between the sizes of subsequent terms and then establishing a dynamic model in discrete-time. Assumption 1 is fairly restrictive, which requires the linearized model to be anti-stable. This assumption is used for the convenience of presentation of the basic concepts shown in this paper. The study is underway to relax this condition. Further studies are also needed on how this presented result can be used for determining the proper order of approximation and on possible domain of attraction observer errors. Note that the domain of attraction of the observer errors are different from the domain of the convergence of the power series for the observer gain design.

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