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Sampled-data Output Feedback Control of a Class of Nonlinear Systems

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Abstract— **In this paper, we study the problem of sampled-data output feedback for a class of nonlinear systems. The main contributions of this work are twofold: (i) we develop a systematic design procedure for the sampled-data observer and the observer-based sampled-data controller by combining the backstepping method and two kinds of discretization approaches; (ii) we provide a theoretical analysis of the sampled-data closed-loop system, which shows that the sampled-data observer error and the state of the plant can be regulated within any given neighborhood of the origin by an appropriate choice of design parameters and sampling period. Finally, an illustrative example is given to demonstrate the effectiveness of the presented method.**

I. INTRODUCTION

Nowadays, modern controllers are typically implemented digitally and this strongly motivates investigation of sampled-data systems that consist of a continuous-time plant controlled by feeding sampled outputs back with analog-to-digital and digital-to-analog devices for interfacing. In general, there are two categories of approaches, extensively developed for sampled-data control of nonlinear systems over the past twenty years. In the first category called "controller emulation", for example [1]-[3], a continuous-time controller is firstly designed for the continuous-time plant and then discretized for digital implementation in discrete time. The second category of approach, for instance [4-6], exploits an approximation discretization model of the plant that ignores the inter-sample behavior. Although this method does not require fast sampling to maintain stability, performance of the sampled-data system is not automatically guaranteed since the inter-sample behavior may be unacceptable.

Results on sampled-data output feedback control and digital observers for nonlinear systems can be found in [7-10]. Sampled-data output feedback control of nonlinear systems using a discretized high-gain observers has been studied in [7]. Closed-loop analysis shows that the sampled-data controller recovers the performance of the continuous-time controller as the sampling frequency and the observer gain become sufficiently large. In [8], a deadbeat observer is designed in the discrete-time to estimate the derivatives of the output. The resulting sampled-data output feedback controller recovers the performance of a stabilizing continuous-time state feedback controller when the sampling period is sufficiently small. Paper [9] shows that in the presence of bounded disturbances, given a sampled-data state feedback controller that achieves stabilization with respect to a closed set, the multi-rate output feedback controller recovers stabilization of the same set provided the measurement sampling rate is sufficiently fast. In addition, sampled-data observer for nonlinear systems is also studied using two approaches consisting of the emulation method and the design based on an approximate discrete-time model of the plant [10].

The main idea of this work generally belongs to the emulation method and is to deal with the problem of stabilization of a class of nonlinear systems with lower triangular structure under sampled-data output feedback control. We firstly introduce the backstepping technique proposed in [11] to design a continuous-time observer for nonlinear systems. We also design an observer-based controller in continuous-time using a constructive method. In the following, the Euler method is employed to discretize the continuous-time observer and the zero-order-hold method is used to discretize the continuous-time observer-based controller, then we obtain the sampled-data observer and sampled-data controller based on the sampled-data observer. We prove that the state of the plant and sampled-data observer error can be driven into any given neighborhood of the origin under the sampled-data output feedback controller. Finally, we provide an illustrative example of Norrbin model which demonstrates the performance of the sampled-data output feedback controller.

II. PROBLEM FORMULATION AND PRELIMINARIES

We consider in this paper a class of SISO nonlinear systems in lower triangular form

$$\begin{cases} \dot{x}_1 = \alpha_1 x_2 + f_1(x_1) \\ \dot{x}_2 = \alpha_2 x_3 + f_2(x_1, x_2) \\ \vdots \\ \dot{x}_{n-1} = \alpha_{n-1} x_n + f_{n-1}(x_1, \cdots, x_{n-1}) \\ \dot{x}_n = f_n(x) + b(x) u \\ y = x_1, \end{cases} \tag{1}$$

where $x = [x_1, \cdots, x_n]^T \in R^n, u \in R, y \in R$ are the system input and output, respectively, $\alpha_1, \cdots, \alpha_{n-1}$ are the known constant parameters, and $f_1, \cdots, f_n, b$ are known smooth nonlinear functions satisfying $f_1(0) = f_2(0,0) = \cdots = f_n(0, \cdots, 0)$ and $b(x) \neq 0$ for any x.

We make the further assumption regarding the system throughout the paper.

Assumption. Let T denote the sampling period, we assume that the sampled-data output of feedback controller is constant during sampling intervals $[kT, (k+1)T)$ and the output $y(t)$ is measured at sampling instants $kT, k = 0, 1, 2, \cdots$.

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The objective of this study is to develop a systematic procedure to design the sampled-data observer

$$\chi(k+1) = \mathscr{F}(\chi(k), y(k), u(k)), \tag{2}$$

and the sampled-data controller

$$u(k) = \mathscr{L}(\chi(k)), \tag{3}$$

based on the sampled-data observer (2), such that the following three statements hold under the sampled-data output feedback controller consisting of (2) and (3):

S1. the state $x(t)$ of the plant (1) is bounded,

S2. for any given constant $\varepsilon_1 > 0$, there exists a finite integer $k_0 > 0$, such that the state $x(t)$ of the plant (1) satisfies

$$\|x(t)\| < \varepsilon_1, \forall t \geq k_0 T. \tag{4}$$

S3. for any given constant $\varepsilon_2 > 0$, the sampled-data observer error satisfies

$$\|x(k) - \chi(k)\| < \varepsilon_2, \forall k \geq k_0. \tag{5}$$

Next, we introduce the following lemmas, which play a key role in the derivation of our main results in Sec.III.

Lemma 1. Consider the continuous-time nonlinear plant

$$\begin{cases} \dot{\xi} = f(\xi, v) \\ \eta = h(\xi), \end{cases} \tag{6}$$

with the dynamic state feedback controller

$$\begin{cases} \dot{\zeta} = g(\xi, \zeta) & (a) \\ v = \kappa(\xi, \zeta), & (b) \end{cases} \tag{7}$$

where $\xi \in R^n, \zeta \in R^r, v \in R^m, \eta \in R^q$ are respectively the state of the plant, the state of the controller, control input and output of the plant. It is assumed that f, g and κ are locally Lipschitz, and satisfy $f(0,0) = 0, g(0,0) = 0$ and $\kappa(0,0) = 0$.

If the following conditions hold:

C1. the closed-loop system (6)-(7) is (V, w)- dissipative, i.e. there exists a continuously differentiable function $V : R^n \times R^r \to R$, called the storage function, and a continuous function $w : R^n \times R^r \to R$, called the dissipation rate, such that for all $\xi \in R^n, \zeta \in R^r$, the following inequity holds

$$\tfrac{dV}{dt} = \tfrac{\partial V}{\partial \xi} f(\xi, \kappa(\xi, \zeta)) + \tfrac{\partial V}{\partial \zeta} g(\xi, \zeta) \leq w(\xi, \zeta), \tag{8}$$

C2. for any given positive real numbers Δ_ξ, Δ_ζ, there exists functions $\rho_1, \rho_2 \in \mathscr{K}_\infty$ and $T^* > 0, M > 0$, such that, for all $T \in (0, T^*)$, $\|\xi\| \leq \Delta_\xi, \|\zeta\| \leq \Delta_\zeta$, the following inequalities hold

$$\begin{array}{l} a) \left\|G_T^a - G_T^{Euler}\right\| \leq T\rho_1(T), \\ b) \|g(\xi, \zeta)\| \leq M, \\ c) \|g(\xi, \zeta_1) - g(\xi, \zeta_2)\| \leq \rho_2(\|\zeta_1 - \zeta_2\|), \end{array} \tag{9}$$

where G_T^a represents an approximate discrete-time model of dynamic system in (7-a), correspondingly, G_T^{Euler} denotes the approximate discrete-time model of dynamic system in (7-a) by using the Euler discretization method.

Then, for any given real $\tau > 0$, there exists $T^* > 0$ such that for all $T \in (0, T^*)$, and $\|\xi\| \leq \Delta_\xi, \|\zeta\| \leq \Delta_\zeta$, under the sampled-data output feedback controller

$$\begin{cases} \zeta(k+1) = G_T^a(\xi(k), \zeta(k)) \\ v(k) = \kappa(\xi(k), \zeta(k)), \end{cases} \tag{10}$$

the following inequality holds for the closed-loop system consisting of (6) and (10):

$$\frac{V(\xi(k+1), \zeta(k+1)) - V(\xi(k), \zeta(k))}{T} \leq w(\xi(k), \zeta(k)) + \tau. \tag{11}$$

Proof. The result can follow straightforwardly from the Theorem 3.1 in [3]. □

Remark 1. It can be further concluded from the proof of the Theorem 3.1 in [3] that T^* is directly proportionally dependent on τ.

Lemma 2. (Gronwall-Bellman Inequality [12]) Let $\phi : [a,b] \to R$ and $\mu : [a,b] \to R$ be continuous and nonnegative. If a continuous function $\gamma : [a,b] \to R$ satisfies

$$\gamma(t) \leq \phi(t) + \int_a^t \mu(s)\gamma(s)\, ds \tag{12}$$

for $a \leq t \leq b$, then on the same interval

$$\gamma(t) \leq \phi(t) + \int_a^t \phi(s)\mu(s) \exp\left[\int_s^t \mu(\vartheta)\, d\vartheta\right] ds. \tag{13}$$

III. MAIN RESULTS

This section will be devoted to present a systematic design procedure to construct the sampled-data output feedback controller with the form (2)-(3) for nonlinear system (1).

Step 1. Design a continuous-time observer for nonlinear system (1).

Motivated by the results of [11], we consider the observer in the following form

$$\begin{cases} \dot{\hat{x}}_1 = \alpha_1 \hat{x}_2 + f_1(\hat{x}_1) + \varphi_1(\hat{x})(y - \hat{y}) \\ \dot{\hat{x}}_2 = \alpha_2 \hat{x}_3 + f_2(\hat{x}_1, \hat{x}_2) + \varphi_2(\hat{x})(y - \hat{y}) \\ \vdots \\ \dot{\hat{x}}_n = f_n(\hat{x}) + b(\hat{x}) \cdot l(\hat{x}) + \varphi_n(\hat{x})(y - \hat{y}), \end{cases} \tag{14}$$

where $\hat{y} = \hat{x}_1$, and $u(t) = l(\hat{x})$ is the observer-based controller to be designed in the next step.

Let $e(t) = x(t) - \hat{x}(t)$ be the observer error, it follows from (1) and (14) that we have the error dynamics

$$\begin{cases} \dot{e}_1 = \alpha_1 e_2 + f_1(x_1) - f_1(\hat{x}_1) - \varphi_1(\hat{x}) e_1 \\ \dot{e}_2 = \alpha_2 e_3 + f_2(x_1, x_2) - f_2(\hat{x}_1, \hat{x}_2) - \varphi_2(\hat{x}) e_1 \\ \vdots \\ \dot{e}_n = f_n(x) - f_n(\hat{x}) + (b(x) - b(\hat{x})) \cdot l(\hat{x}) - \varphi_n(\hat{x}) e_1. \end{cases} \tag{15}$$

Then the problem of continuous-time observer design is converted to find the gain functions $\varphi_1(\hat{x}), \cdots, \varphi_n(\hat{x})$, such that $e(t) \to 0$ as $t \to \infty$.

A simple extension to the result in [11] can be used to construct the gain functions based on the backstepping method. For saving space, we omit the detailed derivation and directly give the following final result.

We choose the gain functions as follows

$$\varphi_{n+1-j}(\hat{x}) = \frac{1}{\gamma_{n+1,j}}\beta_{n+1,j} + f_{n,j}(\hat{x}) + b_{n,j}(\hat{x}), \quad 1 \le j \le n \tag{16}$$

where

$$\begin{aligned}
&\beta_{i,1} = \beta_{i-2,1} - \varphi_{i-3} + c_{i-1}\left(\beta_{i-1,1} - \beta_{i-2,i-2}\varphi_{i-2}\right)' \\
&\quad + \sum_{j=1}^{i-2}\left(\beta_{i-1,j} - \beta_{i-2,i-2}\varphi_{i-j-1}\right)\left(f_{j,1} - \varphi_j\right) \\
&\quad + \beta_{i-1,i-1}f_{i-1,1}, \\
&\beta_{i,j} = \beta_{i-2,j} + c_{i-1}\left(\beta_{i-1,j} - \beta_{i-2,i-2}\varphi_{i-j-1}\right) \\
&\quad - \beta_{j,j}\varphi_{i-j-2} + \left(\beta_{i-1,j} - \varphi_{i-1-j}\right)' \\
&\quad + \sum_{k=j}^{i-2}\left(\beta_{i-1,j} - \varphi_{i-j-1}\right)f_{k,j} + \beta_{i-1,i-1}f_{i-1,j}, \\
&\beta_{i,i-2} = \beta_{i-2,i-2} + c_{i-1}\left(\beta_{i-1,i-2} - \beta_{i-2,i-2}\varphi_1\right) \\
&\quad + \left(\beta_{i-1,i-2} - \varphi_1\right)' + \left(\beta_{i-1,i-2} - \varphi_1\right)' f_{i-2,i-2} \\
&\quad + \beta_{i-1,i-1}f_{i-1,i-2}, \\
&\beta_{i,i-1} = c_{i-1}\beta_{i-1,i-1} + \alpha_{i-2}\beta_{i-1,i-2} + \beta_{i-1,i-1}f_{i-1,i-1}, \\
&\quad \beta_{i,i} = \alpha_{i-1}\beta_{i-1,i-1}, \beta_{1,1} = 1, \\
&\gamma_{i,1} = \beta_{i-1,i-1}, \gamma_{i,j} = \alpha_{j-1}\gamma_{i-1,j-1}, \gamma_{i,i-2} = \alpha_{i-3}\gamma_{i-1,i-3}, \\
&\gamma_{i,i-1} = \alpha_{i-2}\gamma_{i-1,i-2}, \gamma_{1,1} = 0, \gamma_{2,1} = 1, \gamma_{2,2} = 0,
\end{aligned} \tag{17}$$

for $1 \le i \le n, 1 \le j \le n$, where $c_1, \ldots, c_n$ are parameters to be chosen, and the operation $'$ is defined on functions $\varphi(\hat{x}_1, \cdots, \hat{x}_n)$ by

$$\varphi' = \sum_{i=1}^{n-1}\frac{\partial\varphi}{\partial\hat{x}_i}\left(\alpha_i\hat{x}_{i+1} + f_i(\hat{x}_1, \cdots, \hat{x}_i)\right) + \frac{\partial\varphi}{\partial\hat{x}_n}\left(f_n(\hat{x}) + b(\hat{x}) \cdot l(\hat{x})\right).$$

Suppose that Λ is compact, positively invariant set for nonlinear system (1) with the controller $u(t) = l(\hat{x})$. Then, there exist a constant $\delta > 0$, such that it holds that

$$\dot{V}_n \le -\frac{1}{2}\sum_{i=1}^{n} c_i z_i^2 \tag{18}$$

for all $x(0) \in \Lambda$ and $\|e(0)\| < \delta$, where

$$V_n = \frac{1}{2}\sum_{i=1}^{n} z_i^2, \tag{19}$$

and

$$\left\{\begin{array}{l} z_1 = e_1 \\ z_i = \beta_{i,i}e_i + \sum\limits_{p=1}^{i-1}\left(\beta_{i,p} - \gamma_{i,p}\varphi_{i-p}(\hat{x})\right)e_p, \quad i = 2, \cdots, n. \end{array}\right. \tag{20}$$

Step 2. Design a continuous-time controller based on the continuous-time observer (14) for nonlinear system (1).

We firstly design a continuous-time state-feedback controller using the backstepping method to stabilize nonlinear system (1).

Define $\varsigma_1 = x_1$ and $U_1 = \frac{1}{2}\varsigma_1^2$, then we have $\dot{U}_1 = \varsigma_1\dot{\varsigma}_1 = -d_1\varsigma_1^2 + \varsigma_1\varsigma_2$, where $d_1 > 0$ is a parameter to be chosen, and $\varsigma_2 = d_1\varsigma_1 + \dot{\varsigma}_1$. We successively define $U_2 = U_1 + \frac{1}{2}\varsigma_2^2$, then $\dot{U}_2 = -d_1\varsigma_1^2 - d_2\varsigma_2^2 + \varsigma_2\varsigma_3$, where $d_2 > 0$ is a parameter and $\varsigma_2 = d_1\varsigma_1 + d_2\varsigma_2 + \dot{\varsigma}_2$.

After a calculation similar to the above, we define $U_i = U_{i-1} + \frac{1}{2}\varsigma_i^2$, we have

$$\dot{U}_i = \dot{U}_{i-1} + \varsigma_i\dot{\varsigma}_i = -\sum_{j=1}^{i} d_j\varsigma_j^2 + \varsigma_i\varsigma_{i+1}, \tag{21}$$

where $d_j > 0$ is parameter to be designed, and $\varsigma_{i+1} = \varsigma_i + d_i\varsigma_i + \dot{\varsigma}_i$. A successive calculation yields

$$\dot{U}_n = \dot{U}_{n-1} + \varsigma_n\dot{\varsigma}_n = -\sum_{j=1}^{n} d_j\varsigma_j^2 + \varsigma_n\varsigma_{n+1}, \tag{22}$$

where

$$\begin{aligned}\varsigma_{n+1} = \varsigma_{n-1} + d_n\varsigma_n + \dot{\varsigma}_n = \varsigma_{n-1} + d_n\varsigma_n + f_n(x) \\ + b(x)u(t).\end{aligned} \tag{23}$$

From (23), we know that if we choose

$$u(x) = -\frac{1}{b(x)}\left(\varsigma_{n-1}(x) + d_n\varsigma_n(x) + f_n(x)\right), \tag{24}$$

where

$$\left\{\begin{array}{l} \varsigma_1 = x_1 \\ \varsigma_i = \alpha_{i-1}x_i + \pi_{i-1}(x_1, \cdots, x_{i-1}), i = 2, \ldots, n, \end{array}\right. \tag{25}$$

and π_i are smooth functions determined by $f_1, \cdots, f_i$, then the closed-loop system consisting of the plant (1) and the controller (24) satisfies

$$\dot{U}_n \le -\sum_{i=1}^{n} d_i\varsigma_i^2, \tag{26}$$

with the choice of the Lyapunov function

$$U_n = \frac{1}{2}\sum_{i=1}^{n}\varsigma_i^2, \tag{27}$$

where $d_1, \cdots, d_n$ are the parameters to be chosen.

In the following, we replace the state variables $x_1, \cdots, x_n$ in (24) with their estimates $\hat{x}_1, \cdots, \hat{x}_n$ from the observer (14), and we get the continuous-time observer-based controller

$$u(\hat{x}) = -\frac{1}{b(\hat{x})}\left(\varsigma_{n-1}(\hat{x}) + d_n\varsigma_n(\hat{x}) + f_n(\hat{x})\right). \tag{28}$$

Remark 2. After replacing $\hat{x}$ in (20) with $x - e$, it is obvious that the transformations (20) and (25) define a diffeomorphism from $\left[e^T, x^T\right]^T$ to $\left[z^T, \varsigma^T\right]^T$, which satisfies $z = 0, \varsigma = 0$, only when $e = 0$ and $x = 0$.

Theorem 1. For the closed-loop system consisting of the plant (1), the observer (14) and the observer-based controller (28), if $x(0) \in \Lambda, \|e(0)\| < \delta$, both the observer error dynamics (15) and the state of the plant asymptotically converge to zero.

Proof. Choose the Lyapunov function

$$\Xi = V_n^{1/2} + U_n = \frac{\sqrt{2}}{2}\|z\| + \frac{1}{2}\|\varsigma\|^2, \tag{29}$$

where $\varsigma = [\varsigma_1, \cdots, \varsigma_n]^T$, then it follows from (18), (22)-(23) and (28) that

$$\begin{aligned}\dot{\Xi} = \frac{1}{2}V_n^{-1/2}\dot{V} + \dot{U}_n \le -\frac{\sqrt{2}}{4}c\|z\| - d\|\varsigma\|^2 \\ -\varsigma_n \cdot b(x) \cdot (u(x) - u(\hat{x})),\end{aligned} \tag{30}$$

where $c = \min\{c_1, \cdots, c_n\}$ and $d = \min\{d_1, \cdots, d_n\}$.

Since $f_1(x_1), \cdots, f_n(x), b(x)$ and ς_n are smooth functions, there exists three positive constants L_1 ,B, and Q, such that

$$|u(x) - u(\hat{x})| \le L_1\|e\|, \tag{31}$$

and

$$\left\{\begin{array}{l} |b(x)| \le B \\ |\varsigma_n(x)| \le Q \end{array}\right. \tag{32}$$

on the set $D=\{(x,e),x\in\Lambda,\|e\|<\delta\}$. On the other hand, from (20), we have that there exist two constants $L_2,L_3>0$, such that it is satisfied that

$$L_2\|e\|\le\|z\|\le L_3\|e\| \tag{33}$$

for all $(x,e)\in D$.

Substituting (31), (32) and (33) into (30), we obtain

$$\dot{\Xi}\le\left(-\frac{\sqrt{2}}{4}cL_2+BQL_1\right)\|e\|-d\|\varsigma\|^2=-2dU_n+\sigma\|e\|, \tag{34}$$

where $\sigma=-\frac{\sqrt{2}}{4}cL_2+BQL_1$. Then, we always can choose $c_1,\cdots,c_n$ satisfying $c=\min\{c_1,\cdots,c_n\}>\frac{2\sqrt{2}BQL_1}{L_2}$, such that $\sigma<0$. Combining (33) and (34), we further have

$$\dot{\Xi}\le-2dU_n+\frac{\sigma}{L_3}\|z\|\le-2dU_n+\frac{\sqrt{2}\cdot\sigma}{L_3}V_n^{1/2}\le-\lambda\Xi, \tag{35}$$

where $\lambda=\min\left\{2d,\frac{-\sqrt{2}\cdot\sigma}{L_3}\right\}$. It follows from the well-known comparison theorem that

$$\Xi(t)\le\Xi(t_0)e^{-\lambda(t-t_0)}, \tag{36}$$

which implies that both the state of the plant and the observer error asymptotically converge to zero as time tends to infinity. This completes the proof of the theorem 1. □

Remark 3. From the remark 2, it follows that the Lyapunov function Ξ can be represented in the form of z and ς, or, e and x, or, x and $\hat{x}$. Furthermore, $\Xi=0$ holds only when $z=0$ and $\varsigma=0$, or, $e=0$ and $x=0$, or, $x=0$ and $\hat{x}=0$.

Step 3. Discretize the continuous-time observer (14) and continuous-time observer-based controller (28).

For convenience of the following discussion, we simply rewrite (14) and (28) as

$$\begin{cases}\dot{\hat{x}}(t)=\hat{F}(\hat{x},y) & (a)\\ u(\hat{x})=l(\hat{x},y) & (b)\end{cases} \tag{37}$$

where $\hat{F}$ represents the vector field of (14).

Next, we employ the Euler method to discretize the continuous-time observer (37-a) and use the zero-order-hold method to discretize the continuous-time observer-based controller (37-b), then we obtain the following sampled-data output feedback controller

$$\begin{cases}\chi(k+1)=\chi(k)+T\cdot\hat{F}(\chi(k),y(k)) & (a)\\ u(k)=l(\chi(k),y(k)), & (b)\end{cases} \tag{38}$$

where T is the sampling period to be chosen.

Remark 4. It should be noted that $\chi(k)\neq\hat{x}(t)|_{t=kT}$. The computation of $\hat{x}(t)|_{t=kT}$ needs firstly to find an analysis solution to (37-a), and then substitute $t=kT$ into the solution. However, generally speaking, it is more difficult to obtain an exact solution to nonlinear system (37-a). In fact, $\chi(k)$ in (38-a) is an approximate value to $\hat{x}(t)|_{t=kT}$ using the Euler method. In other words, the system (38-a) is the Euler approximate discrete-time model of (37-a).

In the following, we investigate the stability of the overall sampled-data system consisting of the plant (1) and the sampled-data output feedback controller (38).

Theorem 2. Under the assumption, for any given constants $\varepsilon_1>0$ and $\varepsilon_2>0$, there exist a maximum allowable sampling period (MASP) denoted as $T^*>0$ and a sampled-data output feedback controller of the form (38), such that if $(x(0),e(0))\in D$, then, for all $T\in(0,T^*)$, the three statements S1, S2 and S3 in the section II hold for the sampled-data closed-loop system.

Proof. From the remark 3, we know that the derivative of Ξ in (30) can be rewritten as

$$\frac{d\Xi}{dt}=\frac{\partial\Xi}{\partial e}\dot{e}+\frac{\partial\Xi}{\partial x}\dot{x}=\frac{\partial\Xi}{\partial\hat{x}}\dot{\hat{x}}+\frac{\partial\Xi}{\partial x}\dot{x}\le-\lambda\Xi. \tag{39}$$

It is easy to be verified that the condition C1 in the lemma 1 is satisfied, and $\Xi,x,\hat{x}$ are equivalent to V,ξ, and ζ respectively in the formula (8).

In view of (37)-(38), it is obvious that the discrete-time system (38-a) is an Euler approximate discrete-time model of (37-a), therefore, the condition C2-a of the lemma 1 is naturally satisfied. Since $b(x)$ and $f_1\cdots f_n$ in the plant (1) are smooth nonlinear functions, the conditions C2-b and C2-c are also satisfied. Then, it follows from the lemma 1 that for a given constant $\bar{\tau}>0$, there exist $T_1^*>0$, such that the following inequality holds

$$\frac{\bar{\Xi}(k)-\bar{\Xi}(k-1)}{T}\le-\lambda\bar{\Xi}(k)+\bar{\tau}, \tag{40}$$

for $T\in(0,T_1^*)$, where

$$\bar{\Xi}(k)=\frac{1}{2}\|\varsigma(x(k))\|^2+\frac{\sqrt{2}}{2}\|z(\bar{e}(k),x(k)-\bar{e}(k))\| \tag{41}$$

and

$$\begin{cases}x(k)=x(t)|_{t=kT}\\ \bar{e}(k)=x(k)-\chi(k).\end{cases} \tag{42}$$

In the discrete-time Lyapunov function $\bar{\Xi}(k)$, $z(\bar{e}(k),x(k))$ can be achieved by replacing $e(t)$ and $x(t)$ in (20) with $\bar{e}(k)$ and $x(k)$ respectively. Correspondingly, $\varsigma(x(k))$ can be also achieved by replacing $x(t)$ in (25) with $x(k)$. It should be noted that

$$e(k)=x(k)-\hat{x}(t)|_{t=kT}\neq\bar{e}(k),$$

and

$$\Xi(k)=\frac{1}{2}\|\varsigma(x(k))\|^2+\frac{\sqrt{2}}{2}\|z(e(k),x(k)-e(k))\|\neq\bar{\Xi}(k). \tag{43}$$

From (40), we have

$$\bar{\Xi}(k+1)\le(1-\lambda T)\bar{\Xi}(k)+\bar{\tau}T. \tag{44}$$

Since the above formula holds for all $k\ge0$, we then further obtain

$$\begin{aligned}&\bar{\Xi}(k)\le(1-\lambda T)\bar{\Xi}(k-1)+\bar{\tau}T\\&(1-\lambda T)\bar{\Xi}(k)\le(1-\lambda T)^2\bar{\Xi}(k-1)+(1-\lambda T)\bar{\tau}T\\&\vdots\\&(1-\lambda T)^k\bar{\Xi}(1)\le(1-\lambda T)^{k+1}\bar{\Xi}(0)+(1-\lambda T)^k\bar{\tau}T.\end{aligned} \tag{45}$$

From the above inequalities, we obtain

$$\bar{\Xi}(k)\le(1-\lambda T)^k\bar{\Xi}(0)+\bar{\tau}T\cdot\frac{1-(1-\lambda T)^k}{\lambda T}. \tag{46}$$

Combining (40) with (46), we choose T_2^* satisfying $\lambda T<1$ for $T\in(0,T_2^*)$. Then, if $\bar{\Xi}(0)>\frac{\bar{\tau}}{\lambda}$, we can conclude that

$$\bar{\Xi}(k+1)-\bar{\Xi}(k)<0 \tag{47}$$

holds for $k > 0$ until $\bar{\Xi}(k) < \frac{\bar{\tau}}{\lambda}$, which means that $\bar{\Xi}(k)$ will decrease monotonously to the set

$$\Omega = \left\{ (x,\bar{e}) | \tfrac{1}{2} \|\varsigma(x)\|^2 + \tfrac{\sqrt{2}}{2} \|z(\bar{e}, x-\bar{e})\| \leq \tfrac{\bar{\tau}}{\lambda} \right\}, \quad (48)$$

as k increases. Hence, there exists a finite positive integer k_0 such that

$$\bar{\Xi}(k) \leq \tfrac{\bar{\tau}}{\lambda} \quad (49)$$

is satisfied for all $k > k_0$. From (33) and (48), we further have

$$\|\bar{e}(k)\| \leq \frac{\sqrt{2} \cdot \bar{\tau}}{\lambda \cdot L_2}. \quad (50)$$

On the other hand, under the diffeomorphism from x to ς defined by (25), there exist two constants $L_4 > 0$ and $L_5 > 0$, such that

$$L_4 \|x\| \leq \|\varsigma\| \leq L_5 \|x\|. \quad (51)$$

Then, combining (48) and (51) yields

$$\|x(k)\| \leq \tfrac{1}{L_4} \cdot \sqrt{\tfrac{2\bar{\tau}}{\lambda}}. \quad (52)$$

Consider the sampled-data closed-loop system consisting of (1) and (38), the solution of (1) over the sampling period $[kT, (k+1)T]$ is given by

$$\begin{aligned} x(t) &= x(kT) + \textstyle\int_{kT}^{t} F(x(s), u(k))\, ds \\ &= x(kT) + (t-kT) F(x(k), u(k)) \\ &\quad + \textstyle\int_{kT}^{t} \left[F(x(s), u(k)) - F(x(k), u(k))\right] ds, \end{aligned} \quad (53)$$

where F is the vector field of the plant (1). On the set D, according to the lemma 2 and the locally Lipschitz property of F in (1), it follows that

$$\|x(t) - x(k)\| \leq \frac{1}{L_F} \left[e^{L_F(t-kT)} - 1 \right] \|F(x(k), u(k))\| \quad (54)$$

for $t \in [kT, (k+1)T]$, where L_F is a Lipschitz constant of F with respect to x. Since $F(x(k), u(k))$ is bounded over the set D, it is concluded from (54) that the state of the plant is bounded for all $t > 0$, which means that the statement S1 holds.

Note that the parameters λ and T^* or $\bar{\tau}$ are chosen independent on each other, and T^* is directly proportionally dependent on $\bar{\tau}$. There always exist 2-tuple of positive real numbers λ and T_2^* such that the following inequalities hold:

$$\begin{cases} \frac{\bar{\tau}}{\lambda} < \varepsilon \\ \lambda T < 1 \end{cases} \quad (55)$$

for all $T \in (0, T_2^*)$, where $\varepsilon = \min\left\{ \frac{L_4^2 \cdot \varepsilon_1^2}{2}, \frac{\sqrt{2} \cdot L_2 \cdot \varepsilon_2}{2} \right\}$. Therefore, the statement S3 is satisfied.

Finally, denote M_Ω as the maximum of $F(x(k), u(k))$ on the set Ω. It follows from (52) and (54) that

$$\begin{aligned} \|x(t)\| &\leq \|x(k)\| + \tfrac{M_\Omega}{L_F} \left(e^{L_F T} - 1 \right) \\ &\leq \tfrac{1}{L_4} \cdot \sqrt{\tfrac{2\bar{\tau}}{\lambda}} + \tfrac{M_\Omega}{L_F} \left(e^{L_F T} - 1 \right) \end{aligned} \quad (56)$$

for all $t > k_0 T$. Then, if we select T^* as

$$T^* = \min\left\{ \ln\left[\frac{L_F}{M_\Omega} \left(\varepsilon_1 - \frac{1}{L_4} \cdot \sqrt{\frac{2\bar{\tau}}{\lambda}} \right) + 1 \right], T_1^*, T_2^* \right\}, \quad (57)$$

the statement S2 holds.

This completes the proof of the theorem 2. □

Remark 5. One of the important issues in the emulation method for sampled-data controller is a priori estimate of maximum allowable sampling period (MASP) T^*. The sampling period T is a design parameter that control engineers need to choose before implementing the designed controller digitally. However, accurate analytic computation of MASP is very challenging and not carried out in most literatures.

Step 4. Choose the sampling period T.

Instead of analytic computation, we can try to determine appropriate parameters and sampling period by many times experiments with the help of simulation softwares.

IV. Example

Example. In the autopilot design field, the Norrbin model has been used extensively for ship manoeuvring studies involving both deep and confined waters. The structure of the Norrbin ship model can be represented by the following form of the state equations

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -\frac{r_1}{H} x_2(t) - \frac{r_2}{H} x_2^3(t) + u(t) \\ y = x_1, \end{cases} \quad (58)$$

where $u(t) = \frac{K}{H} \cdot \theta(t)$, and $x_1, \theta(t)$ represent the ship course to be controlled and the rudder angle respectively. In the following simulation studies, we borrow a model of a ship called "Yu Long" from [13], which can be described by the following set of parameters

$$H = 208.91, r_1 = 1.0, r_2 = 30, K = 0.4963. \quad (59)$$

Next, we employ the procedure developed in previous studies to design a sampled-data output controller with the form (38).

Step 1. By using the formula (16) in the previous section and choosing $c_1 = c_2 = 2$, we can obtain the continuous-time observer for the plant (58) as follows

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 + \left(3.995 - 0.432\hat{x}_2^2\right)(x_1 - \hat{x}_1) \\ \dot{\hat{x}}_2 = -2\hat{x}_1 - 2\hat{x}_2 + (4.98 - 1.728\hat{x}_1\hat{x}_2 \\ \qquad -3.4517\hat{x}_2^2 + 0.1866\hat{x}_2^4)(x_1 - \hat{x}_1). \end{cases} \quad (60)$$

Step 2. According to (24) and (28), we get the following continuous-time controller based on the state observer (60)

$$u(\hat{x}) = -1.995\hat{x}_2 - 2\hat{x}_1 + 0.114\hat{x}_2^3. \quad (61)$$

Step 3. Discretizing the above continuous-time observer (60) with the Euler method yields

$$\begin{cases} \chi_1(k+1) = \chi_1(k) + T \cdot [\chi_2(k) + (3.995 \\ \qquad -0.432\chi_2^2(k))(x_1(k) - \chi_1(k))] \\ \chi_2(k+1) = \chi_2(k) + T \cdot [-2\chi_1(k) - 2\chi_2(k) \\ \quad + \left(4.98 - 1.728\chi_1(k)\chi_2(k) - 3.4517\chi_2^2(k)\right. \\ \qquad \left. +0.1866\chi_2^4(k)(x_1(k) - \chi_1(k))\right]. \end{cases} \quad (62)$$

Correspondingly, we discretize the continuous-time controller (61) using the zero-order-hold method and obtain

$$u(k) = -1.995\chi_2(k) - 2\chi_1(k) + 0.114\chi_2^3(k). \quad (63)$$

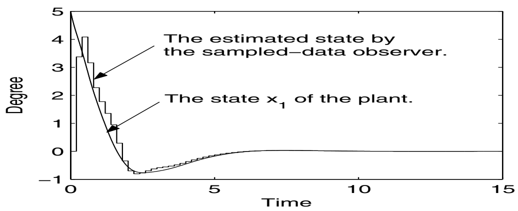

Fig. 1. The trajectories of the state x_1 and the estimate of the sampled-data observer (T=0.2s).

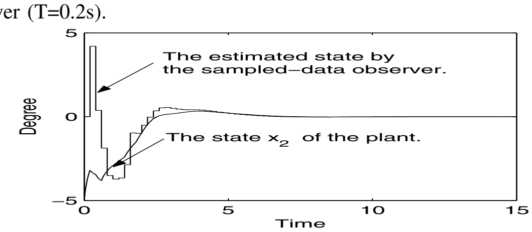

Fig. 2. The trajectories of the state x_2 and the estimate of the sampled-data observer (T=0.2s).

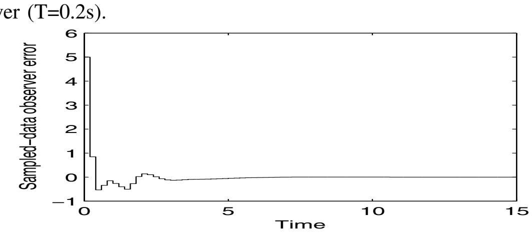

Fig. 3. The sampled-data observer error $\bar{e}_1(k) = x_1(k) - \chi_1(k)$.

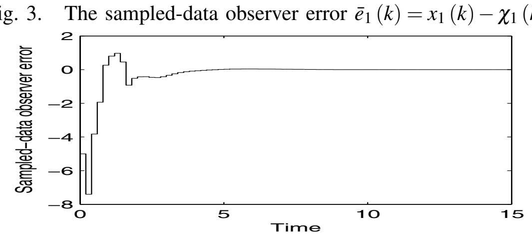

Fig. 4. The sampled-data observer error $\bar{e}_2(k) = x_2(k) - \chi_2(k)$.

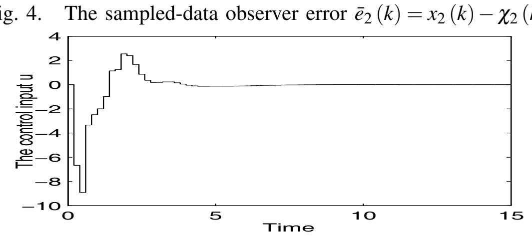

Fig. 5. The control input $u(k)$ of sampled-data controller based on the sampled-data observer (T=0.2s).

Step 4. We take the sampling period $T = 0.2$, which means that the output signal is sampled 5 times per second. The simulations of the sampled-data closed-loop system consisting (58) (62) and (63) were carried out under the initial conditions $x_1(0) = 5, x_2(0) = -5, \chi_1(0) = \chi_2(0) = 0$. The simulation results are given in Fig.1 and Fig.2 for the states of the plant and the estimates of the sampled-data observer, which show that the estimates of the sampled-data observer converge to the states the plant. Correspondingly, the sampled-data observer errors $\bar{e}_1(k) = e_1(k) - \chi_1(k)$ and $\bar{e}_2(k) = e_2(k) - \chi_2(k)$ are also given in Fig.3 and Fig.4 respectively. Based on the estimates of the sampled-data observer, the sampled-data control input is also shown in Fig.5, which remains constant over every sampling period. The simulation results verify that the proposed scheme is effective to cope with the problem of sampled-data output feedback control of nonlinear system (1).

V. CONCLUSIONS AND FUTURE WORKS

The problem of sampled output feedback control has been investigated for a class of nonlinear systems. We have developed the constructive procedures to design the sampled-data output feedback controller using the backstepping method, the zero-order-hold and the Euler discretization method. Closed-loop analysis shows that the sampled-data observer error and the state of the plant can be regulated within any given neighborhood of the origin. A numerical example clearly demonstrates the performance of the sampled-data observer and sampled-data controller.

Several possible extensions remain to be studied. Firstly, one could consider more accurate methods to discretize a controller or an observer. These discretization methods may make ε_1 and ε_2 smaller for the same sampling period, although they also cause an increase in the computation amount. Further, it is natural to try to develop a robust sampled-data output feedback controller for uncertain nonlinear systems.

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