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A Cone-copositive Approach for the Stability of Piecewise Linear Differential Inclusions

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Abstract— **In this paper a cone-copositive approach is proposed for investigating the stability of piecewise linear differential inclusions. From a different perspective the same issue can be viewed as the robust stability problem for uncertain piecewise linear systems. By using piecewise quadratic Lyapunov function the stability problem is formulated as a set of linear matrix inequalities each constrained into a specific cone, i.e. a set of cone-copositive programming problems. A procedure for solving the set of constrained inequalities is presented. The absolute stability problem for Lur'e systems with unknown feedback characteristic belonging to an asymmetric domain, is shown to be tractable as a particular case. Two examples are provided to show that the proposed approach might lead to less conservative estimation of the robust stability region with respect to the classical Circle criterion and to other approaches based on piecewise quadratic Lyapunov function.**

I. Introduction

Piecewise linear (PWL) systems represent a class of hybrid systems characterized by a partition of the state-space into regions where system dynamics can be described by linear models [1]. Unfortunately, mathematical models of practical systems are always affected by uncertainties of various kind. We consider *uncertain* autonomous PWL systems where the state partition consists of convex polyhedral cones and in each cone the uncertain dynamic matrix can be expressed as a convex hull of known matrices. Such class of systems can be viewed as piecewise linear differential inclusions (PWLDIs) [2]. Among others, examples of nonlinear systems which can be embedded in the PWLDI framework are Lur'e systems with possibly asymmetric domain of the feedback characteristic [2] and power electronics converters [3]. A further extension of the PWLDIs class is obtained by considering the piecewise affine slab differential inclusions [4], [5].

Despite their apparent modeling simplicity, PWL systems are very hard to investigate, also when neglecting the uncertainties. Indeed, basic problems such as well-posedness, stability and controllability are far to obtain definite answers, see the good literature review proposed in [6] and [7]. Such issues are highly complex also from a computational point of view [8]. For what concerns stability of PWL systems, several approaches have been proposed in the last years [6], [9], [10]. The simplest way to tackle with the problem consists in using a common quadratic Lyapunov function [2] or a common polynomial Lyapunov function [11], but unfortunately that choice usually implies too much conservative results. To reduce conservativeness one could use the multiple Lyapunov function approach, i.e. to combine Lyapunov functions defined over different regions of the state space, see among others [11], [12]. Within the multiple Lyapunov functions approaches the piecewise quadratic Lyapunov function allows to easily formulate the stability conditions in terms of linear matrix inequalities [13].

In this paper we focus on the stability analysis for autonomous continuous-time PWLDIs, whose results can be alternatively used for solving the robust stability problem of autonomous continuous-time uncertain PWL systems. The stability of linear differential inclusions has been widely analyzed in the literature, see among others [14]–[17]. On the other hand, to the best of our knowledge a limited attention has been dedicated to the case of *piecewise* linear differential inclusions. In [18] and [5] possible solutions exploiting a common quadratic Lyapunov function are proposed. In [2], [19], [20] it is shown that piecewise quadratic Lyapunov functions might be less conservative. Similar conclusions are obtained in [21] through the use of homogeneous polynomial Lyapunov functions. In the above papers the stability problem is formulated by means of a set of constrained inequalities which are transformed into an unconstrained problem by relaxing the linear constraints to quadratic constraints and by applying the $\mathcal{S}$-procedure [22]. In this paper we propose to explicitly consider the conic constraints in order to get less conservative results. Obviously the problem then moves to finding a solution for cone constrained inequalities, which is usually called a cone-copositive problem. Recently several papers have appeared in the mathematical literature dealing with the copositive programming problems, see [23], [24] and the bibliography therein. However, in these specialized papers the focus is on the search for conditions for which a given fixed matrix is cone-copositive, while the stability problem formulated in this paper is more challenging because consists in finding, for each cone of the state partition, an *unknown* matrix that results cone-copositive. The proposed method is obtained by a combination of the piecewise quadratic approach in [2] with the extension of a recently established cone-copositive algorithm for the quadratic stability of certain PWL systems in [25].

We will show that the proposed cone-copositive approach used for the stability analysis of PWLDIs can be successfully applied for the absolute stability analysis of Lur'e systems with asymmetric domains of the feedback characteristic. Two

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numerical examples show that we are able to obtain less conservative results with respect to the classical Circle criterion and to other approaches based on piecewise quadratic Lyapunov function.

II. Preliminaries

In this section we recall some useful definitions for our analysis. For most of them one could refer to [26] and [27].

***Definition** 1:* Given a real vector x and a positive integer p, the p-norm is given by

$$\|x\|_p = \sum_{i=1}^{n} |x_i|^{1/p}. \tag{1}$$

***Definition** 2:* A set $\mathcal{C} \subseteq \mathbb{R}^n$ is a *cone* if for every $x \in \mathcal{C}$ and nonnegative real number θ we have $\theta x \in \mathcal{C}$.

***Definition** 3:* A set $\mathcal{C} \subseteq \mathbb{R}^n$ is a *convex cone* if it is convex and a cone, which means that for any nonnegative real numbers θ_1 and θ_2, and for any x_1 and x_2 belonging to $\mathcal{C}$ we have $\theta_1 x_1 + \theta_2 x_2 \in \mathcal{C}$.

***Definition** 4 ($\mathcal{H}$-representation, [28]):* A set $\mathcal{C} \subseteq \mathbb{R}^n$ is a *polyhedral convex cone* if it is convex cone and there exists a nonzero matrix C such that for each $x \in \mathcal{C}$ it is verified $Cx \succeq 0$, where the symbol '$\succeq$' used with vectors is intended as componentwise inequalities.

***Definition** 5:* Let X be any set in $\mathbb{R}^n$ and $N > 1$ an integer. A family $\mathcal{P} = \{X_1, \dots, X_N\}$ of nonempty sets satisfying

$$X = X_1 \cup \dots \cup X_N \tag{2a}$$

$$\text{int } X_i \neq \varnothing, \quad \forall i \tag{2b}$$

$$\text{int } X_i \cap \text{int } X_j = \varnothing, \quad \text{for } i \neq j \tag{2c}$$

is called a *partition* of X.

If the sets X_i are cones, the family, say $\mathcal{P}_\mathcal{C}$, is called a *conic partition*.

***Definition** 6:* We say that a matrix M is copositive with respect to a cone $\mathcal{C}$ (or cone-copositive), which will be denoted by $M \succeq^{\mathcal{C}} 0$, if $x^T M x \geqslant 0$ for any $x \in \mathcal{C}$. If equality only holds for $x = 0$, then M is strictly cone-copositive ($M \succ^{\mathcal{C}} 0$).

***Definition** 7:* Given p points $x_1, \dots, x_p$ belonging to a set $X \subseteq \mathbb{R}^n$, a *convex closure* $\overline{\text{co}}\,\{x_1, \dots, x_p\}$ is defined by the set of all $\bar{x} \in X$ such that there exist p nonnegative scalars $\theta_1, \dots, \theta_p$, with $\sum_{i=1}^{p} \theta_i = 1$, for which $\bar{x} = \sum_{i=1}^{p} \theta_i x_i$.

***Definition** 8:* Given p points $x_1, \dots, x_p$ belonging to a set $X \subseteq \mathbb{R}^n$, which are affinely independent, i.e. the $p-1$ points $x_2 - x_1, \dots, x_p - x_1$ are linearly independent, then the convex closure $\overline{\text{co}}\,\{x_1, \dots, x_p\}$ is called a $(p-1)$-simplex. Clearly in order to define a simplex we need $p \leqslant n+1$.

***Definition** 9:* The convex closure of any nonempty subset of m points of the p points that define a $(p-1)$-simplex, with $m \leqslant p$, is called an $(m-1)$-face of the $(p-1)$-simplex. The 0-faces ($m = 1$) determine the set of so-called *vertices*, say $\mathcal{V}$, which is equal to the set of points that define the simplex. The 1-faces determine the set of so-called *edges*, say $\mathcal{E}$, which is equal to the set of convex closures of any $m = 2$ points among the simplex vertices.

If the sets X_i in Def. 5 are simplices, the family say $\mathcal{P}_\mathcal{S}$ is called a *simplicial partition* or *triangularization* of X [25], [26]. We denote by $\mathcal{V}_\mathcal{P}$ the set of all vertices of simplices in $\mathcal{P}_\mathcal{S}$, and by $\mathcal{E}_\mathcal{P}$ the set of all edges of simplices in $\mathcal{P}_\mathcal{S}$.

For our analysis it is useful to consider the intersection between a polyhedral convex cone $\mathcal{C}$ and the set

$$\mathcal{B}_1 = \{x \in \mathbb{R}^n : \|x\|_1 = 1\}. \tag{3}$$

It is always possible to find a simplicial partition of the set defined by $\mathcal{C} \cap \mathcal{B}_1$.

Figure 1 shows an example in $\mathbb{R}^2$ of a polyhedral convex cone (gray area), the set $\mathcal{B}_1$, and their intersection partitioned into two simplices. The vertices of the two simplices are $\mathcal{V}_1 = \{P_1, P_3\}$ and $\mathcal{V}_2 = \{P_3, P_2\}$ and the edges are $\mathcal{E}_1 = \{\overline{P_1P_3}\}$ and $\mathcal{E}_2 = \{\overline{P_2P_3}\}$, respectively. The suggested simplicial partition of $\mathcal{C} \cap \mathcal{B}_1$ is characterized by the set of vertices $\mathcal{V}_\mathcal{P} = \{\mathcal{V}_1, \mathcal{V}_2\}$ and edges $\mathcal{E}_\mathcal{P} = \{\mathcal{E}_1, \mathcal{E}_2\}$. Note that the segment $\overline{P_1P_2}$ does not belong to $\mathcal{E}_\mathcal{P}$..

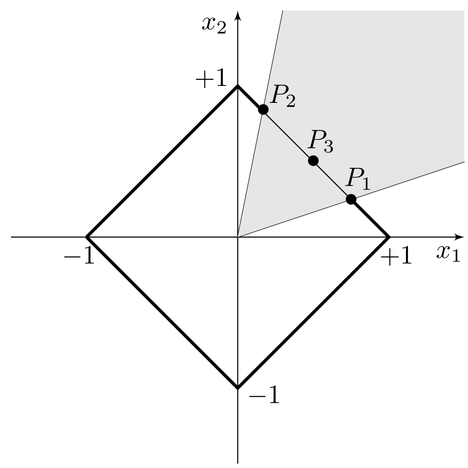

Fig. 1. A simplicial partition in $\mathbb{R}^2$.

The cone-copositivity evaluation of a generic matrix M, with $\mathcal{C}$ being a polyhedral convex cone, can be simplified by the following lemmas [25].

***Lemma** 1:* The following equivalence holds:

$$M \succeq^{\mathcal{C}} 0 \Longleftrightarrow x^T M x \geqslant 0, \ \forall\, x \in \mathcal{C} \cap \mathcal{B}_1. \tag{4}$$

An analogous result holds for $M \succ^{\mathcal{C}} 0$.

***Lemma** 2:* Let M be a symmetric matrix. Let $\mathcal{P}_\mathcal{S}$ be a simplicial partition of $\mathcal{C} \cap \mathcal{B}_1$. If

$$v^T M v \geqslant 0, \quad \forall\, v \in \mathcal{V}_\mathcal{P} \tag{5a}$$

$$u^T M v \geqslant 0, \quad \forall\, \{u, v\} \in \mathcal{E}_\mathcal{P} \tag{5b}$$

then M is cone-copositive. An analogous result holds for strict inequalities.

***Remark** 1:* Clearly if for some of the vertices $v \in \mathcal{V}_\mathcal{P}$ it results $v^T M v < 0$, then M is not cone-copositive.

The opposite result is also of interest. By assuming that $M \succeq^{\mathcal{C}} 0$ the following lemma ensures that it is always possible to find a finite simplicial partition of $\mathcal{C} \cap \mathcal{B}_1$ such that (5) are satisfied.

Lemma 3: Let M be a symmetric matrix and $M \succeq^{\mathcal{C}} 0$. Then there exists $\epsilon > 0$ such that for all finite simplicial partitions $\mathcal{P}_{\mathcal{S}}$ of $\mathcal{C} \cap \mathcal{B}_1$ with maximum diameter of a simplex partition $\delta(\mathcal{P}_{\mathcal{S}})$ not larger than ϵ, i.e.

$$\delta(\mathcal{P}_{\mathcal{S}}) \triangleq \max_{(u,v) \in \mathcal{E}_{\mathcal{P}}} \|u - v\| \leqslant \epsilon, \tag{6}$$

we have

$$v^T M v \geqslant 0, \quad \forall\, v \in \mathcal{V}_{\mathcal{P}} \tag{7a}$$
$$u^T M v \geqslant 0, \quad \forall\, \{u, v\} \in \mathcal{E}_{\mathcal{P}}. \tag{7b}$$

The three lemmas above will be useful for the stability analysis of the class of systems presented in next section.

III. Piecewise linear differential inclusions

Let us consider a partition of $X = \mathbb{R}^n$ into a family of polyhedral convex cones represented by

$$\mathcal{C}_i \triangleq \{x \in X : C_i x \succeq 0\} \tag{8}$$

for $i = 1, \ldots, N$. From Def. 5, since $X = \mathbb{R}^n$, two cones of the partition with $\mathcal{C}_i \cap \mathcal{C}_j \neq \{0\}$ share their common boundary. For such polyhedral cones there exist so-called continuity matrices F_i and F_j such that $F_i x^* = F_j x^*$ for $x^* \in \mathcal{C}_i \cap \mathcal{C}_j$ [2]. Note that the continuity matrices are not unique. For a given partition a possible choice is $F_i = C_i$ for $i = 1, \ldots, N$.

In each ith cone we assume that the system can be written as

$$\dot{x} = \bar{A}_i(t) x, \quad x \in \mathcal{C}_i \tag{9}$$

The system matrix $\bar{A}_i(t)$ is *uncertain* and can be expressed as a time-varying convex combination of known constant matrices:

$$\bar{A}_i(t) = \sum_{k \in K(i)} \alpha_{i,k}(t) A_{i,k} \tag{10}$$

where $K(i)$ is an index set that specifies the matrices used in the combination of the ith cone, $A_{i,k}$ are constant known matrices which will be also called in the sequel as the *extreme matrices* in that cone. For each i and $k \in K(i)$ the function $\alpha_{i,k}(t) : \mathbb{R} \to [0, 1]$ is assumed to be uncertain, continuous and constrained by

$$\sum_{k \in K(i)} \alpha_{i,k}(t) = 1 \tag{11}$$

for all i and t. The function $\alpha_{i,k}(t)$ can be also interpreted as an uncertain time-varying parameter.

We will also rewrite (10) as

$$\bar{A}_i(t) \in \overline{\text{co}}_{k \in K(i)} \{A_{i,k}\} \tag{12}$$

where $\overline{\text{co}}_{k \in K(i)}$ determines the convex closure in each cone.

Thus the overall system (9) can be rewritten

$$\dot{x} \in \overline{\text{co}}_{k \in K(i)} \{A_{i,k} x\}, \quad x \in \mathcal{C}_i. \tag{13}$$

These models are called *piecewise linear differential inclusions* [2].

Let us introduce the solution concept we deal with in this paper:

Definition 10: An absolutely continuous function $x(t)$ is called a solution of (13) with initial condition $x(t_0) = x_0$ if it satisfies (13) for almost all $t \geqslant t_0$.

Note that Def. 10 rules out the so called *sliding behavior* since it is not possible to define the vector field when the state trajectory belongs, for a finite nonzero time interval, to a common boundary between two (or more) cones.

IV. Main result on stability

For system (13) we use the following (strong) stability definitions [14].

Definition 11: We say that the zero solution of (13), i.e. the origin $x \equiv 0$, is asymptotically stable if:

- for every $\epsilon > 0$ there exists a $\delta_\epsilon > 0$ such that for each solution $x(t)$ of (13) the inequality $\|x(t)\| < \epsilon$ holds for all $t \geqslant t_0$ if $\|x(t_0)\| < \delta_\epsilon$;
- there exists a $\Delta > 0$ such that for every solution $x(t)$ of (13) with $\|x(t_0)\| < \Delta$ the relation $\lim_{t \to +\infty} x(t) = 0$ holds.

If the inequality $\|x(t)\| \leqslant c_1 \|x(t_0)\| e^{-c_2(t - t_0)}$ holds for every solution of (13), where the constants $c_1 \geqslant 1$ and $c_2 > 0$ do not depend on the solution $x(t)$ and the initial time t_0, we say that the origin is globally exponentially stable.

In order to prove the stability of the zero solution, we need to prove the following

Lemma 4: The constrained linear matrix inequalities

$$P \succ^{\mathcal{C}} 0 \tag{14a}$$
$$A_k^T P + P A_k \prec^{\mathcal{C}} 0, \quad k \in K, \tag{14b}$$

with P symmetric matrix, A_k known real matrices, K a finite index set, and $\mathcal{C}$ a polyhedral convex cone

a) have a solution P if and only if there exists a finite simplicial partition $\mathcal{P}_{\mathcal{S}}$ of the set $\mathcal{C} \cap \mathcal{B}_1$ such that the set of inequalities

$$v^T P v > 0, \quad \forall\, v \in \mathcal{V}_{\mathcal{P}} \tag{15a}$$
$$u^T P v > 0, \quad \forall\, \{u, v\} \in \mathcal{E}_{\mathcal{P}} \tag{15b}$$
$$v^T (A_k^T P + P A_k) v < 0, \quad \forall\, v \in \mathcal{V}_{\mathcal{P}},\, k \in K \tag{15c}$$
$$u^T (A_k^T P + P A_k) v < 0, \quad \forall\, \{u, v\} \in \mathcal{E}_{\mathcal{P}},\, k \in K \tag{15d}$$

has a solution;

b) have no solution if there exists a finite simplicial partition $\mathcal{P}_{\mathcal{S}}$ of the set $\mathcal{C} \cap \mathcal{B}_1$ such that the set of inequalities

$$v^T P v > 0, \quad \forall\, v \in \mathcal{V}_{\mathcal{P}} \tag{16a}$$
$$v^T (A_k^T P + P A_k) v < 0, \quad \forall\, v \in \mathcal{V}_{\mathcal{P}},\, k \in K \tag{16b}$$

has no solution.

Proof: If (14) hold then from Lemma 1 and Lemma 3 the inequalities (15) are also satisfied. Vice-versa, if (15) hold than from Lemma 2 we get (14). The second part of the proof is straightforward by considering Remark 1. ■

Lemma 4 will be exploited for the stability analysis of the zero solution of (13) by interpreting $\mathcal{C}$ as one of the cones (8) and A_k in (14b) as the corresponding extreme matrices.

We can now proceed with the following

***Lemma** 5:* Consider system (13). If there exist $P_i = F_i^T T F_i$ $i = 1, \ldots, N$, with T symmetric matrix, such that the following constrained linear matrix inequalities

$$P_i \succ^{\mathcal{C}_i} 0 \tag{17a}$$

$$A_{i,k}^T P_i + P_i A_{i,k} \prec^{\mathcal{C}_i} 0, \quad k \in K(i) \tag{17b}$$

are satisfied, then the zero solution of (13) is globally exponentially stable.

Proof: Let us define the candidate piecewise quadratic Lyapunov function

$$V(x(t)) = x(t)^T P_i x(t), \quad x(t) \in \mathcal{C}_i, \tag{18}$$

where $x(t)$ is a solution of (13). The use of the continuity matrices F_i in the definition of P_i allows to conclude that (18) is continuous and piecewise differentiable. Since in each cone the dynamic matrix is an uncertain convex combination of the extreme (known) matrices, the decreasing of (18) is implied by (17). Then by standard Lyapunov stability arguments the proof follows. ■

***Remark** 2:* In order to solve (17) one could transform the constrained inequalities into an unconstrained problem, by relaxing the conic constraints to quadratic constraints and by applying the $\mathcal{S}$-procedure [2]. Instead with our approach the cone constrained inequalities are explicitly taken into account in the problem solution, thus allowing less conservative results.

It is now possible to prove the following main result.

***Theorem** 1:* Given the system (13), if for each $i = 1, \ldots, N$ there exists a finite simplicial partition $\mathcal{P}_{\mathcal{S}_i}$ of the set $\mathcal{C}_i \cap \mathcal{B}_1$ such that the set of inequalities

$$v_i^T P_i v_i > 0, \quad \forall\, v_i \in \mathcal{V}_{\mathcal{P}_i} \tag{19a}$$

$$u_i^T P_i v_i > 0, \quad \forall\, \{u_i, v_i\} \in \mathcal{E}_{\mathcal{P}_i} \tag{19b}$$

$$v_i^T (A_{i,k}^T P_i + P_i A_{i,k}) v_i < 0, \quad \forall\, v_i \in \mathcal{V}_{\mathcal{P}_i},\, k \in K(i) \tag{19c}$$

$$u_i^T (A_{i,k}^T P_i + P_i A_{i,k}) v_i < 0, \quad \forall\, \{u_i, v_i\} \in \mathcal{E}_{\mathcal{P}_i},\, k \in K(i) \tag{19d}$$

has a solution with $P_i = F_i^T T F_i$ and T symmetric matrix independent of i, then the zero solution of (13) is globally exponentially stable.

Proof: The proof directly follows from the application of Lemma 4 and Lemma 5. ■

Theorem 1 and Lemma 4.b allow to define a procedure for checking the exponential stability of the zero solution of piecewise linear differential inclusions.

In order to define the PWLDI (13), it is necessary to introduce a conic partition of $\mathbb{R}^n$, with cones given by (8), and to choose the continuity matrices. The system structure might suggest a first conic partition trial, so as shown in the next section dedicate to Lur'e systems. The freedom in the selection of the continuity matrices can be used to increase the flexibility in the choice of candidate Lyapunov functions, so as shown in [2].

For each cone $\mathcal{C}_i$ consider the set $\mathcal{C}_i \cap \mathcal{B}_1$ and choose a corresponding simplicial partition $\mathcal{P}_{\mathcal{S}_i}$. If conditions (19) are satisfied for some T, then the matrices $P_i = F_i^T T F_i$ with $i = 1, \ldots, N$ can be used to define a piecewise quadratic Lyapunov function for the system (13). If we are not able to conclude feasibility of (19), then a more refined simplicial partition of the sets $\mathcal{C}_i \cap \mathcal{B}_1$ with $i = 1, \ldots, N$ can be used. To this aim a computationally convenient procedure for the partition refinement is the so-called *bisection along the longest edge*, which guarantees $\delta(\mathcal{P}_{\mathcal{S}_i}) \to 0$ when the procedure is repeatedly applied [25]. If a solution exists, Lemma 3 ensures the convergence in a finite number of steps. Instead, if we can conclude that (19a) and (19c) have no solution for one or some of the cones, then (see Lemma 4.b) we need to go back to the system definition and use a refined conic partition of the cones where no solution has been found.

V. Application to Lur'e systems

Lur'e systems can be recast in the form (13) by employing the global linearization approach [18]. Let us consider the Lur'e feedback system in Fig. 2 where Σ_d is a linear system with (A, b, c^T) being a minimal state space realization, $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}^n$. The static characteristic $\varphi(\lambda) : \mathbb{R} \mapsto \mathbb{R}$ is a single-input single-output uncertain characteristic belonging to the domain (k_1, k_2, k_3, k_4) with $k_1 < k_2$, $k_3 < k_4$, see Fig. 3. For this class of systems a natural conic partition of the state space $\mathbb{R}^n$ consists of the two half spaces $\mathcal{C}_1 = \{x \in \mathbb{R}^n : c^T x \geqslant 0\}$ and $\mathcal{C}_2 = \{x \in \mathbb{R}^n : -c^T x \geqslant 0\}$, see Fig. 4. In $\mathcal{C}_1$ we can write

$$\varphi(\lambda) = \varphi(c^T x) \in \overline{\text{co}}\{k_1 c^T x,\, k_2 c^T x\} \tag{20}$$

and in $\mathcal{C}_2$

$$\varphi(\lambda) = \varphi(c^T x) \in \overline{\text{co}}\{k_3 c^T x,\, k_4 c^T x\}. \tag{21}$$

By looking at the closed loop system and by using (20)–(21), the extreme matrices in $\mathcal{C}_1$ and $\mathcal{C}_2$ are, respectively

$$A_{1,1} = A - b k_1 c^T, \qquad A_{1,2} = A - b k_2 c^T \tag{22a}$$

$$A_{2,1} = A - b k_3 c^T, \qquad A_{2,2} = A - b k_4 c^T. \tag{22b}$$

The above matrices can be used to check conditions (19).

In order to illustrate the proposed approach we will investigate first a simple second order example and then a more complex third order example. For both examples we obtain results not allowed by the Circle criterion and less conservative with respect to the method in [2].

***Example** 1:* Let us consider the Example at page 121 in [29]. The system matrices in our formulation become

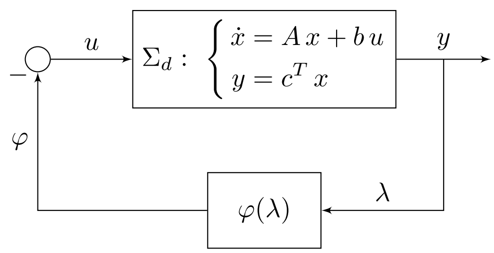

Fig. 2. Block diagram of the Lur'e system.

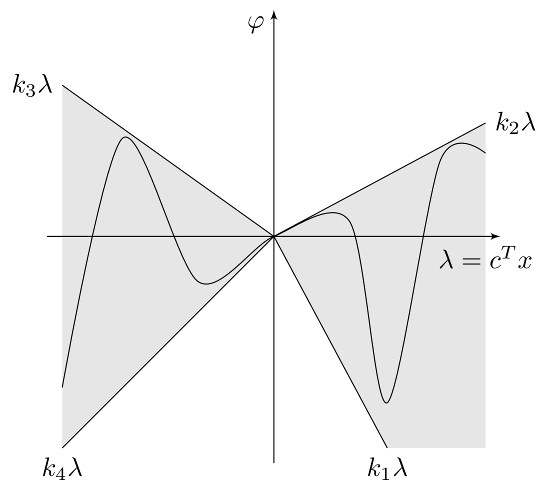

Fig. 3. Asymmetric domain of the feedback characteristic.

$$A = \begin{pmatrix} -5.21 & -4 \\ -2.47 & -2 \end{pmatrix}, \quad b = \begin{pmatrix} -3 \\ -21 \end{pmatrix} \tag{23a}$$

$$c^T = \begin{pmatrix} 1 & 0 \end{pmatrix}. \tag{23b}$$

By using the Circle criterion, or by applying the if and only if quadratic stability conditions for switched systems in the case of two modes with Hurwitz matrices (see [30]), it is possible to prove the absolute stability with respect to the (symmetric) sector $(0, 0.61, 0, 0.61)$. By applying Theorem 4.2 of [2] we were able to prove the stability in the asymmetric domain $(1.9, 2.2, 0, 0.15)$ with a partition in the four quadrants of the state space. However by maintaining the same partition we were not able to enlarge the stability region for lower values of k_1.

For such system, with the same partition of the state space in the four quadrants, by applying Theorem 1, we are able to prove the absolute stability with respect to the asymmetric domain $(0, 2.2, 0, 0.15)$ with the matrices

$$P_1 = \begin{pmatrix} 10.020 & -0.400 \\ -0.400 & 0.950 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0.355 & -0.140 \\ -0.140 & 0.950 \end{pmatrix}$$

$$P_3 = \begin{pmatrix} 0.355 & 2.450 \\ 2.450 & 26.510 \end{pmatrix}, \quad P_4 = \begin{pmatrix} 10.020 & -1.090 \\ -1.090 & 26.510 \end{pmatrix}$$

The cones $\mathcal{C}_i$ are the quadrants taken in the classical counterclockwise order.

***Example** 2:* Let us consider the Example 5.3 in [2]. By neglecting the uncertainty part on the dynamic system, the model corresponds to

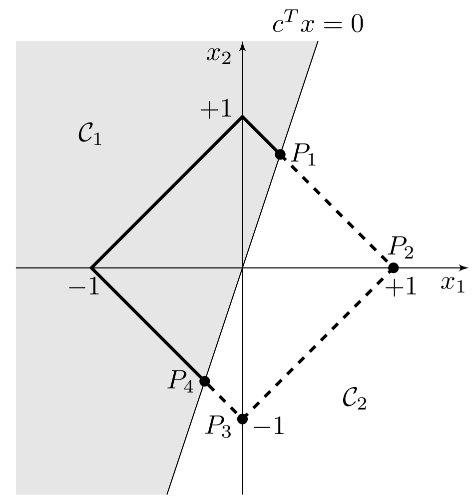

Fig. 4. A state space conic partition ($c^T = (c_1\ c_2)$ with $c_1 < 0$ and $c_2 > 0$) and the set $\mathcal{B}_1$, with their intersections $\mathcal{C}_1 \cap \mathcal{B}_1$ (continuous thick line) and $\mathcal{C}_2 \cap \mathcal{B}_1$ (dashed thick line). Note that the set $\mathcal{C}_2 \cap \mathcal{B}_1$ (so as $\mathcal{C}_1 \cap \mathcal{B}_1$) is not a simplex and it is partitioned into the three simplices with vertices $\mathcal{V}_1 = \{P_1, P_2\}$, $\mathcal{V}_2 = \{P_2, P_3\}$ and $\mathcal{V}_3 = \{P_3, P_4\}$.

$$A = \begin{pmatrix} -3 & 0 & -1 \\ 4 & -1 & 0 \\ 0 & 2 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} \tag{24a}$$

$$c^T = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}. \tag{24b}$$

The feedback characteristic is assumed to belong to an asymmetric domain as in Fig. 3 with $k_3 = 0$, $k_2 > k_1 \geqslant 0$ and $k_4 > 0$. By applying the Circle criterion, or following the approach in [30], one gets the absolute stability for characteristics belonging to the symmetric sector $(0, 1.14, 0, 1.14)$.

Instead by applying Theorem 4.2 of [2] we were able to prove the stability in the asymmetric domain $(0.1, 1.4, 0, 0.2)$ with a partition in the eight orthants of the state space. However, with the arguments in [2] and maintaining the same state space partition we were not able to prove the absolute stability in the asymmetric sector $(0, 1.4, 0, 0.2)$.

By applying Theorem 1 proposed in this paper, still with the eight orthants state space partition, it is possible to prove the absolute stability in the asymmetric domain defined by $(0, 1.4, 0, 0.2)$ with

$$P_1 = \begin{pmatrix} 8.000 & 4.810 & -0.390 \\ 4.810 & 6.700 & 1.510 \\ -0.390 & 1.510 & 3.870 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 6.100 & 1.470 & -0.090 \\ 1.470 & 6.700 & 1.510 \\ -0.090 & 1.510 & 3.870 \end{pmatrix}$$

$$P_3 = \begin{pmatrix} 6.100 & 3.870 & -0.090 \\ 3.870 & 4.400 & 0.660 \\ -0.090 & 0.660 & 3.870 \end{pmatrix}$$

$$P_4 = \begin{pmatrix} 8.000 & 3.140 & -0.390 \\ 3.140 & 4.400 & 0.660 \\ -0.390 & 0.660 & 3.870 \end{pmatrix}$$

$$P_5 = \begin{pmatrix} 8.000 & 4.810 & -1.060 \\ 4.810 & 6.700 & 0.920 \\ -1.060 & 0.920 & 3.350 \end{pmatrix}$$

$$P_6 = \begin{pmatrix} 6.100 & 1.470 & 0.230 \\ 1.470 & 6.700 & 0.920 \\ 0.230 & 0.920 & 3.350 \end{pmatrix}$$

$$P_7 = \begin{pmatrix} 6.100 & 3.870 & 0.230 \\ 3.870 & 4.400 & 1.100 \\ 0.230 & 1.100 & 3.350 \end{pmatrix}$$

$$P_8 = \begin{pmatrix} 8.000 & 3.140 & -1.060 \\ 3.140 & 4.400 & 1.100 \\ -1.060 & 1.100 & 3.350 \end{pmatrix}$$

and $C_i = \mathrm{diag}(\sigma_{i1}, \sigma_{i2}, \sigma_{i3})$ with $\sigma_{i1} = +1$ for $i = 1, 4, 7, 8$ and -1 otherwise, $\sigma_{i2} = +1$ for $i = 1, 2, 5, 6$ and -1 otherwise, $\sigma_{i3} = +1$ for $i = 1, 2, 3, 4$ and -1 otherwise.

***Remark** 3:* Note that by applying Theorem 4.2 of [2], since it uses the $\mathcal{S}$-procedure relaxation, it is possible to conclude that the absolute stability result obtained in the domain (k_1, k_2, k_3, k_4) implies the absolute stability result in the domain (k_3, k_4, k_1, k_2), and it also implies the absolute stability result in the two symmetric sectors (k_1, k_2, k_1, k_2) and (k_3, k_4, k_3, k_4).

VI. Conclusion

A sufficient condition for the stability of piecewise linear differential inclusions has been proposed. The condition is based on a piecewise quadratic Lyapunov function and is formulated as a suitable set of inequalities depending on unknown matrices and constrained to hold on the different cones of the state partition. This cone-copositive programming problem is not easy to be solved, but some possible algorithms have been recently proposed in the mathematical literature. A distinctive contribution of our work is to bring together ideas from several areas of research and presenting them in a unified manner. In this way we were able to obtain less conservative results with respect to existing approaches within the piecewise quadratic stability scenario.

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