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Simultaneous Stabilization and Synchronization for Multi-Agent Systems via A Low Gain Method

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***Abstract*— We study a *simultaneous stabilization and synchronization (SSS)* problem for one class of linear multi-agent systems with non-identical agents and a directed connection graph. The agent dynamics can be different and the orders of agents are not necessarily equal. We show that a single control loop can be designed for each agent to enable some agent states, named as *internal states*, to be stabilized while some other states, named as *external states*, to be synchronized. We design a distributed control law based on local measurements and partial information (external states only) exchanged from neighboring agents to enable SSS. To avoid internal state coupling in the SSS control law, a low gain approach is proposed for control law synthesis. Perturbation analysis, decoupling coordinate transformation, and weighted Laplacian are exploited for convergence and stability analysis. The sufficient conditions to achieve SSS are obtained, followed by specific approaches of designing the control gain matrices.**

I. Introduction

Cooperative control of multi-agent systems has attracted substantial attention over the past decade. One relevant topic is the synchronization problem of dynamic systems (see, e.g., [1]–[7], just name a few) where the agent trajectories in a network converge to each other through distributed local coupling. Specifically, for linear time invariant (LTI) multi-agent systems, Tuna studied the output synchronization problem of identical agents [1] and investigated the synchronizability conditions for coupled linear systems [2] where the number of inputs is equal to the number of states. Scardovi and Sepulchre [3] investigated the synchronization problem of a network of identical linear state-space models using a dynamic output feedback coupling. Li *et al.* [4] introduced a new framework to address the output feedback synchronization problem of a group of LTI systems by introducing the notion of consensus region. Seo *et al.* [5] presented a low gain synchronization approach for designing an output feedback compensator which only used the local output information. Chopra and Spong [6] investigated the output synchronization problem for a class of passive nonlinear systems. Nair and Leonard [7] solved the stable synchronization problem for a network of under-actuated mechanical systems.

In multi-agent systems, the overall behavior of each agent can be determined by its internal dynamics and external dynamics. The internal dynamics govern the behavior of the agent as an individual system while the external dynamics are related to the coordination with the other agents. In some systems (see, e.g., [8], [9]), the internal dynamics are much faster than the external dynamics, so that the internal dynamics can be ignored and the agents are modeled as first-order integrators. For multi-agent systems that need to be represented using more general models, one idea is to use an inner control loop (see, e.g., [10], [11]). Specifically, Fax and Murray [10] proposed an idea to stabilize each agent by closing an inner control loop around its internal dynamics and then closing an outer control loop to achieve the desired formation performance. Arcak [11] assumed that an inner control loop is designed so that the resulting system becomes passive with respect to the external feedback and then proposed a passivity-based method for the coordination purpose. However, as shown in the motivating example in [12], there exist cases where separate control loops are not available for both stabilization of internal dynamics and synchronization of external dynamics.

If no internal control loop is available and the agents have unstable open-loop internal dynamics and/or dynamically coupled internal and external states (see the motivating example in [12]), then the decentralized controller of each agent should perform two tasks simultaneously: 1) stabilize the agent's internal dynamics, and 2) coordinate with other agents to achieve a group behavior. Nair and Leonard [7] developed a new framework for stable synchronization of under-actuated mechanical systems, distinguishing between actuated and under-actuated states. They used an energy shaping method to stabilize the under-actuated states while rendering the actuated states synchronized. It looks encouraging to distinguish the states that are supposed to be stabilized from those that are synchronized through dynamic coupling. This distinction leads to generalizing the existing results for identical LTI systems to non-identical ones in [12].

In our prior work [12], we considered a *simultaneous stabilization and synchronization (SSS)* problem for a group of non-identical linear agents with potentially unstable open-loop dynamics. A single control loop was designed for each agent to enable the internal states to be stabilized and the external states to be synchronized. A distributed SSS protocol was designed based on local measurements and information exchanged from neighboring agents to enable SSS. In [12], the coupling terms in the control law require relative measurements of both the external states and the internal states. It is of more interest to enable SSS using only the external states since the relative measurements of the required internal states from neighboring agents might not be practically available.

In this paper, we revisit the SSS problem. A distributed control law is designed based on local measurements and partial information (external states only) exchanged from neighboring agents to enable SSS. To avoid internal state coupling in the SSS control law, a low gain approach is proposed for control law synthesis. Perturbation analysis, decoupling coordinate transformation, and weighted Laplacian are exploited for control design and stability analysis. The sufficient conditions to achieve SSS are obtained, followed by specific approaches of designing the control gain matrices.

II. Notation and Preliminaries

Graph theory (see, e.g., [13] and [14]**)** is widely used for investigating multi-agent systems. Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ represent a directed graph, and $\mathcal{V} = \{1, ..., N\}$ denote the set of vertices. Every agent is represented by a vertex. The set of edges is denoted as $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. An edge is an ordered pair $(i, j) \in \mathcal{E}$ if agent j can be directly supplied with information from agent i. In this paper, we assume that there is no self loop in the graph, that is, $(i, i) \notin \mathcal{E}$. $\mathcal{N}_i = \{j \in \mathcal{V} \,|(j, i) \in \mathcal{E}\}$ denotes the neighborhood set of vertex i. Graph $\mathcal{G}$ is said to be undirected if for any edge $(i, j) \in \mathcal{E}$, edge $(j, i) \in \mathcal{E}$. Hence, an undirected graph is a special case of a directed graph. A path is referred by the sequence of its vertices. Path $\mathcal{P}$ between two vertices v_0 and v_k is the sequence $\{v_0, ..., v_k\}$ where $(v_{i-1}, v_i) \in \mathcal{E}$ for $i = 1, ..., k$ and the vertices are distinct. The number k is defined as the length of path $\mathcal{P}$. Graph $\mathcal{G}$ is strongly connected if any two vertices are linked with a path in $\mathcal{G}$. Graph $\mathcal{G}$ contains a directed spanning tree if there is a vertex which can reach all the other vertices through a directed path. $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ denotes the adjacency matrix of $\mathcal{G}$, where $a_{ij} > 0$ if and only if $(j, i) \in \mathcal{E}$ else $a_{ij} = 0$. $L = D - \mathcal{A}$ is called Laplacian matrix of $\mathcal{G}$, where $D = [d_{ii}] \in \mathbb{R}^{N\times N}$ is a diagonal matrix with $d_{ii} = \sum_{j=1}^{N} a_{ij}$.

Lemma 1: [13]–[16] Zero is an eigenvalue of L for both directed and undirected graphs. Zero is a simple eigenvalue of L and the associated eigenvector is $\mathbf{1}$ where $\mathbf{1} \in \mathbb{R}^N$ is a unitary column vector, if and only if the undirected graph is connected or if the directed graph has a directed spanning tree. All of the nonzero eigenvalues of L are positive for an undirected graph or have positive real parts for a directed graph.

Kronecker Product: Some properties of the Kronecker product are recalled as below [17]

$$\begin{aligned} (A \otimes B)(C \otimes D) &= AC \otimes BD \\ A \otimes (B + C) &= A \otimes B + A \otimes C \\ (A \otimes B)^T &= A^T \otimes B^T \\ (A \otimes B)^{-1} &= A^{-1} \otimes B^{-1}. \end{aligned} \quad (1)$$

Assume that $\beta_{1i}, i \in \{1, ..., n_1\}$ are the eigenvalues of $A_1 \in \mathbb{R}^{n_1\times n_1}$ and $\beta_{2j}, j \in \{1, ..., n_2\}$ are the eigenvalues of $A_2 \in \mathbb{R}^{n_2\times n_2}$. Eigenvalues of $I_{n_2} \otimes A_1 + A_2 \otimes I_{n_1}$ are $\beta_{1i} + \beta_{2j}$.

Vectorization: [18] The operator $vec(\cdot)$ transfers an $n \times m$ matrix A into an nm dimensional column vector $[a_{11}, \cdots, a_{m1}, a_{12}, \cdots, a_{m2}, \cdots, a_{1n}, \cdots, a_{mn}]^T$. This operator has the following property:

$$\begin{aligned} vec(A_{n\times m}B_{m\times r}) &= (I_r \otimes A)vec(B) \\ &= (B^T \otimes I_n)vec(A). \end{aligned}$$

Right Inverse: Matrix $B_{m\times r}$ is called the right inverse of matrix $A_{r\times m}$ if $AB = I_r$. The necessary condition for the existence of such a matrix B is that $rank(A) = r$.

Lemma 2: [19] The algebraic matrix equation

$$MX + XN = Q, \quad (2)$$

where $M \in \mathbb{R}^{m\times m}$ and $N \in \mathbb{R}^{n\times n}$ are square matrices, has a unique solution if and only if (iff) M and $-N$ don't have any common eigenvalues.

Remark 1: This algebraic matrix equation (2) is very similar to the Sylvester equation. However, in the Sylvester equation, the matrices M and N are square matrices of the same order.

III. Problem Formulation

Consider a multi-agent system of N agents with the following agent dynamics:

$$\begin{aligned} \dot{x}_i &= A_i x_i + B_i u_i \\ \dot{z}_i &= E_i x_i + F z_i, \end{aligned} \quad (3)$$

where $x_i \in \mathbb{R}^{n_i}$ and $z_i \in \mathbb{R}^r$ are the states of agent i, $u_i \in \mathbb{R}^{m_i}$ is the control input of agent i, and $A_i \in \mathbb{R}^{n_i\times n_i}$, $B_i \in \mathbb{R}^{n_i\times m_i}$, $E_i \in \mathbb{R}^{r\times n_i}$, and $F \in \mathbb{R}^{r\times r}$ are constant matrices. In (3), the matrices A_i, B_i, and E_i can be different for the agents and even the dimensions n_i and m_i can be different. However, the dimensions of the states $z_i, i \in \{1, ..., N\}$ and the matrix F are assumed to be the same for all the agents since z_i will be synchronized, as will be discussed later.

Definition 1: (**Simultaneous Stabilization and Synchronization (SSS)**) The objective of the SSS problem is to design a control law u_i for (3) so that the states x_i are stabilized while the states z_i are synchronized, i.e.,

$$\begin{aligned} x_i &\rightarrow 0 \text{ (stabilization)} \\ z_{ij} &= z_i - z_j \rightarrow 0 \text{ (synchronization)}, \end{aligned} \quad (4)$$

as $t \rightarrow \infty$ for $i, j \in \{1, ..., N\}$.

We name the states for synchronization (i.e., z_i) as *external states* and the states for stabilization (i.e., x_i) as *internal states*. In order to facilitate the subsequent analysis, we use **bold** font to represent the block diagonal matrices used in the collective forms. For example, $\mathbf{A}$, defined as $\mathbf{A} \triangleq diag\{A_1, A_2, ...A_N\}$, represents a block diagonal matrix with $A_i, i \in \{1, ..., N\}$ as the diagonal elements. The concatenated vectors X and Z are defined as

$$X \triangleq [x_1^T, ..., x_N^T]^T, \quad Z \triangleq [z_1^T, ..., z_N^T]^T.$$

To facilitate the subsequent design and analysis, we make the following assumptions.

Assumption 1: The connection network has a fixed directed graph $\mathcal{G}$ that contains a directed spanning tree.

Assumption 2: The pair $\{A_i, B_i\}, i \in \{1, ..., N\}$ is stabilizable.

Lemma 3: Under Assumption 1, the states $z_1, ..., z_N \in \mathbb{R}^r$ are synchronized in the sense that [20] $z_1 = \cdots = z_N$, if and only if $\bar{Z} = 0$ where

$$\bar{Z} \triangleq (L \otimes I_r)Z. \quad (5)$$

Remark 2: Lemma 3 indicates that if $\bar{Z}$ is stabilized, then $z_1, ..., z_N$ are synchronized.

IV. SSS via Full State Coupling

A SSS control law of the following form was proposed in [12]

$$u_i = -K_i x_i + G_i \sum_{j\in\mathcal{N}_i} a_{ij}[z_{ij} + (C_i x_i - C_j x_j)]. \quad (6)$$

In (6), a_{ij}'s are the elements of the adjacency matrix of the connection graph, $K_i \in \mathbb{R}^{m_i\times n_i}$, $C_i \in \mathbb{R}^{r\times n_i}$, and $G_i \in \mathbb{R}^{m_i\times r}$ are constant gain matrices to be designed, and $C_j \in \mathbb{R}^{r\times n_j}$ determines the portion of the internal states that agent j provides to the other agents. Under control law (6), the closed-loop dynamics of system (3) are given by

$$\begin{aligned} \dot{x}_i &= A_{ci} x_i + B_i G_i \sum_{j\in\mathcal{N}_i} a_{ij}[z_{ij} + (C_i x_i - C_j x_j)] \quad (7)\\ \dot{z}_i &= E_i x_i + F z_i, \quad (8)\end{aligned}$$

where $A_{ci} \triangleq A_i - B_i K_i \in \mathbb{R}^{n_i\times n_i}$.

V. SSS via Partial State Coupling and Low Gain Control

A. Design of SSS Control Law

In the previous section, the coupling term in the SSS control law requires relative measurements of both the external state (i.e., z_{ij}) and the internal state (i.e., $C_i x_i - C_j x_j$). It is of more interest to enable SSS using only the external states since the relative measurements of the required internal state might not be practically available.

In this section, we will show that SSS can be enabled using only the relative measurements of the external states via a low gain approach. Specifically, the low gain SSS control law is proposed as

$$u_i = -K_i x_i + \varepsilon_i G_i \sum_{j\in\mathcal{N}_i} a_{ij} z_{ij}, \quad (9)$$

where $\varepsilon_i \in \mathbb{R}$ is a positive scalar gain, $K_i \in \mathbb{R}^{m_i\times n_i}$ and $G_i \in \mathbb{R}^{m_i\times r}$ are constant gain matrices to be designed, and a_{ij}'s are the elements of the adjacency matrix of the connection graph.

As will be shown in Section V-D, G_i is designed for each agent only based on the structure of the agent dynamics but not on the connection graph topology. It will be shown later in Section V-E that SSS can be enabled by selecting ε_i less than a threshold with properly designed K_i and G_i.

Let $\epsilon \in \mathbb{R}$ be the maximum of the scalars ε_i, i.e., $\varepsilon_i \le \epsilon,\ i \in \{1, ..., N\}$. Then the new coupling strength can be defined as $a'_{ij} = \frac{\varepsilon_i}{\epsilon} a_{ij},\ i \in \{1, ..., N\},\ j \in \mathcal{N}_i$. We use L' as the new weighted Laplacian matrix corresponding to the coupling strength a'_{ij}. Note that if L contains a directed spanning tree, so does L'.

Based on a'_{ij}, the controller (9) can be rewritten as

$$u_i = -K_i x_i + \epsilon G_i \sum_{j\in\mathcal{N}_i} a'_{ij} z_{ij}. \quad (10)$$

The control law (10) will only be used for subsequent stability analysis. The control law (9) will actually be implemented to enable SSS.

The closed-loop system given by (3) and (10) can be rewritten in the following collective form:

$$\begin{aligned} \dot{X} &= \mathbf{A}_c X + \epsilon \mathbf{B}\mathbf{G}(L' \otimes I_r) Z \quad (11)\\ \dot{Z} &= \mathbf{E}X + (I_N \otimes F) Z. \end{aligned}$$

The **bold** font is used to represent block diagonal matrices (see Section III).

B. Decoupling of Collective Agent Dynamics

To facilitate the design of control gains (K_i, G_i, and ε_i) and the stability analysis, a new coordinate transformation method is proposed to decouple the collective agent dynamics (11).

Define a new state $\eta \in \mathbb{R}^{\Sigma_{i=1}^N n_i}$ as

$$\eta \triangleq X + \epsilon P(L' \otimes I_r) Z, \quad (12)$$

where P satisfies the following algebraic matrix equation:

$$-\mathbf{A}_c P + \mathbf{B}\mathbf{G} + P(I_N \otimes F) - \epsilon P(L' \otimes I_r)\mathbf{E}P = 0. \quad (13)$$

The solvability of (13) will be discussed in Section V-C.

According to (11), (12), and (13), the derivative of η is given by

$$\begin{aligned} \dot{\eta} &= \mathbf{A}_c(\eta - \epsilon P(L' \otimes I_r)Z) + \epsilon\mathbf{B}\mathbf{G}(L' \otimes I_r)Z \\ &\quad + \epsilon P(L' \otimes I_r)\mathbf{E}(\eta - \epsilon P(L' \otimes I_r)Z) \\ &\quad + \epsilon P(L' \otimes I_r)(I_N \otimes F)Z \\ &= [\mathbf{A}_c + \epsilon P(L' \otimes I_r)\mathbf{E}]\,\eta \\ &\quad + \epsilon\,[-\mathbf{A}_c P + \mathbf{B}\mathbf{G} + P(I_N \otimes F) \\ &\quad - \epsilon P(L' \otimes I_r)\mathbf{E}P]\,(L' \otimes I_r)Z \\ &= [\mathbf{A}_c + \epsilon P(L' \otimes I_r)\mathbf{E}]\,\eta. \end{aligned}$$

Similarly, the derivative of Z is given by

$$\begin{aligned} \dot{Z} &= \mathbf{E}(\eta - \epsilon P(L' \otimes I_r)Z) + (I_N \otimes F)Z \\ &= (I_N \otimes F)\,Z - \epsilon\mathbf{E}P(L' \otimes I_r)Z + \mathbf{E}\eta. \end{aligned}$$

Thus, the system dynamics (11) can be written in terms of (η, Z) as

$$\begin{aligned} \dot{\eta} &= [\mathbf{A}_c + \epsilon P(L' \otimes I_r)\mathbf{E}]\,\eta \quad (14)\\ \dot{Z} &= (I_N \otimes F)\,Z - \epsilon\mathbf{E}P(L' \otimes I_r)Z + \mathbf{E}\eta. \end{aligned}$$

Define another new state $\xi \in \mathbb{R}^{Nr}$ as

$$\begin{aligned} \xi &\triangleq (L' \otimes I_r)(I_N \otimes e^{-Ft})Z - Q(t)\eta \\ &= (L' \otimes e^{-Ft})Z - Q(t)\eta, \quad (15) \end{aligned}$$

where $Q(t)$ is a solution of the following matrix differential equation:

$$\begin{aligned} \dot{Q} &= -\epsilon(L' \otimes e^{-Ft})\mathbf{E}P(I_N \otimes e^{Ft})Q \quad (16)\\ &\quad + (L' \otimes e^{-Ft})\mathbf{E} - Q\,[\mathbf{A}_c + \epsilon P(L' \otimes I_r)\mathbf{E}] \end{aligned}$$

with an initial condition satisfying

$$(b'^T \otimes I_r)Q(t_0) = 0. \tag{17}$$

In (17), b'^T is defined as the left eigenvector of L' corresponding to the zero eigenvalue (i.e., $b'^T L' = 0$ and $\mathbf{1}^T b' = 1$). The solvability of (16) will be discussed in Section V-C.

According to (14), (15), and (16), the derivative of ξ is given by

$$\begin{aligned}\dot{\xi} &= -(L' \otimes Fe^{-Ft})Z + (L' \otimes e^{-Ft})(I_N \otimes F)Z \\ &\quad -\epsilon(L' \otimes e^{-Ft})\mathbf{E}P(L' \otimes I_r)Z + (L' \otimes e^{-Ft})\mathbf{E}\eta \\ &\quad -Q[\mathbf{A}_c + \epsilon P(L' \otimes I_r)\mathbf{E}]\eta - \dot{Q}\eta \\ &= -\epsilon(L' \otimes e^{-Ft})\mathbf{E}P(I_N \otimes e^{Ft})(\xi + Q\eta) \\ &\quad +(L' \otimes e^{-Ft})\mathbf{E}\eta - Q[\mathbf{A}_c + \epsilon P(L' \otimes I_r)\mathbf{E}]\eta - \dot{Q}\eta \\ &= -\epsilon(L' \otimes e^{-Ft})\mathbf{E}P(I_N \otimes e^{Ft})\xi \\ &\quad +\{-\epsilon(L' \otimes e^{-Ft})\mathbf{E}P(I_N \otimes e^{Ft})Q + (L' \otimes e^{-Ft})\mathbf{E} \\ &\quad -Q[\mathbf{A}_c + \epsilon P(L' \otimes I_r)\mathbf{E}] - \dot{Q}\}\eta \\ &= -\epsilon(L' \otimes I_r)\left[(I_N \otimes e^{-Ft})\mathbf{E}P(I_N \otimes e^{Ft})\right]\xi.\end{aligned}$$

Thus, the dynamics (11) with coupling terms are decoupled as

$$\dot{\eta} = [\mathbf{A}_c + \epsilon P(L' \otimes I_r)\mathbf{E}]\eta \tag{18}$$

$$\dot{\xi} = -\epsilon(L' \otimes I_r)\left[(I_N \otimes e^{-Ft})\mathbf{E}P(I_N \otimes e^{Ft})\right]\xi \tag{19}$$

C. Solvability of (13) and (16) for the Decoupling Coordinate Transformations

In this section, we will investigate whether the coordinate transformations in (12) and (15) are feasible. This question boils down to the solvability of 1) the algebraic matrix equation (13) and 2) the matrix differential equation (16).

Lemma 4: Suppose that the matrices A_{ci}'s have no common eigenvalue with F. Then, the unperturbed equation of (13), presented as

$$-\mathbf{A}_c P + \mathbf{BG} + P(I_N \otimes F) = 0, \tag{20}$$

has a solution of the form $\mathbf{P}_0 = diag\{P_{0i}\}$, where P_{0i} satisfies

$$-A_{ci}P_{0i} + P_{0i}F + B_iG_i = 0. \tag{21}$$

Proof: Based on Lemma 2, if the matrices A_{ci}'s have no common eigenvalue with F, then the equation $-A_{ci}P_{0i} + P_{0i}F + B_iG_i = 0$ has a unique solution. Substituting $\mathbf{P}_0 = diag\{P_{0i}\}$ into (13) with $\epsilon = 0$ gives

$$\begin{aligned}&-\mathbf{A}_c diag\{P_{0i}\} + \mathbf{BG} + diag\{P_{0i}\}(I_N \otimes F) \\ &\qquad = diag\{-A_{ci}P_{0i} + P_{0i}F + B_iG_i\} = 0.\end{aligned}$$

Therefore, $\mathbf{P}_0 = diag\{P_{0i}\}$ satisfies the unperturbed equation (20). ■

Theorem 2: There exists $\epsilon_1^* > 0$ such that for every $\epsilon \in (0, \epsilon_1^*)$ a solution exists for the equation (13) if the matrices A_{ci}'s have no common eigenvalue with F. Furthermore, the solution has the following form:

$$P = \mathbf{P}_0 + \mathcal{O}(\epsilon), \tag{22}$$

where $\mathcal{O}(\epsilon)$ represents a function of ϵ satisfying $\mathcal{O}(\epsilon) \to 0$ as $\epsilon \to 0$.

Proof: Differentiating both sides of (13) with respect to ϵ and rearranging the terms gives

$$\begin{aligned}&[-\mathbf{A}_c - \epsilon P(L' \otimes I_r)\mathbf{E}]\frac{dP}{d\epsilon} \\ &+ \frac{dP}{d\epsilon}[(I_N \otimes F) - \epsilon(L' \otimes I_r)\mathbf{E}P] - P(L' \otimes I_r)\mathbf{E}P = 0.\end{aligned} \tag{23}$$

Denote P_1 as the derivative of P with respect to ϵ at $\epsilon = 0$, i.e., $P_1 \triangleq \frac{dP}{d\epsilon}\big|_{\epsilon=0}$. Set $\epsilon = 0$ in (23). Then, we have

$$-\mathbf{A}_c P_1 + P_1(I_N \otimes F) - \mathbf{P}_0(L' \otimes I_r)\mathbf{E}\mathbf{P}_0 = 0. \tag{24}$$

If the matrices A_{ci}'s have no common eigenvalue with F, then $\mathbf{A}_c$ has no common eigenvalue with $I_N \otimes F$. Based on Lemma 2, this ensures that the equation (24) has a solution. Thus, the derivative term $\frac{dP}{d\epsilon}$ at $\epsilon = 0$ exists. Hence, the solution of P can locally be expressed as [21] $P = \mathbf{P}_0 + \epsilon P_1 + \mathcal{O}(\epsilon^2)$.

Since $\mathbf{A}_c$ has no common eigenvalue with $I_N \otimes F$, based on the continuation of $\mathbf{A}_c + \epsilon P(L' \otimes I_r)\mathbf{E}$ with respect to ϵ, there exists an open set $\mathcal{D}_P \times \mathcal{D}_\epsilon$ with $(\mathbf{P}_0, 0) \in \mathcal{D}_P \times \mathcal{D}_\epsilon$ so that $\mathbf{A}_c + \epsilon P(L' \otimes I_r)\mathbf{E}$ has no common eigenvalue with $(I_N \otimes F) - \epsilon(L' \otimes I_r)\mathbf{E}P$ in this set. For every $(P, \epsilon) \in \mathcal{D}_P \times \mathcal{D}_\epsilon$, the derivative term $\frac{dP}{d\epsilon}$ can be uniquely determined in terms of P and ϵ based on (23) and Lemma 2. Since $(\mathbf{P}_0, 0) \in \mathcal{D}_P \times \mathcal{D}_\epsilon$, based on the Peano existence theorem [22], there exists $\epsilon_1^* > 0$ such that differential equation (23) with initial value $P(0) = \mathbf{P}_0$ has a solution for $\epsilon \in (-\epsilon_1^*, \epsilon_1^*)$. Hence, a solution exists for algebraic equation (13) for every $\epsilon \in (-\epsilon_1^*, \epsilon_1^*)$. ■

Remark 3: Theorem 2 indicates that the solution of (13) exists and has the form of (22). However, the solution may not be unique.

In the following theorem, the solvability of matrix differential equation (16) is investigated. In addition, a property of the solution is presented, which will be used in the subsequent stability analysis.

Theorem 3: The matrix differential equation (16) with an initial condition satisfying (17) has a unique solution. Moreover, the solution satisfies $(b'^T \otimes I_r)Q(t) = 0$.

Proof: By using the vectorization operator $vec(.)$ described in Section II, (16) becomes

$$\begin{aligned}\frac{d}{dt}vec(Q) &= -\{I \otimes \epsilon(L' \otimes e^{-Ft})\mathbf{E}P(I_N \otimes e^{Ft}) \\ &\quad + [\mathbf{A}_c + \epsilon P(L' \otimes I_r)\mathbf{E}]^T \otimes I\}vec(Q) \\ &\quad +vec\{(L' \otimes e^{-Ft})\mathbf{E}\}.\end{aligned} \tag{25}$$

System (25) is a first-order non-autonomous linear time-varying ordinary differential equation of the form $\dot{y} = p(t)y + q(t)$, which is well-known to have a closed-loop solution. Thus, a solution exists for (25) for $t \in [t_0, \infty)$.

Define a new state $s(t) \triangleq (b'^T \otimes I_r)Q(t)$. Multiplying $(b'^T \otimes I_r)$ on both sides of (16) gives

$$\dot{s}(t) = -s(t)[\mathbf{A}_c + \epsilon P(L' \otimes I_r)\mathbf{E}], \tag{26}$$

with the help of the fact that $(b'^T \otimes I_r)(L' \otimes e^{-Ft}) = b'^T L' \otimes e^{-Ft} = 0$. Based on (17), the initial condition satisfies

$$s(t_0) = (b'^T \otimes I_r)Q(t_0) = 0. \quad (27)$$

The only solution of (26) with initial condition (27) is $s(t) \equiv 0$ for $t \in [t_0, \infty)$. Thus, $(b'^T \otimes I_r)Q(t) = 0$. ■

D. Design of Control Gain Matrices

According to Sections V-C and V-E, K_i will be designed so that $A_{ci} = A_i - B_i K_i$ is Hurwitz and has no common eigenvalue with F, and G_i will be designed so that the solution of (21), P_{0i}, satisfies $E_i P_{0i} = I_r$. Based on Assumption 2, it is not difficult to design desirable K_i's.

The following lemma discusses the existence of desirable gain matrices G_i's and a strategy to design them.

Theorem 4: There exists a solution for P_{0i} and G_i in the following set of algebraic matrix equations:

$$\begin{aligned} -A_{ci}P_{0i} + P_{0i}F + B_i G_i &= 0 \\ E_i P_{0i} &= I_r, \end{aligned} \quad (28)$$

provided that:

1) F and A_{ci} have no common eigenvalue, and

2) a solution $vec(G_i)$ exists for $\Omega_i vec(G_i) = vec(I_r)$, where $\Omega_i \triangleq \left[(I_r \otimes E_i)(I_r \otimes A_{ci} - F^T \otimes I_{n_i})^{-1}(I_r \otimes B_i)\right]$.

Proof: The method of vectorization is applied in solving the algebraic matrix equations in (28). The equations in (28) can be written in the vectorized form as

$$(-I_r \otimes A_{ci})vec(P_{0i}) + (F^T \otimes I_{n_i})vec(P_{0i}) + (I_r \otimes B_i)vec(G) = 0_{nr\times 1} \quad (29)$$

$$(I_r \otimes E_i)vec(P_{0i}) = vec(I_r). \quad (30)$$

Provided that F and A_{ci} have no common eigenvalue, the matrix $I_r \otimes A_{ci} - F^T \otimes I_{n_i}$ has no zero eigenvalue. Thus, it is invertible. Based on (29), we have

$$vec(P_{0i}) = (I_r \otimes A_{ci} - F^T \otimes I_{n_i})^{-1}(I_r \otimes B_i)vec(G_i). \quad (31)$$

Plugging (31) into (30) gives

$$\Omega_i vec(G_i) = vec(I_r). \quad (32)$$

Provided that (32) is solvable, G_i can be determined. ■

Remark 4: Equation (32) has a solution if and only if

$$rank([\Omega_i, vec(I_r)]) = rank(\Omega_i) \leq m_i r.$$

Since $\Omega_i \in \mathbb{R}^{r^2 \times m_i r}$, the above condition can only be satisfied if $m_i \geq r$. This means the number of inputs must be greater than or equal to the number of external states.

Remark 5: As a special case, if $F = 0$, then $P_{0i} = A_{ci}^{-1} B_i G_i$ based on (28). The gain matrix G_i can be selected as the right inverse of $E_i A_{ci}^{-1} B_i$.

E. Stability Analysis

In this section, we will present several lemmas that will be exploited in the subsequent closed-loop stability analysis.

Lemma 5: Under Assumption 1, suppose that $b^T \in \mathbb{R}^{1\times N}$ is the normalized left eigenvector of L corresponding to the zero eigenvalue (i.e., $b^T L = 0$ and $\mathbf{1}^T b = 1$). Then, the system

$$\dot{\delta} = -(L \otimes I_r)\delta \quad (33)$$

is exponentially stable on the surface

$$(b^T \otimes I_r)\delta \equiv 0. \quad (34)$$

Proof: Equation (33) represents a first-order integrator multi-agent system. It is well-known that if the directed connection graph contains a directed spanning tree (Assumption 1), $(b^T \otimes I_r)\delta(t)$ is invariant and the states of each agent will exponentially reach to $(b^T \otimes I_r)\delta(t_0)$. Thus, if the initial conditions are selected such that $(b^T \otimes I_r)\delta(t_0) = 0$, system (33) will exponentially reach zero and will always remain on the manifold (34). That is, system (33) is exponentially stable on manifold (34). ■

Lemma 6: Suppose that $\mathbf{A}_c$ is Hurwitz. Then, there exists a ϵ_2^* satisfying $0 < \epsilon_2^* \leq \epsilon_1^*$ such that η in (18) is exponentially stable for all $\epsilon \in (-\epsilon_2^*, \epsilon_2^*)$.

Proof: Since $\epsilon_2^* \leq \epsilon_1^*$, a solution P exists for the equation (13). For small values of ϵ, the matrix $\mathbf{A}_c + \epsilon P(L' \otimes I_r)\mathbf{E}$ is Hurwitz because $\mathbf{A}_c$ is Hurwitz. Therefore, η in (18) is exponentially stable. ■

Lemma 7: Under Assumptions 1 and 2, suppose that 1) the gain matrix K_i is designed so that $A_{ci} = A_i - B_i K_i$ is Hurwitz and A_{ci} doesn't have a common eigenvalue with F, and 2) the gain matrix G_i is designed according to Theorem 4 so that $\mathbf{E}\mathbf{P}_0 = I$. Then, there exists a ϵ_3^* satisfying $0 < \epsilon_3^* \leq \epsilon_1^*$ such that for every $\epsilon \in (-\epsilon_3^*, \epsilon_3^*)$ the state ξ defined in (15) with the dynamics (19) is exponentially stable and $\mu(t)$, defined as

$$\mu(t) \triangleq Q(t)\eta(t), \quad (35)$$

is also exponentially stable.

Proof: Multiplying both sides of (15) by $(b'^T \otimes I_r)$ gives

$$(b'^T \otimes I_r)\xi = (b'^T L' \otimes e^{-Ft})Z - (b'^T \otimes I_r)Q(t)\eta.$$

According to the definition of b' and Theorem 3, we have

$$(b'^T \otimes I_r)\xi \equiv 0. \quad (36)$$

Thus, the trajectory of (19) is restricted to the surface (36).

If the gain matrices G_i's are designed according to Theorem 4 so that $\mathbf{E}\mathbf{P}_0 = I$, then we can rewrite $\mathbf{E}P$ as $\mathbf{E}P = I + \epsilon\left(EP_1 + \frac{E\mathcal{O}(\epsilon^2)}{\epsilon}\right) \triangleq I + \epsilon R$. Therefore, the differential equation (19) will become

$$\dot{\xi} = -\epsilon(L' \otimes I_r)\xi - \epsilon^2(L' \otimes I_r)R'(t)\xi, \quad (37)$$

where $R'(t) \triangleq (I_N \otimes e^{-Ft})R(I_N \otimes e^{Ft})$.

Scale the time variable t using $\tau = \epsilon t$. Then the differential equation (37) can be expressed as a slowly time-varying system [23] as

$$\frac{d\xi}{d\tau} = -(L' \otimes I_r)(I - \epsilon R^{'})\xi. \quad (38)$$

Thus, there exists $\epsilon_3^* > 0$ such that the system (38) is exponentially stable for all $\epsilon \in (-\epsilon_3^*, \epsilon_3^*)$. This conclusion is based on the fact that the unperturbed system of (37), i.e., $\frac{d\xi}{d\tau} = -(L' \otimes I_r)\xi$, is exponentially stable according to Lemma 5 because it is always on the manifold (36).

The time derivative of $\mu(t)$ defined in (35) is

$$\begin{aligned} \dot{\mu} &= \dot{Q}\eta + Q\dot{\eta} \\ &= -\epsilon(L' \otimes e^{-Ft})\mathbf{E}P(I_N \otimes e^{Ft})Q\eta \\ &\quad +(L' \otimes e^{-Ft})\mathbf{E}\eta - Q\left[\mathbf{A}_c + \epsilon P(L' \otimes I_r)\mathbf{E}\right]\eta \\ &\quad +Q\left[\mathbf{A}_c + \epsilon P(L' \otimes I_r)\mathbf{E}\right]\eta \\ &= -\epsilon(L' \otimes e^{-Ft})\mathbf{E}P(I_N \otimes e^{Ft})\mu + (L' \otimes e^{-Ft})\mathbf{E}\eta. \end{aligned}$$

Similarly, the unperturbed system $\dot{\mu} = -\epsilon(L' \otimes e^{-Ft})\mathbf{E}P(I_N \otimes e^{Ft})\mu$ can be proved to be exponentially stable. Since $\eta(t)$ is exponentially stable according to Lemma 6, $\mu(t)$ is also exponentially stable. ■

Theorem 5: Consider a system of N non-identical agents with dynamics (3). Under Assumptions 1 and 2, the control law (10) enables SSS in the sense of (4) provided that the control gains K_i, G_i, and ε_i are designed as below.

1) K_i is designed so that $A_{ci} = A_i - B_i K_i$ is Hurwitz and A_{ci} has no common eigenvalue with F.

2) G_i is designed so that the conditions in Theorem 4 are satisfied.

3) ε_i is selected so that $\varepsilon_i \leq \epsilon^*$, where $\epsilon^* = \min\{\epsilon_1^*, \epsilon_2^*, \epsilon_3^*\}$ and $\epsilon_1^*, \epsilon_2^*, \epsilon_3^*$ are demonstrated in Theorem 2, Lemma 6, and Lemma 7, respectively.

Proof: Under Assumption 2, it is not difficult to design desirable K_i's so that A_{ci} is Hurwitz and it has no common eigenvalue with F using pole placement methods. Selecting $\varepsilon_i \leq \epsilon^*$ ensures that $\epsilon = \max\{\varepsilon_i\} \leq \min\{\epsilon_1^*, \epsilon_2^*, \epsilon_3^*\}$. Provided that G_i is selected according to Theorem 4, then all the conditions in Lemmas 6 and 7 are satisfied which indicate that $\eta(t)$, $\xi(t)$, and $Q(t)\eta(t)$ are exponentially stable.

Based on the transformation (15), $(L' \otimes I_r)(I_N \otimes e^{-Ft})Z$ is exponentially stable. According to Lemma 3, $e^{-Ft}z_i$ and consequently z_i are synchronized. Therefore, $(L' \otimes I_r)Z$ is also exponentially stable. Moreover, according to the transformation (12), X is exponentially stable. Hence, SSS is enabled in the sense of (4). ■

VI. Conclusions

In this paper, we studied the problem of *simultaneous stabilization and synchronization (SSS)* for one class of non-identical multi-agent systems with linear dynamics and directed connection topology. The agent dynamics can be different and the orders of agents are not necessarily equal. We showed that a single control loop can be designed for each agent to enable the *internal states* to be stabilized and the *external states* to be simultaneously synchronized. We proposed a distributed control law using a low gain approach, which require only external states (internal states are not required) exchanged from neighboring agents and local measurements to enable SSS. Under mild stabilizability and connectivity assumptions, we showed that SSS can be enabled by properly designing the control gain matrices and selecting a small enough low gain which ensures stability and a fast enough convergence rate.

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