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H_∞ Robust Design of PID Controllers for Arbitrary-order LTI Systems with Time Delay

Linlin Ou, Peidong Zhou, Weidong Zhang and Li Yu

***Abstract*—For a given arbitrary-order LTI (linear time-invariant) plant with time delay, we propose a directly parametric design method of the H_∞ PID controller in an analytical manner. The design problem of the PID controllers satisfying the H_∞ norm requirement is first cast into simultaneous stabilization problem of a family of complex quasipolynomials and the characteristic equation. Then, the linear programming characterization of the PID controllers that can ensure the stability of the complex quasipolynomials are developed on the basis of the extended Hermite-Biehler Theorem.Finally, The admissible set of the H_∞ PID controllers is presented by combining such the results with the stabilizing set of the PID controllers. The results reveal that the set of the integral and derivative gains is a union of convex sets for a fixed proportional gain. The proposed scheme works without any approximation and enable one to find a set of the PID parameters satisfying the H_∞ norm requirement conveniently.**

I. Introduction

TIME delay often occurs in systems of engineering, biology and ecology, especially in process control[1]. The existence of time delay can degrade the achievable performance and even induce the system instability [2]. Since the time-delay terms cause an infinite number of roots of the characteristic equation, they make the system difficult to design with the classical methods. Thus, such problems are often solved indirectly by using rational approximation. However, the rational approximation constitutes a limitation in accuracy and can lead to the instability of the actual system. Although the controllers can be designed for the system with time delay by using the Lyapunov framework or algebraic Riccati equations, these methods require complex formulations, and lead to conservative results and possibly redundant control. More seriously, these control strategies for systems with time delay cannot present the simple low-order controllers.

It is well known that the simple low-order controllers, especially PID controllers, are the most widely-used control strategy in various engineering application fields[3]. Due to this, a lot of efforts have been made to design PID controllers for the system with time delay. Most of these methods are based on the first-order or second-order plus dead-time model[4,5]. For high-order plants with time delay, some design methods of low-order controllers rely on model reduction methods or controller reduction methods[6-8], which will inevitably lead to degradation of system performance. Hence, it is desired to develop an effective approach for PID controller design to satisfy these requirements simultaneously: 1) It is applicable to a broad set of plants with time delay; 2) It is simple to understand and easy to implement; 3) It can deal with robustness under the unavoidable exogenous disturbances in practice. Such a design problem has not been well resolved in current literature.

The controller design for robustness can be cased into the computation and minimization of H_∞ norm of a prescribed transfer function of the system[9], and the H_∞ control theory has been developed to the control system synthesis. Unfortunately, the H_∞ optimal controller cannot be directly applied to the systems with time delay. Even if the system with time delay is converted to the approximated one with rational transfer function, the order of the resultant H_∞ controller is always larger or equal than the order of the plant. Recently, the analytical design methods of the H_∞ PID and first-order controllers were developed for systems without time delay[10,11], but the design is not applicable to the systems with time delay. To the author's knowledge, only a few literatures present the low-order H_∞ controller design method for arbitrary-order systems with time delay. A fixed-order H_∞ controllers for a class of time-delay systems was proposed based on a non-smooth, non-convex optimization method[12]. The method needs largely numerical computation. For a given derivative or integral gain, the graphically design method of the robust PID controllers was proposed based on the D-decomposition technique[13]. Therefore, it remains an open question to design a H_∞ PID controller directly in an analytical way.

For a given arbitrary-order LTI plant with time delay, we proposed an algorithm to determine the complete stabilizing set of PID controllers [14], which can be viewed as the first step for PID controller design. Based on the result, this paper presents a direct parametric design method in an analytical manner for H_∞ PID controllers. The controller design problem is first translated into simultaneous stabilization problem of a family of complex quasipolynomials and then the entire set of all PID gains that can guarantee the stability of these quasipolynomials and the closed-loop characteristic equation is found. The proposed method shows that the set of the integral and derivative gains is a union of convex sets for

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Linlin Ou, Peidong Zhou and Li Yu are with the department of Automation, Zhejiang University of Technology, Hangzhou 310023, China, e-mail: linlinou@zjut.edu.cn.

Weidong Zhang is with the department of Automation, Shanghai Jiaotong University, Shanghai 200240, China, wdzhang@sjtu.edu.cn

a fixed proportional gain, which is similar to the results for the plant free of time delay[10]. The method is applicable to arbitrary-order LTI systems with time delay.

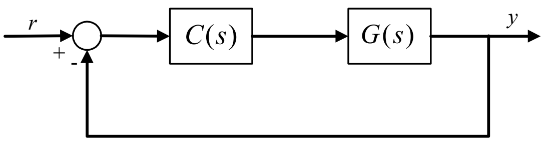

Fig. 1. Block diagram of the unity feedback control system

II. Problem Statement

The SISO LTI plant with time delay can be described as

$$G(s) = \frac{N(s)}{D(s)} e^{-\theta s}, \tag{1}$$

where θ is the time delay, and $N(s)$ and $D(s)$ are coprime polynomials in s, defined as

$$N(s) = v_m s^m + v_{m-1} s^{m-1} + \cdots + v_1 s + v_0$$
$$D(s) = s^n + u_{n-1} s^{n-1} + \cdots + u_1 s + u_0$$

Here, $v_0, v_1, \cdots v_m$ and $u_0, u_1, \cdots u_{n-1}$ are real numbers, and $n > m$. Consider the feedback control system shown in Fig.1, where $G(s)$ is a plant with the transfer function (1) and $C(s)$ is a PID controller with the form

$$C(s) = k_p + \frac{k_i}{s} + k_d s \tag{2}$$

The goal of the paper is to determine the set of (k_p, k_i, k_d) for which the closed-loop system is stable and satisfies the following H_∞ performance index

$$\left\| W(s) T(s, k_p, k_i, k_d) \right\|_\infty < \gamma$$

for a given number $\gamma > 0$, where $T(s, k_p, k_i, k_d)$ can be sensitivity function, complementary sensitivity function or input sensitivity function and $W(s)$ is a stable weighting function to specify the performance requirements[10, 15]. Since the cases for sensitivity function, complementary sensitivity function and input sensitivity function are similar, in this paper we only consider the case that $T(s, k_p, k_i, k_d)$ is the complementary sensitivity function, i.e.

$$T(s, k_p, k_i, k_d) = \frac{C(s)G(s)}{1 + C(s)G(s)} \tag{3}$$

Let the weighting function $W(s) = W_n(s)/W_d(s)$, where $W_n(s)$ and $W_d(s)$ are coprime polynomials and $W_d(s)$ is stable. Substituting (1) and (2) into (3) and then multiply (3) by $W(s)$, we have

$$W(s)T(s,k_p,k_i,k_d) = \frac{W_n(s)(k_d s^2 + k_p s + k_i)N(s)}{sW_d(s)D(s)e^{\theta s} + W_d(s)(k_d s^2 + k_p s + k_i)N(s)} \tag{4}$$

In order to find the k_p, k_i, k_d values satisfying $\left\| W(s)T(s,k_p,k_i,k_d) \right\|_\infty < \gamma$ in an analytical way, a feasible approach is to convert the synthesis problem of H_∞ controllers to the quasipolynomial stabilization problem first and then determine the k_p, k_i, k_d values by solving the stabilization problem. According to the result for the proper rational function[16], the following lemma is given:

Lemma 1 Assume that

$$F(s) = \frac{A(s)}{B(s)e^{\tau s} + E(s)} \tag{5}$$

is stable where $A(s)$, $B(s)$ and $E(s)$ are respectively the polynomials with $\deg[A(s)] = q$, $\deg[B(s)] = p$ and $\deg[E(s)] = r$, $q \le p$, $q \le r$, b_p, e_r and a_q are the highest -order coefficient of $B(s)$, $E(s)$ and $A(s)$, respectively. The inequality $\left\| F(s) \right\|_\infty < 1$ holds if and only if

(1) $|b_p| > |a_q|$ if $p > r$, $|e_r| > |a_q|$ if $p < r$, or $|b_p + e_r| > |a_q|$ if $p = r$;

(2) $\left[B(s)e^{\tau s} + E(s) \right] + e^{j\varphi} A(s)$ is stable for all φ in $[0, 2\pi)$.

Proof of Lemma 1 is similar to the case for the rational transfer function in [16] and is omitted here due to the space limit.

The closed-loop characteristic function is given by

$$\delta(s, k_p, k_i, k_d) = sD(s)e^{\theta s} + (k_d s^2 + k_p s + k_i)N(s) \tag{6}$$

and the quasipolynomial $\upsilon(s, k_p, k_i, k_d, \varphi)$ is defined as

$$\upsilon(s, k_p, k_i, k_d, \varphi) = sD(s)W_d(s)e^{\theta s} + (k_d s^2 + k_p s + k_i)N(s)\left[W_d(s) + e^{j\varphi} W_n(s)/\gamma \right] \tag{7}$$

From Lemma 1, it is seen that for a given number $\gamma > 0$, the necessary and sufficient conditions that the PID gains satisfy $\left\| W(s)T(s, k_p, k_i, k_d) \right\|_\infty < \gamma$ are presented as follows:

(1) $\delta(s, k_p, k_i, k_d)$ in (6) is stable;

(2) $\upsilon(s, k_p, k_i, k_d, \varphi)$ in (7) is stable for all φ in $[0, 2\pi)$;

(3) $\left| W(\infty)T(\infty, k_p, k_i, k_d) \right| < \gamma$

As a result, the synthesis of H_∞ PID controllers is cast into simultaneous quasipolynomial stabilization problem. The set of (k_p, k_i, k_d) for which $\delta(s, k_p, k_i, k_d)$ is stable can be determined using the result in [14]. However, such a result is not applicable to $\upsilon(s, k_p, k_i, k_d, \varphi)$ since $\upsilon(s, k_p, k_i, k_d, \varphi)$ is a complex quasipolynomial. In the following sections, the necessary and sufficient conditions for the stabilization of the complex quasipolynomial will be first given in a simple manner in terms of the extended Hermite-Biehler theorem; based on the result the approach to determine the values of (k_p, k_i, k_d) for which $\upsilon(s, k_p, k_i, k_d, \varphi)$ is stable is given.

III. Preliminary Knowledge For Stability Of The Complex Quasipolynomial

We first state the extended Hermite-Biehler Theorem. Consider the quasipolynomial

$$f(s) = \sum_{k=0}^{M} \sum_{j=0}^{N} a_{kj} s^j e^{s\theta_k}, \tag{8}$$

where $a_{MN} \ne 0$, M and N are positive integers, a_{kj} is real or complex number, and $\theta_0, \theta_1, \cdots, \theta_M$ are real numbers satisfying $0 < \theta_1 < \theta_2 < \cdots < \theta_M$. The term $a_{MN} s^N e^{s\theta_M}$ is called the principle term. For the stability of quasipolynomial $f(s)$ in (8), the extended Hermite-Biehler Theorem is described as follows:

Theorem 1[17,18] $f(s)$ in (8) is stable if and only if

(i) $f_r(\omega)$ and $f_i(\omega)$ have only real zeros and these zeros interlace;

(ii) $f_i'(\omega) f_r(\omega) - f_i(\omega) f_r'(\omega) > 0$ for some $\omega \in (-\infty, +\infty)$.

Here, $f_r(\omega)$, $f_i(\omega)$, $f_r'(\omega)$ and $f_i'(\omega)$ denote the real and imaginary parts of $f(j\omega)$ and their first derivatives with respect to ω, respectively.

To ascertain that $f_r(\omega)$ and $f_i(\omega)$ have only real zeros, the following theorem is given:

Theorem 2[17, 18] Assume that all θ_k's values in (8) are the integers and let η be a constant so that the coefficients of the highest degree terms in $f_r(\omega)$ and $f_i(\omega)$ do not vanish at $\omega=\eta$. The necessary and sufficient condition under which $f_r(\omega)=0$ or $f_i(\omega)=0$ has only real roots is that, in the interval $-2l^*\pi+\eta\le\omega\le 2l^*\pi+\eta$, $f_r(\omega)$ or $f_i(\omega)$ has exactly $4l^*\theta_M+N$ real zeros starting with a sufficiently large integer l^*, respectively.

$\upsilon(s,k_p,k_i,k_d,\varphi)$ in (7) is a complex quasipolynomial with the following general form:

$$H(s)=d(s)e^{\theta s}+n(s) \tag{9}$$

where $n(s)$ and $d(s)$ are the polynomials given by

$$n(s)=(\alpha_g+j\beta_g)s^g+(\alpha_{g-1}+j\beta_{g-1})s^{g-1}+\cdots+(\alpha_0+j\beta_0)$$

$$d(s)=s^h+d_{n-1}s^{h-1}+\cdots+d_1s+d_0$$

Here, $\alpha_0,\alpha_1,\cdots,\alpha_g,\beta_0,\beta_1,\cdots,\beta_g$ and $d_0,d_1,\cdots,d_{h-1}$ are the real numbers and $h>g$.

Based on the preceding theorems, we can obtain the following necessary and sufficient condition for the stability of quasipolynomial $H(s)$.

Theorem 3 If $h>g$, $H(s)$ in (9) is stable if and only if

(i) $H_i(z)$ has exactly $4l^*+h$ real zeros in $[-2l^*\pi-\xi,2l^*\pi-\xi]$;

(ii) All the zeros of $H_i(z)$ in $[-2l^*\pi-\xi,2l^*\pi-\xi]$ interlace with those of $H_r(z)$.

Here $z=\theta\omega$, l^* is a sufficiently large integer, $H_r(z)$ and $H_i(z)$ are, respectively, the real and imaginary parts of $H(jz/\theta)$, and ξ is given by

$$\xi=\begin{cases}-\pi/2 & \text{if } h \text{ even and } \alpha_g\beta_g=0\\ 0 & \text{if } h \text{ odd and } \alpha_g\beta_g=0\\ \arctan(-\beta_m/\alpha_m) \text{ or } \pi+\arctan(-\beta_m/\alpha_m) & \text{if } h+g \text{ even}\\ \arctan(\alpha_m/\beta_m) \text{ or } \pi+\arctan(\alpha_m/\beta_m) & \text{if } h+g \text{ odd}\end{cases}$$

in the interval $[-\pi/2,\pi/2]$ in order to present the values of the PID gains for which $\upsilon(s,k_p,k_i,k_d,\varphi)$ is stable.

Proof: Substituting $s_1=\theta s$ into (9), it is observed that the highest powers of s_1 and e^{s_1} for the new quasipolynomial $H(s_1)$ are h and 1, respectively. Thus, Condition (i) is the necessary and sufficient condition that $H_i(z)$ has only real roots. Similar to the proof of Lemma 2 in [14], it can be proved that: (1) in the case $h>g$, the real zeros of $H_r(z)$ and $H_i(z)$ always interlace for $z\ge z^*$, where z^* is a sufficiently large value; (2) in the case $h>g$, the inequality $H_i'(z)H_r(z)-H_i(z)H_r'(z)>0$ always holds for some $z\in(-\infty,+\infty)$. With Theorem 2, it is easy to see that $H(s)$ in (9) is stable if and only if Conditions (i) and (ii) hold.

IV. PID Control Gains Ensuring The Stability Of The Complex Quasipolynomial

By taking

$$L(s)=sD(s)W_d(s)$$

and

$$M(s)=N(s)\left[W_d(s)+e^{j\varphi}W_n(s)/\gamma\right],$$

the quasipolynomial (7) is transformed into

$$\upsilon(s,k_p,k_i,k_d,\varphi)=L(s)e^{\theta s}+(k_ds^2+k_ps+k_i)M(s) \tag{10}$$

It is seen that $L(s)$ is a real polynomial and $M(s)$ is a complex polynomial. Then $L(s)$ and $M(s)$ can be written as

$$L(s)=s^e+c_{e-1}s^{e-1}+\cdots+c_1s+c_0$$

$$M(s)=(a_f+jb_f)s^f+(a_{f-1}+jb_{f-1})s^{f-1}+\cdots+(a_0+jb_0)$$

where $a_0,a_1,\cdots,a_f,b_0,b_1,\cdots,b_f$ and $c_0,c_1,\cdots,c_{e-1}$ are all real, $a_f+jb_f\ne 0$ and $e>f+2$. Let $s=jz/\theta$. We have

$$L(jz/\theta)=L_r(z)+jL_i(z)$$

$$M(jz/\theta)=M_r(z)+jM_i(z)$$

It is observed from (10) that both the real and imaginary parts of $\upsilon(jz/\theta,k_p,k_i,k_d,\varphi)$ depend on all the three gains, k_p, k_i and k_d, and this causes difficulty when using Theorem 3 to test the stability of $\upsilon(s,k_p,k_i,k_d,\varphi)$. To overcome this problem, we construct a new quasipolynomial in which the imaginary part depends only on k_p and the real part depends only on k_i and k_d. Multiplying two sides of (10) by $M(-s)$, we have

$$\upsilon(jz/\theta,k_p,k_i,k_d,\varphi)M(-jz/\theta)=p(z,k_i,k_d)+jq(z,k_p) \tag{11}$$

where

$$p(z,k_i,k_d)=p_1(z)+(k_i-k_dz^2/\theta^2)[M_r^2(z)+M_i^2(z)] \tag{12}$$

$$q(z,k_p)=q_1(z)+zk_p[M_r^2(z)+M_i^2(z)]/\theta \tag{13}$$

Here

$$\begin{aligned}p_1(z)=&[L_r(z)M_r(z)+L_i(z)M_i(z)]\cos(z)\\&-[L_i(z)M_r(z)-L_r(z)M_i(z)]\sin(z)\end{aligned} \tag{14}$$

$$\begin{aligned}q_1(z)=&[L_i(z)M_r(z)-L_r(z)M_i(z)]\cos(z)\\&+[L_r(z)M_r(z)+L_i(z)M_i(z)]\sin(z)\end{aligned} \tag{15}$$

In order to derive the values of the PID gains for which $\upsilon(s,k_p,k_i,k_d,\varphi)$ is stable, some definitions are first given:

Definition 1 Let $\underline{Z}=-2l^*\pi-\xi$ and $\overline{Z}=2l^*\pi-\xi$, where

$$\xi=\begin{cases}-\pi/2 & \text{if } e \text{ even and } a_fb_f=0\\ 0 & \text{if } e \text{ odd and } a_fb_f=0\\ \arctan(-b_f/a_f) \text{ or } \pi+\arctan(-b_f/a_f) & \text{if } e+f \text{ even}\\ \arctan(a_f/b_f) \text{ or } \pi+\arctan(a_f/b_f) & \text{if } e+f \text{ odd}\end{cases}$$

For a given value of k_p, let $z_1<z_2<\cdots<z_{c-1}$ be the real and distinct zeros of $q(z,k_p)$ in (13) in the interval $(\underline{Z},\overline{Z})$, and assume $z_0=\underline{Z}$ and $z_c=\overline{Z}$. Denote $\varsigma_f=a_f+jb_f$ as the leading coefficient of $M(s)$ and define i_t as follows:

$$i_t=\operatorname{sgn}\left[p(z_t,k_i,k_d)\right]=\begin{cases}0 & \text{if } M(-jz_t/\theta)=0\\ -1 \text{ or } 1 & \text{if } M(-jz_t/\theta)\ne 0\end{cases}$$

where $t=0,1,2,\cdots,c$.

Definition 2 Let $I=\{i_0,i_1,\cdots i_c\}$ or $\{i_1,i_2,\cdots i_{c-1}\}$. Then, the signature $\sigma(I)$ is denoted by

$$\sigma(I)=\begin{cases}\frac{1}{2}\left\{i_0+2\sum_{t=1}^{c-1} i_t\cdot(-1)^t+(-1)^c i_c\right\}\cdot(-1)^{c-1}\operatorname{sgn}\left[q(z_{c-1}^+)\right] \\ \quad \text{if } f \text{ is odd and } \varsigma_f \text{ is not purely imaginary} \\ \quad \text{or } f \text{ is even and } \varsigma_f \text{ is not purely real} \\ \frac{1}{2}\left\{2\sum_{t=1}^{c-1} i_t\cdot(-1)^t\right\}\cdot(-1)^{c-1}\operatorname{sgn}\left[q(z_{c-1}^+)\right] \\ \quad \text{if } f \text{ is odd and } \varsigma_f \text{ is purely imaginary} \\ \quad \text{or } f \text{ is even and } \varsigma_f \text{ is purely real}\end{cases}$$

Theorem 4 Let $l(M)$ and $r(M)$ denote the numbers of left half-plane and right half-plane zeros of $M(s)$, respectively. For a fixed k_p, if there exists one string I satisfying

$$\sigma(I)=(4l^*+e)-[l(M)-r(M)], \tag{16}$$

the set of (k_d,k_i) ensuring the stability of $\upsilon(s,k_p,k_i,k_d,\varphi)$ is the intersection of the following inequalities:

$$\left[k_i-A(z_t)k_d+B(z_t)\right]i_t>0 \quad \text{for } \forall i_t\in I \text{ and } i_t\neq 0 \ . \tag{17}$$

Here, $A(z_t)=z_t^2/\theta^2$ and $B(z_t)=p_1(z_t)/[M_r^2(z_t)+M_i^2(z_t)]$. If the strings $I_1,I_2,\cdots I_h$ all satisfy (16), then the set of (k_d,k_i) is the union of the regions of (k_d,k_i) satisfying (17) for $I_1,I_2,\cdots I_h$.

Proof: We first present the condition for which $\upsilon(jz/\theta,k_p,k_i,k_d,\varphi)M(-jz/\theta)$ must guarantee that $\upsilon(s,k_p,k_i,k_d,\varphi)$ is stable. According to Theorem 3, $\upsilon(s,k_p,k_i,k_d,\varphi)$ is stable if and only if $\upsilon_i(z)$ has $4l^*+e$ real zeros in $(\underline{Z},\overline{Z})$ and the zeros of $\upsilon_i(z)$ interlace with those of its real part $\upsilon_r(z)$. The foregoing necessary and sufficient condition for the stability of $\upsilon(s,k_p,k_i,k_d,\varphi)$ is equivalent to the condition that the net phase angle of $\upsilon(jz/\theta,k_p,k_i,k_d,\varphi)$ for z changing from z_1 to z_{c-1} satisfies

$$\Delta_{z_1}^{z_{c-1}}\phi=\pi(4l^*+e-1), \tag{18}$$

The net phase angles of $\upsilon(jz/\theta,k_p,k_i,k_d,\varphi)$ for z changing from $\underline{Z}$ to z_1 and for z changing from z_{c-1} to $\overline{Z}$ are, respectively, given to be

$$\Delta_{\underline{Z}}^{z_1}\phi=\begin{cases}\pi-\tan^{-1}\left|\upsilon_i(\underline{Z})/\upsilon_r(\underline{Z})\right| & \text{if } \upsilon_r(z_1)\upsilon_r(\underline{Z})<0\\ \tan^{-1}\left|\upsilon_i(\underline{Z})/\upsilon_r(\underline{Z})\right| & \text{if } \upsilon_r(z_1)\upsilon_r(\underline{Z})>0\end{cases} \tag{19}$$

$$\Delta_{z_{c-1}}^{\overline{Z}}\phi=\begin{cases}\pi-\tan^{-1}\left|\upsilon_i(\overline{Z})/\upsilon_r(\overline{Z})\right| & \text{if } \upsilon_r(z_{c-1})\upsilon_r(\overline{Z})<0\\ \tan^{-1}\left|\upsilon_i(\overline{Z})/\upsilon_r(\overline{Z})\right| & \text{if } \upsilon_r(z_{c-1})\upsilon_r(\overline{Z})>0\end{cases} \tag{20}$$

When $z\to\infty$ and $e>f+2$, it can be readily be obtained that

$$\upsilon_r(z)=\sqrt{L_r^2(z)+L_i^2(z)}\sin[z+\varphi(z)]$$
$$\upsilon_i(z)=\sqrt{L_r^2(z)+L_i^2(z)}\cos[z+\varphi(z)]$$

where

$$\varphi(z)\to\begin{cases}0 \text{ or } \pi & \text{for } e \text{ even}\\ \pi/2 \text{ or -}\pi/2 & \text{for } e \text{ odd}\end{cases}.$$

In each case for $\varphi(z)=0,\pi,\pi/2$ and $-\pi/2$, the inequality $\nu_r(z_1)\nu_r(z_{c-1})<0$ always holds due to the periodicity property of $\nu_r(z)$ and $\nu_i(z)$ when $z\to\infty$. Furthermore,

$$\nu_r(\underline{Z})\nu_r(\overline{Z})>0 \text{ and } \tan^{-1}\left|\nu_i(\underline{Z})/\nu_r(\underline{Z})\right|=\tan^{-1}\left|\nu_i(\overline{Z})/\nu_r(\overline{Z})\right| \tag{21}$$

From (19)-(21) and $\nu_r(z_1)\nu_r(z_{c-1})<0$, it is easy to obtain that if $e>f+2$,

$$\Delta_{\underline{Z}}^{z_1}\phi+\Delta_{z_{c-1}}^{\overline{Z}}\phi=\pi+O_1 \tag{22}$$

since l^* is a sufficiently large integer, where O_1 represents a very small approximation error. The value of $|O_1|$ is sufficiently small if l^* is a sufficiently large value. Then, from (18) and (22), it is seen that the necessary and sufficient condition for the stability of $\upsilon(s,k_p,k_i,k_d,\varphi)$ is equivalent to the condition that the net phase angle $\Delta_{\underline{Z}}^{\overline{Z}}\vartheta$ of $\upsilon(jz/\theta,k_p,k_i,k_d,\varphi)M(-jz/\theta)$ satisfies

$$\Delta_{\underline{Z}}^{\overline{Z}}\vartheta=(4l^*+e)\pi-\pi[l(M)-r(M)]+O_1 \tag{23}$$

Next, determine whether $\underline{Z}$ and $\overline{Z}$ are the approximate zeros of $q(z,k_p)$ in (13) and $p(z,k_i,k_d)$ in (12), and then establish the feasible string set by using the signature $\sigma(I)$ given in Definition 2 according to the net change condition in (23). $q(z,k_p)$ can be written as

$$\begin{aligned}q(z,k_p)=&\sqrt{[L_r^2(z)+L_i^2(z)][M_r^2(z)+M_i^2(z)]}\sin[z+\psi(z)]\\&+zk_p[M_r^2(z)+M_i^2(z)]/\theta\end{aligned} \tag{24}$$

where

$$\psi(z)=\begin{cases}\arctan\left(\dfrac{L_i(z)M_r(z)-L_r(z)M_i(z)}{L_r(z)M_r(z)+L_i(z)M_i(z)}\right)\\ \quad \text{if } L_r(z)M_r(z)+L_i(z)M_i(z)>0\\ \pi+\arctan\left(\dfrac{L_i(z)M_r(z)-L_r(z)M_i(z)}{L_r(z)M_r(z)+L_i(z)M_i(z)}\right)\\ \quad \text{if } L_r(z)M_r(z)+L_i(z)M_i(z)<0\end{cases} \tag{25}$$

Taking $q(z,k_p)=0$ yields

$$\sin[\psi(z)+z]=-\frac{zk_p[M_R^2(z)+M_I^2(z)]}{\theta\sqrt{[L_R^2(z)+L_I^2(z)][M_R^2(z)+M_I^2(z)]}} \tag{26}$$

It is observed from (25) that, as $z\to+\infty$,

$$\psi(z)\to\begin{cases}\arctan(-\frac{b_f}{a_f}) \text{ or } \pi+\arctan(-\frac{b_f}{a_f}) & \text{if } e\text{+}f \text{ is even}\\ \arctan(\frac{a_f}{b_f}) \text{ or } \pi+\arctan(\frac{a_f}{b_f}) & \text{if } e\text{+}f \text{ is odd}\end{cases} \tag{27}$$

and

$$\frac{zk_p[M_R^2(z)+M_I^2(z)]}{\theta\sqrt{[L_R^2(z)+L_I^2(z)][M_R^2(z)+M_I^2(z)]}}\to 0 \tag{28}$$

Since $\psi(z)$ depends on the values of a_f and b_f, the following cases will be discussed in order to determine whether there exist zeros of $q(z,k_p)$ and $p(z,k_i,k_d)$ close to $\underline{Z}$ and $\overline{Z}$.

1) $a_f\neq 0$ and $b_f=0$

When $a_f\neq 0$ and $b_f=0$ (i.e. the leading coefficient of $M(s)$ is purely real), from (27) we have

$$\psi(z)\to\begin{cases}0 \text{ or } \pi & \text{if } e\text{+}f \text{ is even}\\ \pi/2 \text{ or} -\pi/2 & \text{if } e\text{+}f \text{ is odd}\end{cases} \tag{29}$$

This implies that, as $z\to\infty$, the real zeros of $q(z,k_p)$ tend to those of $\cos(z)=0$ for e+f odd or those of $\sin(z)=0$ for $e+f$ even. Similarly, as $z\to\infty$, the real zeros of $p(z,k_i,k_d)$ in (12) tend to those of $\cos(z)=0$ for e+f even or those of $\sin(z)=0$ for $e+f$ odd. Moreover, it is known from Definition 1 that $\underline{Z}=-2l^*\pi$ and $\bar{Z}=2l^*\pi$ for e odd and $\underline{Z}=-2l^*\pi+\pi/2$ and $\bar{Z}=2l^*\pi+\pi/2$ for e even. Thus, as $z\to\infty$, $\underline{Z}$ and $\bar{Z}$ can be regarded as the zeros of $q(z,k_p)$ for f odd, while for f even, $\underline{Z}$ and $\bar{Z}$ can be regarded as the zeros of $p(z,k_i,k_d)$

2) $b_f\neq 0$ and $a_f=0$

If $b_f\neq 0$ and $a_f=0$, the leading coefficient of $M(s)$ is purely imaginary. Following the similar lines as that in Case 1), it can be obtained that as $z\to\infty$, $\underline{Z}$ and $\bar{Z}$ are the approximate zeros of $q(z,k_p)$ for f even, and otherwise, they are the approximate zeros of $p(z,k_i,k_d)$.

3) $a_f\neq 0$ and $b_f\neq 0$

If $a_f\neq 0$ and $b_f\neq 0$, the leading coefficient of $M(s)$ is complex. From (26)-(28), it is seen that the zeros of $q(z,k_p)$ tend to those of $\sin[\psi(z)+z]=0$ as $z\to\infty$. $\underline{Z}=-2l^*\pi-\xi$ and $\bar{Z}=2l^*\pi-\xi$ are presented in Definition 1 in the case $a_f\neq 0$ and $b_f\neq 0$, where

$$\xi=\begin{cases}\arctan(-\dfrac{b_f}{a_f})\text{ or }\pi+\arctan(-\dfrac{b_f}{a_f}) & \text{if } e+f \text{ even}\\ \arctan(\dfrac{a_f}{b_f})\text{ or }\pi+\arctan(\dfrac{a_f}{b_f}) & \text{if } e+f \text{ odd}\end{cases}\quad(30)$$

Thus, $\underline{Z}$ and $\bar{Z}$ can be regarded as the zeros of $q(z,k_p)$.

If $\underline{Z}$ and $\bar{Z}$ are approximate zeros of $q(z,k_p)$, then from the result in [17], it can be derived that the net phase angle $\Delta_{\underline{Z}}^{\bar{Z}}\vartheta$ of $\upsilon(jz/\theta,k_p,k_i,k_d,\varphi)M(-jz/\theta)$ is

$$\begin{aligned}\Delta_{\underline{Z}}^{\bar{Z}}\vartheta=\sum_{t=0}^{c}\Delta_{z_{t-1}}^{z_t}\vartheta=\frac{\pi}{2}\Big\{&\operatorname{sgn}\left[p(z_0)\right]+2\sum_{t=1}^{c-1}\operatorname{sgn}\left[p(z_t)(-1)^t\right]\\&+(-1)^c\operatorname{sgn}[p(z_c)]\Big\}\cdot(-1)^{c-1}\operatorname{sgn}\left[q(z_{c-1}^+)\right]+O_2\end{aligned}\quad(31)$$

If $\underline{Z}$ and $\bar{Z}$ are approximate zeros of $p(z,k_i,k_d)$, $\Delta_{\underline{Z}}^{\bar{Z}}\vartheta$ is

$$\Delta_{\underline{Z}}^{\bar{Z}}\vartheta=\frac{\pi}{2}\left\{2\sum_{t=1}^{c-1}\operatorname{sgn}\left[p(z_t)(-1)^t\right]\right\}\cdot(-1)^{c-1}\operatorname{sgn}\left[q(z_{c-1}^+)\right]+O_2\quad(32)$$

Combining (23), (31) and (32), taking $\operatorname{sgn}[p(z_t,K)]=i_t$ for $t=0,1,2,\cdots,c$, and using Definition 2, we have

$$4l^*+e-\left[l(M)-r(M)\right]-\sigma(I)=\left(O_2-O_1\right)/\pi\quad(33)$$

A sufficiently large value of l^* can always be found to make $|O_2-O_1|<\pi$. Since $4l^*+e-\left[l(M)-r(M)\right]$ and $\sigma(I)$ are both integers, Equation (16) can be derived. This means that $\upsilon(s,k_p,k_i,k_d,\varphi)$ is stable if and only if (16) holds.

Finally, for a fixed k_p, we determine the set of (k_d,k_i) for which $\upsilon(jz/\theta,k_p,k_i,k_d,\varphi)$ is stable. Each feasible string I can be found by using (16). As in the proof of the main results in [14], if $M(jz_t/\theta)=0$, the corresponding i_t belonging to the feasible string I is pre-determined and is independent of k_i and k_d. The definition of I given in Definition 1 covers such special case. Now consider the case $M(jz_t/\theta)\neq 0$. In this case, $i_t=\operatorname{sgn}[p(z_t,k_i,k_d)]$, which is actually equivalent to $p(z_t,k_i,k_d)i_t>0$. Combining (12) and $p(z_t,k_i,k_d)i_t>0$, the inequality (17) can be obtained. Thus, the stabilizing set of (k_d,k_i) is the union of stabilizing regions of (k_d,k_i) satisfying (17) for all feasible strings $I_1,I_2,\cdots I_h$.

V. Algorithm for Synthesis of H_∞ PID Controller

Based on the results given in Section II and [14], and Theorem 4, the entire set of the PID control parameters that guarantee $\|W(s)T(s,k_p,k_i,k_d)\|_\infty<\gamma$ can be derived. The detailed algorithm is presented as follows:

Step 1: Determine the allowable range of k_p over which the partitioning needs to be carried out by using Theorem 6 in [14].

Step 2: Pick up a grid point of k_p^* in the resultant range.

Step 3: Determine the stabilizing set of (k_d,k_i) based on Theorem 7 in [14] and denote it as $S_{(1,k_p^*)}$.

Step 4: For a fixed value φ^*, set $L(s)=sD(s)W_d(s)$ and $M(s)=N(s)[W_d(s)+e^{j\varphi^*}W_n(s)/\gamma)$. By using Theorem 4 and solving a linear programming problem, present the set of (k_d,k_i) for which $\upsilon(s,k_p,k_i,k_d,\varphi^*)$ is stable and denote it as $S_{(2,k_p^*,\varphi^*)}$. By sweeping over $\varphi\in[0,2\pi)$, determine the set of (k_d,k_i) such that each $\upsilon(s,k_p,k_i,k_d,\varphi^*)$ is stable and let this set be defined as $S_{(2,k_p^*)}=\bigcap_{\varphi\in[0,2\pi)}S_{(2,k_p^*,\varphi^*)}$.

Step 5: Present the admissible set of (k_d,k_i) for which $|W(\infty)T(\infty,k_p^*,k_i,k_d)|<\gamma$ and define it as $S_{(3,k_p^*)}$

Step 6: Determine the entire set of (k_d,k_i) for which $\|W(s)T(s,k_p,k_i,k_d)\|_\infty<\gamma$ by finding the intersection of $S_{(1,k_p^*)}$, $S_{(2,k_p^*)}$ and $S_{(3,k_p^*)}$.

Step 7: Go to Step 2 with another grid point of k_p till all the grid points are considered.

In order to check the validity of the above-mentioned algorithm, the following example is given.

Example 1 Consider the time-delayed plant

$$G(s)=\frac{s+2}{s^3+5s^2+7s+3}e^{-0.5s}$$

The weight $W(s)$ is chosen as $W(s)=(s+0.1)/(s+1)$, which is the same as that in [10]. The problem is to determine the set of PID control parameters so that $\|W(s)T(s)\|_\infty<1$

The complementary sensitivity function is given as

$$T(s)=\frac{(k_ds^2+k_ps+k_i)(s+2)}{s(s^3+5s^2+7s+3)e^{0.5s}+(k_ds^2+k_ps+k_i)(s+2)}$$

and $\gamma=1$. From (6) and (7), we have

$$\delta(s,k_p,k_i,k_d)=s(s^3+5s^2+7s+3)e^{0.5s}+(k_ds^2+k_ps+k_i)(s+2)$$

$$\begin{aligned}\upsilon(s,k_p,k_i,k_d,\varphi)=&\,s(s^3+5s^2+7s+3)(s+1)e^{0.5s}\\&+(s+2)[s+1+e^{j\varphi}(s+0.1)]\end{aligned}$$

It is known that the PID gains satisfy $\|W(s)T(s)\|_\infty<1$ if and only if the following three conditions hold:

(1) $\delta(s,k_p,k_i,k_d)$ is stable;

(2) $\upsilon(s,k_p,k_i,k_d,\varphi)$ is stable for all $\varphi\in[0,2\pi)$;

(3) $|W(\infty)T(\infty)|=0<1$.

Since Condition (3) always holds, we only need consider Conditions (1) and (2). By using Theorem 6 in [14], it is obtained that the allowable range of k_p is from -1.364 to 3.782. Then, the stabilizing region of (k_d,k_i) is determined for a fixed $k_p\in(-1.364,3.782)$, for example, $k_p=1$. When $k_p=1$, the stabilizing set $S_{(1,1)}$ is derived based on Theorem 7 in [14]. By Setting $L(s)=s(s^3+5s^2+7s+3)(s+1)$ and $M(s)=(s+2)[s+1+e^{j\varphi}(s+0.1)]$ and sweeping over $\varphi\in[0,2\pi)$, the admissible set $S_{(2,1)}$ can be presented on the basis of Theorem 4. Thus, for $k_p = 1$, the set of (k_d, k_i) for which $\|W(s)T(s)\|_\infty < 1$ is the intersection of $S_{(1,1)}$ and $S_{(2,1)}$, which is sketched in Fig. 2. The values of $\|W(s)T(s)\|_\infty$ corresponding to different (k_d, k_i) values are presented in Table I, which shows that all the values of $\|W(s)T(s)\|_\infty$ are less than 1 for the (k_d, k_i) values inside the region in Fig. 2, while all of them are larger than 1 for those outside the region.By repeatedly using Steps 3, 4, 5 and 6 for each k_p value in $(-1.364, 3.782)$, we can obtain the set of (k_p, k_d, k_i) for which $\|W(s)T(s)\|_\infty < 1$. The admissible set of is shown (k_p, k_d, k_i) in Fig. 3.

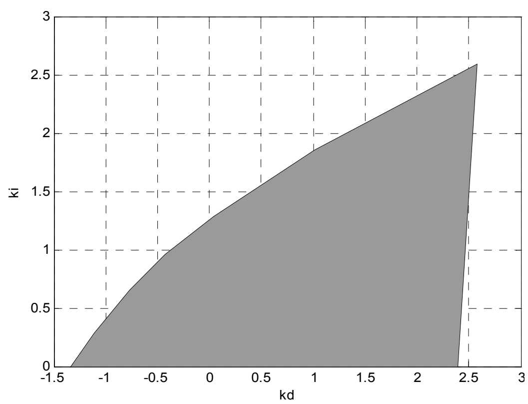

Fig. 2. The set of $(k_d,\ k_i)$ satisfying $\|W(s)T(s)\|_\infty < 1$ for $k_p = 1$

TABLE I
THE H_∞ NORM VALUES FOR DIFFERENT (k_d, k_i)

(k_d, k_i) values outside the set	$\|W(s)T(s)\|_\infty$	(k_d, k_i) values inside the set	$\|W(s)T(s)\|_\infty$
(0.5, 2)	1.687	(-0.5,0.5)	0.6101
(3, 1)	1.51	(1.5,1)	0.4642
(1, 2.5)	2.011	(0.5,0.6)	0.3229

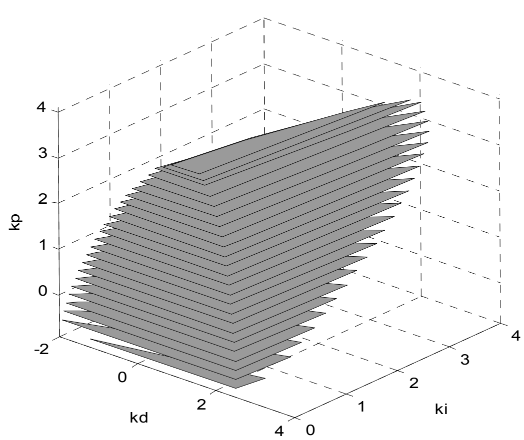

Fig. 3. The set of (k_p, k_i, k_d) for which $\|W(s)T(s)\|_\infty < 1$

VI. Conclusion

In this paper, we propose a parametric design method of the H_∞ PID controller for the general LTI plant with time delay. It is shown that the design problem to the H_∞ PID controller can be converted to simultaneous stabilization problem of the complex quasipolynomials and the characteristic equation, and thus, the characterization of the PID gain values for which the complex quasipolynomial is stable is presented in terms of the extended Hermite-Biehler Theorem. Based on the result, the algorithm to determine the set of the PID controller is provided both to meet H_∞ norm requirement and to ensure the stability of the system. The characterization for H_∞ PID controllers involves a procedure of solving a linear programming problem, which allows the simple and effective computation. The proposed method is a systematic method and applicable to an arbitrary-order system with time delay. Given the frequency occurrence of time delay and the widespread use of PID controllers in the industrial practice, it is expected that the results of this paper will contribute to the development of the practical control system design.

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