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Distribution of Agents in Heterogeneous Multiagent Systems

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***Abstract*— Heterogeneous multiagent systems are useful for performing various complex distributed tasks. The effectiveness and scope of service such systems provide, can be attributed to the distribution of agents of different types through the network. This paper deals with the development of methods and techniques to analyse heterogeneity distributions in such networks, both qualitatively and quantitatively. These developed tools are used to establish information pertaining to the roles and significance of individual agents. Moreover, the notion of heterogeneity is formalized in terms of the underlying network topology.**

I. INTRODUCTION

Heterogeneous multiagent systems can provide solutions to various complex group level tasks that cannot be accomplished by teams of homogeneous agents alone as pointed out in [1]. Several applications of such heterogeneous systems have been studied in areas ranging from multirobot systems [2], task allocation schemes and sensor networks [3], including power efficient sensor networks [4], better coverage [5], stability and efficiency in distributed systems [6], just to name a few. But heterogeneity in multiagent systems brings with it complexities in terms of the problem formulation, intricate communication schemes and more involved network topologies as pointed out in [7]. Here, we provide a way to characterize heterogeneity in multiagent systems, based upon the topological properties of the underlying network.

The distribution of agents in such systems makes certain nodes more crucial and significant than others, in the sense that an abnormality in their functionality adversely affects the overall behavior of the system. Similar is the case with the communication links among the agents, where certain links have a greater significance over the other. Thus, a mechanism is required to quantify the significance of the nodes and the links between them, in the context of distribution of heterogeneous agents. This paper aims to achieve this goal along with providing other information regarding distribution of agents in heterogeneous multiagent systems.

This investigation turns out to be helpful not only in the analysis, but also for the design of heterogeneous multiagent and multirobot systems. Consider an example of such a system where each agent in the network belongs to one of the following types, α, β and γ. Moreover, each agent is expected to accomplish a task by interacting locally with all other types of agents, i.e. if a node in the network is of type α, then it needs to interact with at least one node of type β and γ to complete a task. Similarly, every node of type β must interact with at least one node from both α and γ types. In such a scenario, the underlying network along with the location of agents in it, needs to be analysed to figure out if there are nodes, that are not capable of doing the required heterogeneous task. Understanding the effect of a certain node failure or a communication link failure on the overall functionality is also an important step in the design process of such systems. Further, these notions can be extended to the more general heterogeneous systems where a task completion by an agent requires interactions with agents of other types, that may not be found in the immediate neighborhood, but within a certain distance from that node.

The underlying inter-connection infrastructure of a multiagent system can be modelled by a graph $G(V,E)$, where the set of vertices, V, represents the agents and the set of edges, E, models the communication links among the agents. The heterogeneity in a multiagent system is attributed to the "difference" among the agents. This difference can be in their functional capabilities, communication methodologies, control laws they implement, complexities, power consumptions, hardware and software, or any other aspect that plays a significant role in the overall behavior of a system, e.g. [6]. By letting agents belong to different types, the resulting structure can be modelled by a *graph coloring* notion from graph theory, where the vertices (or edges) in the graph are partitioned into various classes based on some constraints. Each class in the partition is assigned a color and all the vertices in one class have the same color. Depending on the conditions and coloring constraints, many variants of graph coloring problem exist and have been extensively studied, e.g. [8].

This paper is organized as follows: In Section II, the notion of heterogeneous coloring is given. Section III provides a method for analysing a distribution of agents in a heterogeneous multiagent system. Various applications of this method, including an algorithm for figuring out the most important communication links in the network, are given in Section IV. The notion of heterogeneous coloring is generalized in Section V to deal with more realistic and practical scenarios. The methods of Section III are extended in Section VI for the generalized heterogeneous coloring, followed by the concluding remarks in Section VII.

II. PRELIMINARIES

In a heterogeneous multiagent system, an agent interacts locally with other agents of different types, to perform a certain task. In terms of the network topology based heterogeneity, the extent and capacity of such a task is determined by the number of heterogeneous components involved in the

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local interactions among the agents of various types, that is to say how many different types of agents are present in the neighborhood of an agent. In a network with C_H different types of agents, if an agent can interact with all C_H types in its closed neighborhood, then the heterogeneity of the task performed by that agent will be maximal. This is so because it can exploit all C_H different functionalities available in its closed neighborhood. If all the nodes in a network are capable of performing a maximally heterogeneous task, that is every node has all C_H types of nodes in its closed neighborhood, then, the network will be maximally heterogeneous with C_H types, in terms of the heterogeneity distribution, based on the underlying network topology. This can be modelled by the notion of heterogeneous coloring defined below.

Throughout this paper, by a graph $G(V,E)$, we mean an undirected graph having a vertex set represented by V, and an edge set given by E. Also, we use vertex and node interchangeably.

Definition 2.1: (Open and Closed Neighborhoods): The open neighborhood of a vertex $v \in V(G)$, denoted by $\mathscr{N}(v)$ is the set of vertices adjacent to v. Its closed neighborhood, denoted by $\mathscr{N}[v]$, is $\mathscr{N}(v) \cup \{v\}$.

Definition 2.2: (Heterogeneous Coloring): Given a graph $G(V,E)$ and a set of colors $C_H = \{1,2,\cdots,H\}$. A heterogeneous coloring of $G(V,E)$ is an assignment of a color from C_H to every $v \in V(G)$, such that the closed neighborhood of every $v \in V(G)$ contains every color from C_H.

The set C_H is called the *Coloring Set*. All vertices having the same color belong to the same *color class* and will be denoted by $V_i \subseteq V(G)$, where $i \in C_H$.

Another way to state the above coloring is that, given a graph $G(V,E)$ and a coloring set $C_H = \{1,2,\cdots,H\}$, a *Heterogeneous Coloring* is a function $c : V(G) \to C_H$ such that $c(\mathscr{N}[v]) = C_H, \ \ \forall v \in V(G)$.

It should be noted that not all graphs can be heterogeneously colored by a given set of colors. In fact, the number of colors that can heterogeneously color a given G is bounded by the heterogeneous chromatic number for that G.

Definition 2.3: (Heterogeneous Chromatic Number): The heterogeneous chromatic number of G, denoted by $\chi_h(G)$, is the maximum number of colors that can heterogeneously color a given graph G.

If a network is not maximally heterogeneous, then it will always contain nodes that do not have all node types available in their closed neighborhood. We refer to such nodes as the deficient nodes and their deficiencies are defined as following.

Definition 2.4: (Deficiency of the node and the network): The deficiency of a node $v \in V(G)$ denoted by $\mathtt{d}(v)$, is the number of colors from the coloring set C_H that are missing from the $\mathscr{N}[v]$, i.e. $\mathtt{d}(v) = \mid C_H \mid - \mid c(\mathscr{N}[v]) \mid$, where $c(\mathscr{N}[v])$ is the set of colors available in the closed neighborhood of v.

Deficiency of the network, denoted by $\mathscr{D}$, is the sum of deficiencies of all the nodes in the network.

A node having a deficiency $\mathtt{d}(v) > 0$ is referred to as a *deficient node*.

The inter-connection infrastructure among agents in a heterogeneous multiagent system, plays a vital role in the overall heterogeneity distribution in the system. In fact, some links tend to have a greater impact on overall heterogeneity of the system.

Definition 2.5: (Redundant and Crucial Edge): An edge $e \in E(G)$ is a redundant edge if its removal from a network does not increase the deficiency of any node in a network. An edge is crucial if its deletion increases the deficiency of at least one node in that network.

Throughout this paper, an undirected edge between vertices v_i and v_j will be denoted by (v_i, v_j). Also, it should be noted here that the notions of deficiency and redundancy are in the context of heterogeneous coloring.

Definition 2.6: (Completely Heterogeneous Graph): A graph $G(V,E)$ is completely heterogeneous under a given coloring of V, if none of $v \in V$ is a deficient node.

Remark: It should be pointed out here, that the notion of heterogeneous coloring has been adapted from the concept of domatic partition in the theory of domination in graphs. A domatic partition of a graph is a partitioning of its vertex set into a maximum number of disjoint dominating sets. Interested readers are referred to [9] and [10] for details.

Example:

An industrial process requires a monitoring of various manufacturing parameters for its successful completion. Let us, in particular, consider a manufacturing locality where a specific climatic condition $\mathscr{C}(t,h,p,l)$, depending on the temperature (t), humidity (h), air pressure (p), and light (l) availability, needs to be maintained. The value of $\mathscr{C}(t,h,p,l)$ is monitored by deploying four types of sensors (temperature, humidity, air pressure and light sensors), at diverse locations called data collection points, such that, only one sensor is located at each data collection point.

The network and the distribution of sensors within the network, are designed to ensure the availability of all sensor types in the closed neighborhood of every data collection point. The $\mathscr{C}(t,h,p,l)$ at every such point, can then be obtained by the local coordinations of every sensor with its neighbors. An example of such a sensor configuration is shown in the Fig. 1. Another configuration, having some deficient nodes (data collection points) is also shown.

III. Analyzing a Network for the Heterogeneous Coloring of its Nodes

In this section, we provide a method to analyse a distribution of heterogeneous agents in a network in terms of the most deficient agent type, deficiency of the nodes and the network. Here, a network is modelled by a heterogeneous colored graph with each color representing an agent type.

We start with the adjacency matrix A of the given graph G with n nodes. Thus $A \in \mathbf{R}^{n \times n}$. Every vertex $v \in G$ is colored with a unique color from the coloring set $C_H = \{1,2,\cdots,h\}$

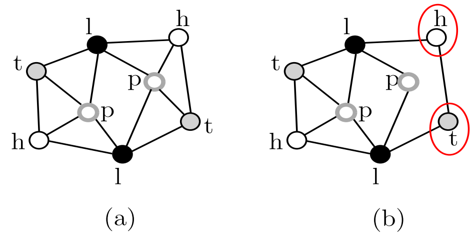


Fig. 1. Sensor networks having four types of sensors, {t,h,p,l}. In (a), every node has all four sensor types in its closed neighborhood. In (b), the circled nodes are deficient as they do not have a sensor of type {p} in their closed neighborhoods.

with $\mid C_H \mid = h$. Now, a matrix C called a color matrix is constructed. We define C as follows,

Definition 3.1: (Color Matrix): $C \in \mathbf{R}^{n\times h}$ is a color matrix of a graph $G(V,E)$ whose vertices are colored from a coloring set C_H, where,

$$[C]_{ij} = \begin{cases} 1 & \text{if } c(v_i) = j, \text{ where } j \in C_H \\ 0 & \text{otherwise.} \end{cases}$$

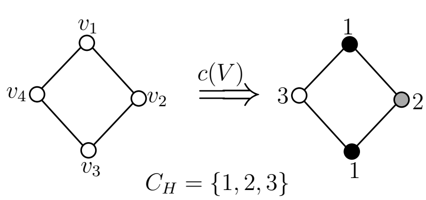
It is to be noted here that the column j of C matrix represents vertices with the color j from C_H, as only those enteries in the j^{th} column of C are 1 that correspond to the vertex indices with the color j. Also, every element in a row of C will be 0 except one. Fig 2 illustrates an example of constructing C for the given coloring of a graph G.

Now we define a color distribution matrix containing information about the distribution of colors in the closed neighborhood of any vertex $v \in V(G)$.

Definition 3.2: (Color Distribution Matrix Φ)

$$\Phi = AC + C$$

where, A is the adjacency matrix and C is the color matrix. Here $\Phi \in \mathbf{R}^{n\times h}$.



$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Phi = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

Fig. 2. An example illustrating a color matrix C and a color distribution matrix Φ.

It can be seen in Fig. 2 that v_2 is missing color $\{3\}$ in $\mathcal{N}[v_2]$, so it is a deficient node with a deficiency $\mathtt{d}(v_2) = 1$. Similarly, v_4 is also a deficient node as color $\{2\}$ is missing from $\mathcal{N}[v_4]$. It turns out that the color distribution matrix Φ contains a complete information about the available colors in the closed neighborhood of any vertex in a given graph as stated in the Lemma 3.1

Lemma 3.1: The $(i,j)^{th}$ entry of Φ matrix, denoted by $[\Phi]_{ij}$ is the number of vertices with color j in the closed neighborhood of v_i

Proof: The entries in the i^{th} column of A, denoted by A_i, are 1 only for the vertices in $\mathcal{N}(v_i)$, and 0 otherwise. The entries in the j^{th} column of C, denoted by C_j, are 1 only for the vertices with color j and 0 otherwise. So, $A_i^T C_j$ is the number of vertices in the open neighborhood of v_i and with color j. Now, $[\Phi]_{ij} = A_i^T C_j + C_{ij}$ is the total number of vertices with color j in the *closed* neighborhood of v_i. This is true $\forall v \in V(G)$. ■

Corollary 3.2: A graph G with the given coloring is completely heterogeneous if and only if $[\Phi]_{ij} \neq 0, \forall i,j$. ■

In terms of the color matrix and the color distribution matrix, the problem of finding the heterogeneous chromatic number of a graph $G(V,E)$ with n vertices is,

$$\max_{C} h \quad \text{s.t.} \quad \begin{cases} C \in \mathbf{R}^{n\times h} & \text{is a color matrix and} \\ [\Phi]_{ij} \geq 1, \ \forall\, i,j \end{cases} \tag{1}$$

Here, h will be the heterogeneous chromatic number χ_h of the given G.

Another representation for the number of vertices of a particular color in the closed neighborhood of some $v \in V(G)$, can be given as,

$$\tilde{\Phi} = \Phi - \mathbf{1}_{(n\times h)}$$

Here, $\mathbf{1}$ is an $n \times h$ matrix with all 1's. $[\Phi]_{ij} = 0 \Rightarrow [\tilde{\Phi}]_{ij} < 0$, thus negative $[\tilde{\Phi}]_{ij}$ means v_i is deficient of the color j. Similarly, $[\Phi]_{ij} > 1 \Rightarrow [\tilde{\Phi}]_{ij} > 0$, thus implying that $\mathcal{N}[v_i]$ has $[\tilde{\Phi}]_{ij}$ *extra* vertices of color j. Finally, $[\Phi]_{ij} = 1 \Rightarrow [\tilde{\Phi}]_{ij} = 0$, meaning that v_i has exactly one vertex of color j in $\mathcal{N}[v_i]$. Thus, the sign of $[\tilde{\Phi}]_{ij}$ is sufficient to check for the deficiency status of any node in a network.

IV. Information from the Color Distribution Matrix, Φ

In this section, it is shown how the color distribution matrix Φ, can be be used to gather useful information about the coloring related notions of the overall network and its nodes.

A. *Redundant and Crucial Edges*

Φ contains information of both the redundant and the crucial edges and an algorithm is presented here that separates the redundant and crucial edges for the given heterogeneous coloring. An important observation here is that $[\Phi]_{ij} \geq 2$ means that v_i has more than one vertices of color j in its closed neighborhood. If $c(v_i) \neq j$ [1], then among the edges between v_i and its j colored neighbors, there will be at least one crucial edge that will ensure the presence of a vertex with color j in $\mathcal{N}(v_i)$. Rest of the edges may be redundant and their redundancy can be figured out as follows. Look at the the vertices $V_i^j = \{v \in \mathcal{N}(v_i) \text{ s.t. } c(v) = j\}$, if

[1]Here, $c(v_i)$ means the color of vertex v_i

all the vertices in V_i^j have v_i as the only vertex in their closed neighborhood with color $c(v_i)$, then clearly the edges between them and v_i are not redundant. Also, if $c(v_i) = j$, then all the edges between v_i and its j colored neighbors are redundant. Algorithm I deletes all the redundant edges for the given coloring of G from its adjacency matrix A.

The algorithm below is initialized with Φ_{11}. Here, V_i^j is the set of indices of the vertices that are in $\mathcal{N}(v_i)$ and have color j. Every $v \in V(G)$ is assigned a distinct color from a coloring set $C_H = \{1, 2 \cdots, h\}$.

Algorithm I

Require: Update the adjacency matrix A of a graph $G(V,E)$ by deleting redundant edges for a given coloring.

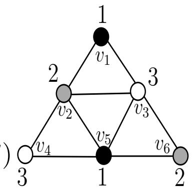
1 : $\Phi = AC + C$
2 : **for all** $[\Phi]_{ij} \geq 2$ **do**
3 :   **let** V_i^j = set of indices of $\{v \in \mathcal{N}(v_i)$ s.t $c(v) = j\}$
4 :   **let** $c(v_i) = \tau$
5 :   **if** $\tau = j$
6 : $a_{i\kappa} = a_{\kappa i} = 0$; $[\Phi]_{\kappa\tau} = [\Phi]_{\kappa\tau} - 1$; $\forall \kappa \in V_i^j$
7 : $[\Phi]_{ij} = 1$
8 :   **elseif** at least one but not all $[\Phi]_{\kappa\tau} \geq 2$, $\forall \kappa \in V_i^j$
9 : $a_{i\kappa} = a_{\kappa i} = 0$; iff $[\Phi]_{\kappa\tau} \geq 2$
10 : $[\Phi]_{\kappa\tau} = [\Phi]_{\kappa\tau} - 1$; iff $[\Phi]_{\kappa\tau} \geq 2$
11 : $[\Phi]_{ij} = [\Phi]_{ij} - \mid \{[\Phi]_{\kappa\tau} : [\Phi]_{\kappa\tau} \geq 2\} \mid$
12 :   **elseif** $[\Phi]_{\kappa\tau} \geq 2, \forall \kappa \in V_i^j$
13 :   repeat lines (9-11) $\forall \kappa \in V_i^j \backslash \{1^{st}$ elt. of $V_i^j\}$
14 :   **end if**
15 : **end for**
16 : **return** A

At each step of the algorithm, the status of the edges between v_i and the vertices of a certain color j in $\mathcal{N}(v_i)$ is evaluated for redundancy, and if found redundant, the edges are removed by making the corresponding entries 0 in the adjacency matrix. The final result is the new adjacency matrix containing only the crucial edges. If A_{new} is the adjacency matrix after the algorithm and A is the original adjacency matrix, then $[A_{redundant}]_{xy}$=1, where $A_{redundant} = A - A_{new}$, will indicate a redundant edge between the vertices v_x and v_y in the original G.

These observations can be summarized in the Proposition 4.1.

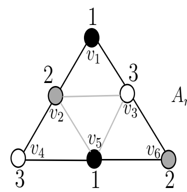
Proposition 4.1: For the given adjacency matrix A and the color matrix C, Algorithm I updates A such that $a_{ij} = a_{ji} = 1$, iff (v_i, v_j) is a crucial edge in the given graph $G(V,E)$.

For illustration, consider a network shown in the Fig. 3 with a given C. The updated adjacency matrix with only crucial edges and the new color distribution matrix obtained after the application of an algorithm I, are shown in the Fig. 4. Looking at one iteration of the above scheme, consider $\Phi_{21} = 2$, which means that v_2 has two vertices of the color 1 (v_1 and v_5) in its closed neighborhood. Since, $c(v_2) = 2$, so $\tau = 2$. Now $V_i^j = V_2^1 = \{1, 5\}$. From here, we get $\Phi^\tau = \{\Phi_{12}, \Phi_{52}\} = \{1, 2\}$. Φ_{12} shows that v_1 has only v_2 as the vertex with the color $\tau = 2$ in its neighborhood, so (v_1, v_2) is a crucial edge. On the other hand, $\Phi_{52} = 2$ shows the existence of more than one vertices with color 2 in $\mathcal{N}(v_5)$, thus, (v_2, v_5) is a redundant edge. So, in adjacency matrix A, we can make $a_{25} = a_{52} = 0$. This change will also be updated in the Φ matrix, by making $\Phi_{21} = \Phi_{52} = 1$.



$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \Phi = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

Fig. 3. Adjacency matrix A, Color matrix C and Φ matrix for a given heterogeneous coloring of the graph G.



$$A_{new} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & \boxed{0} & 1 & \boxed{0} & 0 \\ 1 & \boxed{0} & 0 & 0 & \boxed{0} & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & \boxed{0} & \boxed{0} & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \Phi_{new} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

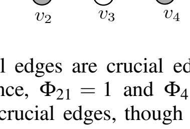
Fig. 4. Adjacency, Color and Φ matrices of G after the application of an algorithm. The redundant edges are shown in grey, while the crucial edges are in black.

It should be observed that if $\Phi = \mathbf{1}$, i.e. $[\Phi]_{ij} = 1$, $\forall i, j$, then clearly every edge $e \in E(G)$ is a crucial edge. But the existence of $[\Phi]_{ij} > 1$ in the color distribution matrix does not necessarily mean the existence of redundant edges. In fact,

Lemma 4.2: If $[\Phi]_{ij} > 1$ and also $[\Phi]_{ab} = 1$ $\forall a, b$, where $b = c(v_i)$ and a = index of a vertex from the set $V_i^j = \{v \in \mathcal{N}(v_i) : c(v) = j\}$, then there is no redundant edge between v_i and $v \in V_i^j$.

Proof: $[\Phi]_{ab} = 1$ implies that all the vertices $v \in V_i^j$ has only v_i as the b colored vertex in their respective closed neighborhoods. So, (v_i, v), where $v \in V_i^j$ are all the crucial edges. Since, all $v \in V_i^j$ have a color j and they are also the neighbors of v_i, thus $[\Phi]_{ij} > 1$. ■

This is shown in the example in Fig. 5.



$$\Phi = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Fig. 5. All edges are crucial edges. $\Phi_{12} > 1$ and $c(v_i) = 1$. Also, $V_1^2 = \{v_2, v_4\}$. Since, $\Phi_{21} = 1$ and $\Phi_{41} = 1$, thus, by lemma 4.2, (v_1, v_2) and (v_1, v_4) are crucial edges, though $\Phi_{12} > 1$.

B. Most Deficient Color in a Network

Definition 4.1: The most deficient color in a network is the one that is missing from the closed neighborhood of maximum number of vertices in $V(G)$.

The j^{th} column of Φ tells about the availability of the color j in the closed neighborhood of all the vertices in G. By Lemma 3.1, $[\Phi]_{ij} = 0$ means v_i does not have a color j in $\mathscr{N}[v_i]$. So, the column index of Φ with the *maximum number of zeros* will be the most deficient color in the given coloring of G.

C. Extra Edges to Make G Completely Heterogeneous

Number of extra edges required to make G completely heterogeneous, is equal to the number of 0's in the color distribution matrix Φ. $[\Phi]_{ij} = 0$ implies that, to get a complete heterogeneous coloring, v_i must be directly connected to some vertex v_x such that $c(v_x) = j$. This x can be any index such that $[C]_{xj} = 1$, as the j^{th} column of the color matrix C contain indices of the vertices with color j.

D. Deficiency of the Nodes and the Network

In terms of the Φ matrix, the number of zeros in the i^{th} row of Φ will give the deficiency of the vertex v_i. The deficiency of the whole network will be the sum of deficiencies of all the vertices or equivalently, it will be the cardinality of the set $\{[\Phi]_{ij} : [\Phi]_{ij} = 0\}$.

E. Critical Node

A *critical node* is the one whose removal from the network increases the deficiency of the network by most. If a network represented with a graph $G(V,E)$ is colored with C_H colors, let $G_{new}(V,E')$ be a graph with the same vertex set V as G, and edge set $E' \subseteq E$ containing only the crucial edges. E' can be obtained by the application of algorithm I in Section IV-A

Definition 4.2: (Critical Node): It is a node $v \in V$, that will maximize the deficiency of the network $(G_{new}/\{v\})$ upon its deletion from G, i.e. $v \in V : \mathscr{D}(G_{new}/\{v\})$ is maximium.

From Φ_{new}, obtained from Φ, the critical node can be figured out. The i^{th} row sum of Φ_{new} is equal to the number of vertices that have only v_i as the $c(v_i)$ colored vertex, in their respective closed neighborhoods in G_{new}. Thus, deleting v_i from G_{new} will increase the deficiency of the network by an amount equal to the [(i^{th} row sum of Φ_{new})-1]. So, it implies that the vertex v_i is the critical node, when i is the index of the row in Φ_{new} with the maximum row sum.

V. Generalizing Heterogeneous Coloring

There are many scenarios where all the colors from the coloring set C_H are not available in the closed neighborhood of a vertex v. In such situations, the notion of neighborhood can be generalized and extended to ensure that all the colors are available within some specific distance from the vertex v in G, $\forall v \in V$. If δ is the minimum degree of a given graph G, then clearly $\chi_h \leq \delta + 1$, implying that G can never be heterogeneously colored with more than $\delta + 1$ colors. But with this extended notion of neighborhood, it can be ensured that more than $\delta + 1$ colors are available to every $v \in V$ within a specific distance from v.

As an example of such a situation, consider a group of villages that are connected through a network of roads (the interconnection network). It is intended to provide certain facilities (e.g. hospitals, banks, universities, etc.) to these villages with the condition that every village can be provided with only one facility. The goal is to distribute these facilities among them in such a way that every village has an access to every facility in its closed neighborhood (i.e. at a maximum distance of one path length). But, if this is not possible due to the underlying network topology (of roads), the next best thing is to distribute these facilities such that every village gets an access to all of them at a maximum distance of two path lengths from it. This is illustrated in the Fig. 6.

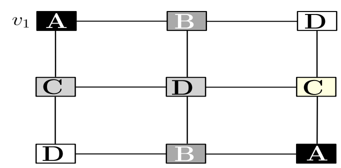


Fig. 6. The facilities from the set $\mathscr{F} = \{A,B,C,D\}$ are distributed such that every village gets exactly one from the set, with an access to all others at a maximum distance of two path lengths from it. For example, village v_1 has $\mathscr{F}_1 = \{A,B,C\}$ in its closed neighborhood, while $\mathscr{F}_2 = \{D\}$ at a distance of two path lengths from it.

This extended neighborhood notion is useful in heterogeneous multiagent systems from their design perspectives, specifically in the scenarios where greater number of agent types need to be distributed throughout the network. This leads towards the generalization of the notion of heterogeneous coloring to the $k-heterogeneous$ coloring defined below.

Definition 5.1: (Distance between Vertices): Let $v, v' \in V(G)$, then the distance $d(v,v')$ between v and v' is the length of the shortest path from v to v' in G.

Definition 5.2: (Open and Closed k-Neighborhood): The open k-neighborhood of a vertex $v' \in V(G)$, denoted by $\mathscr{N}_k(v')$, is the set of vertices $\{v \in V : d(v',v) \leq k\}$. The closed k-neighborhood, denoted by $\mathscr{N}_k[v']$, is $\mathscr{N}_k(v') \cup \{v'\}$.

Based on this k-neighborhood notion, heterogeneous coloring can be generalized to k-heterogeneous coloring.

Definition 5.3: (k-Heterogeneous Coloring): Given a graph $G(V,E)$ and a set of colors $C_H = \{1,2,\cdots,H\}$. A k- heterogeneous coloring of $G(V,E)$ is an assignment of a color from C_H to every $v \in V(G)$, such that the closed k-neighborhood of every $v \in V(G)$ contains every color from C_H.

The maximum number of colors that can k-heterogeneously color the given G is the $k-heterogeneouschromaticnumber$.

It is clear that the k-heterogeneous coloring is exactly same as the heterogeneous coloring defined in Section II, for $k = 1$. Also, based on these definitions, the notion of deficiency in Definition 2.4 can also be extended to the k-deficiency.

Definition 5.4: (k-Deficiency of a node and a network): The k-deficiency of a node $v \in V(G)$, denoted by $\mathtt{d}_k(\mathtt{v})$, is the number of colors from the coloring set C_H that are missing from the k-closed neighborhood $\mathscr{N}_k[v]$ of v. k-deficiency of a network, $\mathscr{D}_k$, is the sum of k-deficiencies of all the nodes in that network.

VI. Analyzing a Network for the k-Heterogeneous Coloring of its Nodes

Here, we will present a way of analyzing a distribution of colors in the k-heterogeneous coloring, through a matrix operation similar to the one introduced in Section III. This will give a systematic way of collecting information about colors available in the closed k-neighborhood of all the vertices in V.

It is known that $(i,j)^{th}$ entry of the k^{th} power of the adjacency matrix, A^k, gives the the number of k length paths between v_i and v_j. Thus, in order to know if there is a k length path between two distinct vertices in given G, we define

Definition 6.1: If A^k is the k^{th} power of an adjacency matrix of the graph G, then $\mathscr{A}^k$ is

$$[\mathscr{A}^k]_{ij} = \begin{cases} 1 & \text{if } [A^k]_{ij} > 0 \text{ and } i \neq j \\ 0 & \text{otherwise.} \end{cases}$$

Here, $\mathscr{A}^k$ tells if there is a k length path between v_i and v_j, $\forall i \neq j$. Also, note that $A = \mathscr{A}^1$.

Lemma 6.1: If $\oplus$ is an element-wise *or* operation and

$$\mathscr{Q}(k) \;=\; \mathscr{A}^1 \oplus \mathscr{A}^2 \oplus \mathscr{A}^3 \oplus \cdots \mathscr{A}^k \tag{2}$$

then,

$$[\mathscr{Q}(k)]_{ij} = \begin{cases} 1 & \text{iff } d(v_i, v_j) \leq k, \ \forall i \neq j \\ 0 & \text{otherwise.} \end{cases}$$

Proof: Whenever $d(v_i,v_j) \leq k$, $\forall i \neq j$, $\exists \mathscr{A}^r$, $r \in \{1,2,\cdots k\}$, such that $(i,j)^{th}$ entry of $\mathscr{A}^r$, denoted by $[\mathscr{A}^r]_{ij}$, is 1. Also, by the definition of $\mathscr{A}^r$, $[\mathscr{A}^r]_{ij} = 0$, whenever $d(v_i,v_j) > k$ or $i \neq j$, $\forall r \in \{1,2,\cdots k\}$. ∎

Thus $[\mathscr{Q}(k)]_{ij} = 1$ iff v_i and v_j are in the closed k-neighborhood of each other.

Now, similar to the color distribution matrix, we define a k-color distribution matrix.

Definition 6.2: (k-Color Distribution Matrix $\Theta(k)$):

$$\Theta(k) \;=\; \mathscr{Q}(k)C \;+\; C$$

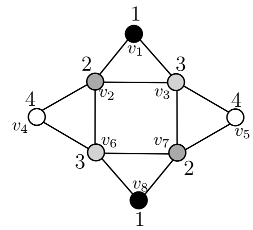
where, $\mathscr{Q}(k)$ is defined in (2) and C is the color matrix.

This k-color distribution matrix contains exactly the similar information related to the k-heterogeneous coloring, as the color distribution matrix contain about the heterogeneous coloring. Also, note that $\Theta(1) = \Phi$.

Lemma 6.2: $[\Theta(k)]_{ij}$ is the number of vertices with color j in the closed k-neighborhood of v_i

Proof: The proof is exactly similar to that of Lemma 3.1, with the only change that closed k-neighborhood is used here instead of the closed neighborhood. ∎

An example in the Fig. 7 demonstrates an application of $\Theta(k)$ in knowing a distribution of colors among the vertices in a graph. In Fig. 7, v_1 does not have any vertex with the color $\{4\}$ in its closed neighborhood, so $\Phi_{14} = 0$. But there are two vertices with the color $\{4\}$ (v_4 and v_5) at a maximum distance of two path lengths (since $k = 2$) from v_1, thus $\Theta_{14} = 2$.



$$\Theta(2) = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 1 & 2 & 2 & 2 \end{bmatrix}; \Phi = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Fig. 7. Θ and Φ matrices for 2-heterogeneous coloring and heterogeneous coloring respectively, for the given graph.

Thus, Θ allows us to figure out k-deficiencies of the nodes and the network, the most k-deficient color, crucial and redundant edges with respect to the k-heterogeneous coloring, in a similar way as Φ is used in the Section IV to get all this information for the heterogeneous coloring.

VII. Conclusions

The distribution of agents in heterogeneous multiagent systems is investigated from a network topology view point. The importance of certain nodes and communication links in such networks is analysed and an algorithm is presented to find these significant nodes and links within a network. Graph coloring notions are used to characterize heterogeneity in multiagent systems. This characterization provides a systematic way to exploit the capabilities of different agents in a network for accomplishing complex distributed tasks. This framework also captures the capability of a network topology to incorporate various heterogeneous entities, thus, giving a useful information for designing heterogeneous multiagent systems.

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