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Worst-case Analysis Based Adaptive Control Design for SISO Linear Systems with Plant and Actuation Uncertainties

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***Abstract*— We study the adaptive controller design for SISO linear systems subjected to plant and actuation uncertainties simultaneously. We first formulate the actuation and plant components of the linear system as two subsystems sequentially interconnected with additional feedback, and then convert the robust adaptive control problem as a nonlinear H^∞ control problem under imperfect state measurement. We derive the estimators and identifiers of the two subsystems using *cost-to-come analysis*, and then apply integrator backstepping methodology to obtain the control law. The controller guarantees the boundedness of closed-loop signals with bounded exogenous disturbances, and achieves desired disturbance attenuation level with respect to the unmeasured exogenous disturbance inputs and arbitrary positive or zero disturbance attenuation level with respect to the measured disturbance inputs. In addition, for the measured disturbances that the controller can achieve zero disturbance attenuation level, the asymptotic tracking objective is achieved even if they are only uniformly bounded without being of finite energy.**

***Index Terms*— Nonlinear H^∞ control; cost-to-come function analysis; measured disturbances; adaptive control.**

I. Introduction

Adaptive control attracted a lot of research attention in control theory since 1970s. The classic adaptive control design based on the certainty equivalence approach leads to structurally simple adaptive controllers[1] [2], and its effectiveness for linear systems with or without stochastic disturbance inputs has been demonstrated when long term asymptotic performance is considered [3]. However, early designs based on this approach were not robust to exogenous disturbance inputs and unmodeled dynamics[4]. Then, the stability and the performance of the closed-loop system becomes an important issue, which motivated the study of robust adaptive control in the 1980s and 1990s.

The robustness of closed-loop adaptive systems was studied intensely in late 1980s and early 1990s. Various adaptive controllers were modified to render the closed-loop systems robust [5]. Despite their successes, they still fell short of directly addressing the disturbance attenuation property of the closed-loop system.

Worst-case analysis based adaptive control design was motivated by the success of the game-theoretic approach to H^∞-optimal control problems [6] in late 1990s, which addresses the disturbance attenuation property directly. This design paradigm has been applied to worst-case parameter identification problems [7], which has led to new classes of parametrized identifiers for linear and nonlinear systems. It has also been applied to adaptive control problems [8], [9], [10], [11], and offered a promising tool to system subjected to uncertainties.

Most of the control applications today are implemented based on digital controllers. Driver is one critical component of control system as well as actuator, and usually is or can be approximated as a SISO linear system with zero relative degree. The driver performance can be impacted by uncertainties such as poor linearity properties, environmental and thermal issues. Moreover, the plant output will intendedly or unintendedly feedback to the driver in some applications, such as the back-emf voltage in the motor control applications. Nevertheless, the controller in practice are usually designed with the assumption that the control command can be applied on the plant directly by ignoring the actuation uncertainty induced from driver and actuator. To improve system performance, we need to take both plant and actuation uncertainties into account in the controller design. The above driver, actuator and plant components are essentially in a sequentially interconnected structure as Figure 1, and it is the real plant in the practical control system design.

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In this paper, we study the adaptive control design for linear systems under simultaneous driver, plant and actuation uncertainties. We view the linear system as two subsystems, actuation subsystem(which includes the driver and actuation blocks in Figure 1) and plant subsystem, sequentially interconnected with noisy output measurement and partially measured disturbance, and we assume that they satisfy the assumptions as [12] and [11], respectively. Under these assumptions, we can transform the above two subsystems into the models which are linear in all of the uncertainties. We then formulate the robust adaptive control problem as a nonlinear H^∞ control problem under imperfect state measurements, and apply the cost-to-come function analysis to derive the worst-case identifier and state estimator. The control design of the plant subsystem follows [12], and the adaptive controller can be obtained by the integrator backstepping methodology. The control design for the actuation subsystem can be completed in one step in view of the last backstepping design step for plant subsystem and the equivalent cost function. The robust adaptive controller achieves asymptotic tracking if the disturbances are bounded and of finite energy, and guarantees the stability of the closed-loop system with respect to the bounded disturbance inputs and the initial conditions. Furthermore, the closed-loop system admits a guaranteed disturbance attenuation level with respect to the exogenous disturbance inputs, where ultimate lower bound for the achievable attenuation performance level is only related to the noise intensity in the measurement channel of the plant subsystem, and zero or arbitrary positive distance attenuation level with respect to the measured disturbances. It further leads to a stronger asymptotic tracking property for the measured disturbances that the controller can achieve disturbance attenuation level zero with respect to, namely, the asymptotic tracking objective is achieved when the above measured disturbances are only bounded, without requiring it to be of finite energy.

The balance of the paper is organized as follows. In Section II, we list the notations used in the paper. In Section III, we present the formulation of the adaptive control problem and discuss the general solution methodology. In Section IV, we first obtain parameter identifier and state estimator using the *cost-to-come function* analysis in Subsection IV-A, then we derive the adaptive control law and present the main results on the robustness of the system in Subsection IV-B. The paper ends with some concluding remarks in Section V.

II. NOTATIONS

We denote $\mathbb{R}$ to be the real line; $\mathbb{R}_e$ to be the extended real line; $\mathbb{N}$ to be the set of natural numbers. For a function f, we say that it belongs to $\mathcal{C}$ if it is continuous; we say that it belongs to $\mathcal{C}_k$ if it is k-times continuously (partial) differentiable. For any matrix A, A' denotes its transpose. For any $b \in \mathbb{R}$, $\text{sgn}(b) = \begin{cases} -1 & b < 0 \\ 0 & b = 0 \\ 1 & b > 0 \end{cases}$. For any vector $z \in \mathbb{R}^n$, where $n \in \mathbb{N}$, $|z|$ denotes $(z'z)^{1/2}$. For any vector $z \in \mathbb{R}^n$, and any $n \times n$-dimensional symmetric matrix M, where $n \in \mathbb{N}$, $|z|_M^2 = z'Mz$. For any matrix M, the vector $\overrightarrow{M}$ is formed by stacking up its column vectors. For any symmetric matrix M, $\overleftarrow{M}$ denotes the vector formed by stacking up the column vector of the lower triangular part of M. For $n \times n$-dimensional symmetric matrices M_1 and M_2, where $n \in \mathbb{N}$, we write $M_1 > M_2$ if $M_1 - M_2$ is positive definite; we write $M_1 \geq M_2$ if $M_1 - M_2$ is positive semi-definite. For $n \in \mathbb{N}$, the set of $n \times n$-dimensional positive definite matrices is denoted by $\mathcal{S}_{+n}$. For $n \in \mathbb{N} \cup \{0\}$, I_n denotes the $n \times n$-dimensional identity matrix. For any matrix M, $\|M\|_p$ denotes its p-induced norm, $1 \leq p \leq \infty$. $\mathcal{L}_2$ denotes the set of square integrable functions and $\mathcal{L}_\infty$ denotes the set of bounded functions. For any n, $m \in \mathbb{N} \cup \{0\}$, $\mathbf{0}_{n \times m}$ denotes the $n \times m$-dimensional matrix whose elements are zeros. For any $n \in \mathbb{N}$ and $k \in \{1, \cdots, n\}$, $e_{n,k}$ denotes $\left[\begin{array}{ccc} 0_{1\times(k-1)} & 1 & 0_{1\times(n-k)} \end{array}\right]'$.

III. PROBLEM FORMULATION

We consider the robust adaptive control problem for the uncertainty system which is described by the block diagram in Figure 1. We call the plant as subsystem $\mathbf{S_1}$ and assume the system dynamics

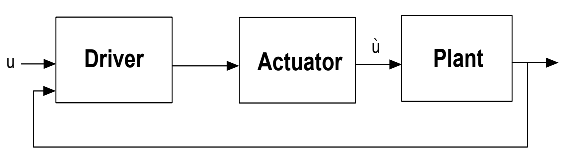

Fig. 1. Block diagram of uncertainty system.

for plant are as below,

$$\begin{aligned} \dot{x}_1 &= A_1 x_1 + B_1 \grave{u} + \check{D}_1 \check{w}_1 + D_1 w_1 \\ &\quad + (y_1 A_{1,21} + \grave{u} A_{1,22} + \sum_{i=1}^{\check{q}_1} \check{w}_{1,i} A_{1,23i})\theta_1; \quad (1a) \\ y_1 &= C_1 x_1 + E_1 w_1 \quad (1b) \end{aligned}$$

where x_1 is the n_1-dimensional state vectors with initial condition $x_1(0) = x_{1,0}$, $n_1 \in \mathbb{N}$; $\grave{u}$ is the scalar control input; y_1 is the scalar measurement output; w_1 is q_1-dimensional unmeasured disturbance input vector, $q_1 \in \mathbb{N}$; $\check{w}_1$ is $\check{q}_1$-dimensional measured disturbance input vector, $\check{q}_1 \in \mathbb{N}$; θ_1 is σ_1-dimensional unknown parameter vector, $\sigma_1 \in \mathbb{N}$; the matrices A_1, B_1, D_1, $\check{D}_1$, $A_{1,21}$, $A_{1,22}$, $A_{1,23}$, C_1, and E_1 are of appropriate dimensions and completely known.

To simplify the illustration, we combine the actuator and driver blocks in Figure 1 as subsystem $\mathbf{S_2}$, and we assume the system dynamics of $\mathbf{S_2}$ are given by,

$$\begin{aligned} \dot{x}_2 &= A_2 x_2 + B_2 u + A_{2,y} y_1 + \check{D}_2 \check{w}_2 + D_2 w_2 + (y_2 A_{2,21} \\ &\quad + u A_{2,22} + \sum_{i=1}^{\check{q}_2} \check{w}_{2,i} A_{2,23i} + y_1 A_{2,24})\theta_2; \quad (2a) \\ y_2 &= C_2 x_2 + (\bar{C}_{2,0}\theta_2 + b_{2,p0}) u + E_2 w_2 \quad (2b) \end{aligned}$$

where x_2 is the n_2-dimensional state vectors with initial condition $x_2(0) = x_{2,0}$, $n_2 \in \mathbb{N}$; u is the scalar control input; y_2 is the scalar measurement output and $y_2 = \grave{u}$; w_2 is q_2-dimensional unmeasured disturbance input vector, $q_2 \in \mathbb{N}$; $\check{w}_2$ is $\check{q}_2$-dimensional measured disturbance input vector, $\check{q}_2 \in \mathbb{N}$; θ_2 is σ_2-dimensional unknown parameter vector, $\sigma_2 \in \mathbb{N}$; the matrices A_2, B_2, $A_{2,y}$, D_2, $\check{D}_2$, $A_{2,21}$, $A_{2,22}$, $A_{2,23}$, $A_{2,24}$, C_2, $\bar{C}_{2,0}$, $b_{2,p0}$ and E_2 are of appropriate dimensions and completely known. In addition, the high frequency gain, $b_{2,0}$, of the transfer function from u to y_2 is equal to $b_{2,p0} + \bar{C}_{2,0}\theta_2$.

We assume that $\mathbf{S_1}$ satisfies the assumptions in [12], and $\mathbf{S_2}$ satisfies the assumptions in [11]. To make this paper more readable, we will summarize the assumption as follows,

Assumption 1: $\mathbf{S_1}$ and $\mathbf{S_2}$ are observable. The transfer function of $\mathbf{S_1}$ is known to have relative degree $r_1 \in \mathbb{N}$, and the transfer function of $\mathbf{S_2}$ is known to have relative degree zero. Moreover, both subsystems are strictly minimum phase. □

Based on the above assumptions, we have

$$A_1 = (a_{1,jk})_{n_1 \times n_1}; \begin{cases} a_{1,j\,(j+1)} = 1 & 1 \leq j \leq r_1 - 1 \\ a_{1,jk} = 0 & 1 \leq j \leq r_1 - 1, k > j + 1 \end{cases}$$

$$\bar{A}_{1,22} = \left[\begin{array}{ccc} \mathbf{0}_{\sigma_1 \times (r_1 - 1)} & \bar{A}'_{1,22\,0} & \bar{A}'_{1,22\,r_1} \end{array}\right]'; C_1 = \left[\begin{array}{cc} 1 & \mathbf{0}_{1 \times (n_1 - 1)} \end{array}\right]$$

$$B_1 = \left[\begin{array}{cccc} \mathbf{0}_{1 \times (r_1 - 1)} & b_{1,p0} & \cdots & b_{1,p(n_1 - r_1)} \end{array}\right]'$$

where $A_{1,22\,0}$ is a row vector, $b_{1,pj}$ $j = 0, 1, \cdots, n_1 - r_1$ are constants. Then the high frequency gain of the transfer function of $\mathbf{S_1}$ and $\mathbf{S_2}$ are $b_{1,0} = b_{1,p0} + \bar{A}_{1,22\,0}\theta_1$ and $b_{2,0} = b_{2,p0} + \bar{C}_{2,0}\theta_2$, respectively.

The subsystems may be uncontrollable, but the uncontrollable part satisfies the following assumption,

Assumption 2: The uncontrollable parts with respect to $\grave{u}$ of (1) and u of (2) are stable in the sense of Lyapunov. Any uncontrollable mode corresponding to an eigenvalue of the matrix A_1 and A_2 on the $j\omega$-axis is uncontrollable from w_1, $\check{w}_1$, and y_1, w_2, $\check{w}_2$, respectively. □

Since we consider the adaptive control design for systems with noisy output measurements, we have the following assumption,

Assumption 3: The matrices E_i are such that $E_i E_i' > 0$, for $i = 1, 2$. □

and we define $\zeta_i := 1/(E_i E_i')^{\frac{1}{2}}$ and $L_i := D_i E_i'$, for $i = 1, 2$.

To guarantee the stability of the closed-loop system and the boundedness of the estimate of θ_i, for $i = 1, 2$, we make the following assumption.

Assumption 4: The sign of $b_{i,0}$ is known, and without loss of generality, assume $b_{i,0} > 0$; there exists a known smooth nonnegative radially-unbounded strictly convex function $P_i : \mathbb{R}^{\sigma_i} \to \mathbb{R}$, such that the true value $\theta_i \in \Theta_i := \{\bar{\theta}_i \in \mathbb{R}^{\sigma_i} \mid P_i(\bar{\theta}_i) \leq 1\}$. □

Since we consider a trajectory tracking control design problem, we make the following assumption about the reference signal y_d.

Assumption 5: The reference trajectory, y_d, is r_1 times continuously differentiable. Define vector $Y_d := [y_d^{(0)}, \cdots, y_d^{(r_1)}]'$, where $y_d^{(0)} = y_d$, and $y_d^{(j)}$ is the jth order time derivative of y_d, $j = 1, \cdots, r_1$; define $Y_{d0} := [y_d^{(0)}(0), \cdots, y_d^{(r_1 - 1)}(0)]' \in \mathbb{R}^{r_1}$. The signal Y_d is available for feedback. □

Our objective is to derive a control law, which is generated by the following mapping,

$$u(t) = \mu(\omega_m) \quad (3)$$

where $\mu : \mathcal{C} \times \mathcal{C} \times \mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathbb{R}$, such that $x_{1,1}$ can asymptotically track the reference trajectory y_d, while rejecting the uncertainty

$(\omega_1, \omega_2) \in \mathcal{W}_1 \times \mathcal{W}_2$, and keeping the closed-loop signals bounded, where $\omega_m \in \mathcal{W}_m$ is the measurement signal of the system

$$\begin{aligned} \omega_m &:= (y_{1[0,\infty)}, y_{2[0,\infty)}, \breve{w}_{1[0,\infty)}, \breve{w}_{2[0,\infty)}, Y_{d0}, y_{d[0,\infty)}^{(r_1)}) \\ \mathcal{W}_m &:= \mathcal{C} \times \mathcal{C} \times \mathcal{C} \times \mathcal{C} \times \mathbb{R}^{r_1} \times \mathcal{C} \end{aligned}$$

$\omega_i \in \mathcal{W}_i$ is the uncertainty of subsystem $\mathbf{S_i}$ for $i = 1, 2$,

$$\begin{aligned} \omega_i &:= (x_{i,0}, \theta_i, w_{i[0,\infty)}, \breve{w}_{i[0,\infty)}, Y_{d0}, y_{d[0,\infty)}^{(r_1)}) \\ \mathcal{W}_i &:= \mathbb{R}^{n_i} \times \Theta_i \times \mathcal{C} \times \mathcal{C} \times \mathbb{R}^{r_1} \times \mathcal{C} \end{aligned}$$

The control law μ must also satisfy that, $\forall(\omega_1, \omega_2) \in \mathcal{W}_1 \times \mathcal{W}_2$, there exists a solution $x_{1[0,\infty)}$ and $x_{2[0,\infty)}$ to the system (1) and (2) , which yields a continuous control signal $u_{[0,\infty)}$. We denote the class of these admissible controllers by $\mathcal{M}_\mu$.

Next, we introduce the following assumption about the measured disturbance $\breve{w}_1$ before we define the disturbance attenuation level to capture the control objectives by optimizing a game-theoretic cost function.

Assumption 6: The measured disturbance $\breve{w}_1$ can be partitioned as: $\breve{w}_1 = \begin{bmatrix} \breve{w}_{1,a}' & \breve{w}_{1,b}' & \breve{w}_{1,c}' \end{bmatrix}'$, where $\breve{w}_{1,a}$ is $\breve{q}_{1,a}$ dimensional, $\breve{q}_{1,a} \in \mathbb{N} \cup \{0\}$, and the transfer function from each element of $\breve{w}_{1,a}$ to y_1 has relative degree strictly less than r_1; $\breve{w}_{1,b}$ is $\breve{q}_{1,b}$ dimensional, $\breve{q}_{1,b} \in \mathbb{N} \cup \{0\}$, and the transfer function from each element of $\breve{w}_{1,b}$ to y_1 has relative degree r_1. □

Definition 1: A controller $\mu \in \mathcal{M}_\mu$ is said to achieve *disturbance attenuation level* γ with respect to disturbance $\begin{bmatrix} w_1' & w_2' \end{bmatrix}'$, *arbitrary disturbance attenuation level* $\breve{\gamma}$ with respect to $\breve{w}_{1,a}$, and *disturbance attenuation level zero* with respect to disturbance $\begin{bmatrix} \breve{w}_{1,b}' & \breve{w}_{1,c}' & \breve{w}_2 \end{bmatrix}'$, if there exists functions $l_1(t, \theta_1, x_1, y_{1[0,t]}, Y_{d[0,t]})$, $l_2(t, \theta_2, x_2, y_{2[0,t]}, Y_{d[0,t]})$, and a known nonnegative constant l_0, such that

$$\sup_{w_1 \in \mathcal{W}_1, w_2 \in \mathcal{W}_2} J_{1,\gamma t_f} + J_{2,\gamma t_f} \leq 0; \quad \forall t_f \geq 0 \tag{4}$$

where

$$\begin{aligned} J_{1,\gamma t_f} &:= \int_0^{t_f} ((C_1 x_1 - y_d)^2 + l_1 - \breve{\gamma}^2 |\breve{w}_{1,a}|^2 - \gamma^2 |w_1|^2) \mathrm{d}\tau \\ &\quad -\gamma^2 \left| \begin{bmatrix} \theta_1' - \breve{\theta}_{1,0}' & x_{1,0}' - \breve{x}_{1,0}' \end{bmatrix}' \right|^2_{\bar{Q}_{1,0}} - l_0 \quad (5) \\ J_{2,\gamma t_f} &:= \int_0^{t_f} (l_2 - \gamma^2 |w_2|^2) \mathrm{d}\tau \\ &\quad -\gamma^2 \left| \begin{bmatrix} \theta_2' - \breve{\theta}_{2,0}' & x_{2,0}' - \breve{x}_{2,0}' \end{bmatrix}' \right|^2_{\bar{Q}_{2,0}} \quad (6) \end{aligned}$$

In the equation above, for $i = 1, 2$, $\bar{Q}_{i,0} > 0$ is a $(n_i + \sigma_i) \times (n_i + \sigma_i)$-dimensional weighting matrix, quantifying the level of confidence in the estimate $\begin{bmatrix} \breve{\theta}_{i,0}' & \breve{x}_{i,0}' \end{bmatrix}'$; $\bar{Q}_{i,0}^{-1}$ admits the structure $\begin{bmatrix} Q_{i,0}^{-1} & Q_{i,0}^{-1}\Phi_{i,0}' \\ \Phi_{i,0}Q_{i,0}^{-1} & \Pi_{i,0} + \Phi_{i,0}Q_{i,0}^{-1}\Phi_{i,0}' \end{bmatrix}$, $Q_{i,0}$ and $\Pi_{i,0}$ are $\sigma_i \times \sigma_i$- and $n_i \times n_i$-dimensional positive definite matrices, respectively.

Clearly, when the inequality (4) is achieved, the squared $\mathcal{L}_2$ norm of the output tracking error $C_1 x_1 - y_d$ is bounded by γ^2 times the squared $\mathcal{L}_2$ norm of the unmeasured disturbance input $\begin{bmatrix} w_1' & w_2' \end{bmatrix}'$, plus $\breve{\gamma}^2$ times the squared $\mathcal{L}_2$ norm of the measured disturbance input $\breve{w}_{1,a}$ and some constant. When the $\mathcal{L}_2$ norm of w_1, w_2, $\breve{w}_1$, and $\breve{w}_2$ are finite, the squared $\mathcal{L}_2$ norm of $C_1 x_1 - y_d$ is also finite, which implies $\lim_{t\to\infty}(C_1 x_1(t) - y_d(t)) = 0$, under additional assumptions.

The worst-case optimization of the cost function (4) can be carried out in two steps as depicted in the following equations.

$$\sup_{\omega_1 \in \mathcal{W}_1,\ \omega_2 \in \mathcal{W}_2} J_{\gamma t_f} \leq \sup_{\omega_m \in \mathcal{W}_m} \left(\sum_{i=1}^{2} \sup_{\omega_i \in \mathcal{W}_i | \omega_m \in \mathcal{W}_m} J_{i,\gamma t_f} \right) \tag{7}$$

The inner supremum operators will be carried out first. We maximize over ω_i given that the measurement ω_m is available for estimator design, $i = 1, 2$. In this step, the control input, u, is a function only depended on ω_m, then u is an open-loop time function and available for the optimization. Using *cost-to-come* function analysis, we derive the dynamics of the estimators for subsystem $\mathbf{S}_1$ and $\mathbf{S}_2$ independently.

The outer supremum operator will be carried out second. In this step, we use a backstepping procedure to design the controller μ.

This completes the formulation of the robust adaptive control problem.

IV. Adaptive Control Design

In this section, we present the adaptive control design, which involves estimation design and control design. First, we discuss estimation design.

A. Estimation Design

In this subsection, we present the estimation design for the adaptive control problem formulated.

To be able to apply *cost-to-come function* analysis to design a stabilizing controller, we first expand the system dynamics (1) and (2) by including θ_1 and θ_2 as part of the the expanded state vector $\xi_1 = \begin{bmatrix} \theta_1' & x_1' \end{bmatrix}'$ and $\xi_2 = \begin{bmatrix} \theta_2' & x_2' \end{bmatrix}'$. The expanded system dynamics are given as (8).

We skip the estimation design for $\mathbf{S_1}$ due to page limitation, and the derivation can be found in [12].

The estimation design of $\mathbf{S_2}$ generally follows [11], but the identifier dynamics are significantly different due to the feedback input y_1 and measured disturbance $\breve{w}_2$ in (2). In this step, the measurements waveform, y_1, $\breve{w}_2$ and Y_d are assumed to be known. We ignore terms considered to be constant in the estimation design step, and set l_2 in (6) to be $|\xi_2 - \hat{\xi}_2|^2_{\bar{Q}_2} + 2(\xi_2 - \breve{\xi}_2)' l_{2,2} + \breve{l}_2$. The equivalent cost function of (6) is then given by,

$$\begin{aligned} J_{2,\gamma t_f} &:= \int_0^{t_f} (|\xi_2 - \hat{\xi}_2|^2_{\bar{Q}_2} + 2(\xi_2 - \breve{\xi}_2)' l_{2,2} + \breve{l}_2 - \gamma^2 |w_2|^2) \mathrm{d}\tau \\ &\quad -\gamma^2 |\tilde{\xi}_{2,0}|^2_{\bar{Q}_{2,0}} \end{aligned} \tag{9}$$

where $\bar{Q}_2$ is a matrix-valued weighting function, $\hat{\xi}_2$ is the worst-case estimates for the expanded state ξ_2, $l_{2,2}$ is a design function, and $\breve{l}_2$ is considered to be constant in the estimation design step. The cost function of subsystem $\mathbf{S_2}$ is then of a linear quadratic structure, and the robust adaptive control problem for $\mathbf{S_2}$ becomes an H^∞ control of affine quadratic problem, which admits a finite dimensional solution.

We introduce the value function $W_2 = |\xi_2 - \breve{\xi}_2|^2_{\bar{\Sigma}_2^{-1}}$, and treat y_1 as the measured disturbance of $\mathbf{S_2}$, and we then can obtain the dynamics of state estimator $\breve{\xi}_2$ and worst-case covariance matrix $\bar{\Sigma}_2$ by the *cost-to-come function* methodology. However, it is difficult to analyze $\bar{\Sigma}_2$ directly, we thus partition $\bar{\Sigma}_2$, and define Φ_2 and Π_2 as shown below,

$$\begin{aligned} \bar{\Sigma}_2 &= \begin{bmatrix} \Sigma_2 & \bar{\Sigma}_{2,12} \\ \bar{\Sigma}_{2,21} & \bar{\Sigma}_{2,22} \end{bmatrix}; \\ \Pi_2 &:= \gamma^2 (\bar{\Sigma}_{2,22} - \bar{\Sigma}_{2,21} \Sigma_2^{-1} \bar{\Sigma}_{2,12}); \\ \Phi_2 &:= \bar{\Sigma}_{2,21} \Sigma_2^{-1}; \end{aligned}$$

$$\dot{\xi}_1 = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ y_1\bar{A}_{1,21} + \grave{u}\bar{A}_{1,22} + \sum_{j=1}^{\check{q}_1} \check{w}_{1,j}\bar{A}_{1,23\,j} & A_1 \end{bmatrix} \xi_1 + \begin{bmatrix} \mathbf{0} \\ B_1 \end{bmatrix} \grave{u} + \begin{bmatrix} \mathbf{0} \\ \check{D}_1 \end{bmatrix} \check{w}_1 + \begin{bmatrix} \mathbf{0} \\ D_1 \end{bmatrix} w_1$$

$$=: \bar{A}_1\xi_1 + \bar{B}_1 u + \bar{\check{D}}_1\check{w}_1 + \bar{D}_1 w_1 \quad (8a)$$

$$y_1 = \begin{bmatrix} \mathbf{0}_{1\times\sigma_1} & C_1 \end{bmatrix} \xi_1 + E_1 w_1 =: \bar{C}_1\xi_1 + E_1 w_1; \quad (8b)$$

$$\dot{\xi}_2 = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ y_2\bar{A}_{2,21} + u\bar{A}_{2,22} + \sum_{j=1}^{\check{q}_2} \check{w}_{2,j}\bar{A}_{2,23\,j} + y_1\bar{A}_{2,24} & A_2 \end{bmatrix} \xi_2 + \begin{bmatrix} \mathbf{0} \\ B_2 \end{bmatrix} u + \begin{bmatrix} \mathbf{0} \\ A_{2,y} \end{bmatrix} y_1 + \begin{bmatrix} \mathbf{0} \\ \check{D}_2 \end{bmatrix} \check{w}_2 + \begin{bmatrix} \mathbf{0} \\ D_2 \end{bmatrix} w_2$$

$$=: \bar{A}_2\xi_2 + \bar{B}_2 u + \bar{A}_{2,y} y_1 + \bar{\check{D}}_2\check{w}_2 + \bar{D}_2 w_2 \quad (8c)$$

$$y_2 = \begin{bmatrix} u\bar{C}_{2,0} & C_2 \end{bmatrix} \xi_2 + b_{2,p0} u + E_2 w_2 =: \bar{C}_2\xi_2 + b_{2,p0} u + E_2 w_2; \quad (8d)$$

then the weighting matrix $\bar{\Sigma}_2$ is positive definite if and only if Σ_2 and Π_2 are positive definite. To guarantee the boundedness of Σ_2, we choose the weighing matrix $\bar{Q}_2$ as follows,

$$\bar{Q}_2 = \begin{bmatrix} -\Phi_2' \\ I_{n_2} \end{bmatrix} \gamma^4 \Pi_2^{-1} \Delta_2 \Pi_2^{-1} \begin{bmatrix} -\Phi_2' \\ I_{n_2} \end{bmatrix}' + \begin{bmatrix} \epsilon_2(t)(\bar{C}_{2,0}u + C_2\Phi_2)'\gamma^2\zeta_2^2(\bar{C}_{2,0}u + C_2\Phi_2) & \mathbf{0}_{\sigma_2\times n_2} \\ \mathbf{0}_{n_2\times\sigma_2} & \mathbf{0}_{n_2\times n_2} \end{bmatrix}$$

where $\Delta_2(t)$ is an $n_2 \times n_2$- dimensional positive-definite matrix, and ϵ_2 is a scalar function, which can be defined by either of two possibilities,

$$\epsilon_2(t) := \mathrm{Tr}(\Sigma_2(t))^{-1}/K_{2,c} \quad \forall t \in [0, t_f] \quad (10a)$$

$$\text{or } \epsilon_2(t) := 1 \quad \forall t \in [0, t_f] \quad (10b)$$

where $K_{2,c} \geq \gamma^2 \mathrm{Tr}(Q_{22,0})$ is a design constant, $Q_{2,0}$ is an $\sigma_2\times\sigma_2$-dimensional positive-definite matrix. The dynamics of Σ_2, Φ_2, and Π_2 are summarized as follows,

$$\dot{\Sigma}_2 = (\epsilon_2 - 1)\Sigma_2(\bar{C}_{2,0}u + C_2\Phi_2)'\gamma^2\zeta_2^2(\bar{C}_{2,0}u + C_2\Phi_2)\Sigma_2; \quad \Sigma_2(0) = \gamma^{-2}Q_{2,0}^{-1} \quad (11a)$$

$$\dot{\Phi}_2 = (A_2 - \zeta_2^2 L_2 C_2 - \zeta_2^2 \Pi_2 C_2' C_2)\Phi_2 + y_2\bar{A}_{2,21} + u(\bar{A}_{2,22} - \zeta_2^2 L_2\bar{C}_{2,0} - \Pi_2\zeta^2 C_2'\bar{C}_{2,0}) + \sum_{j=1}^{\check{q}_2} \check{w}_{2,j}\bar{A}_{2,23\,j} + y_1\bar{A}_{2,24}; \quad \Phi_2(0) = \Phi_{2,0} \quad (11b)$$

$$\dot{\Pi}_2 = (A_2 - \zeta_2^2 L_2 C_2)\Pi_2 + \Pi_2(A_2 - \zeta_2^2 L_2 C_2)' - \zeta_2^2\Pi_2 C_2' C_2 \Pi_2 + D_2 D_2' - \zeta_2^2 L_2 L_2' + \gamma^2\Delta_2; \quad \Pi_2(0) = \Pi_{2,0} \quad (11c)$$

As described in [9], we have the covariance matrix Σ_2 upper and lower bounded as follows,

$$K_{2,c}^{-1} I_{\sigma_2} \leq \Sigma_2(t) \leq \Sigma_2(0) = \gamma^{-2}Q_{2,0}^{-1}$$
$$\gamma^2 \mathrm{Tr}(Q_{2,0}) \leq \mathrm{Tr}(\Sigma_2(t))^{-1} \leq K_{2,c}$$

whenever Σ_2 exists on $[0, t_f]$ and Φ_2 is continuous on $[0, t_f]$.

To avoid the inversion of Σ_2 online, we define $s_{2,\Sigma}(t) := \mathrm{Tr}((\Sigma_2(t))^{-1})$, and its dynamic is given by,

$$\dot{s}_{2,\Sigma} = \gamma^2\zeta_2^2(1 - \epsilon_2)(\bar{C}_{2,0}u + C_2\Phi_2)(\bar{C}_{2,0}u + C_2\Phi_2)'; \quad s_{2,\Sigma}(0) = \gamma^2 \mathrm{Tr}(Q_{2,0})$$

To guarantee the estimates parameter to be bounded and the estimate of high frequency gain to be bounded away from zero without persistently exciting signals, we introduce the following soft projection design on the parameter estimate.

We first define $\rho_2 := \inf\{P_2(\bar{\theta}_2) \mid \bar{\theta}_2 \in \mathbb{R}^{\sigma_2}, b_{2,p0} + \bar{C}_{2,0}\bar{\theta}_2 = 0\}$, and we have $1 < \rho_2 \leq \infty$ by Lemma 2 in [11]. We then fix any $\rho_{2,o} \in (1, \rho_2)$, and we define the open set $\Theta_{2,o} := \{\bar{\theta}_2 \mid P_2(\bar{\theta}) < \rho_{2,o}\}$. Our soft projection design will guarantee that the estimate $\check{\theta}_2$ lies in $\Theta_{2,o}$, which immediately implies $|b_{2,p0} + \bar{A}_{2,212\,0}\check{\theta}_2| > c_{2,0} > 0$, for some $c_{2,0} > 0$. Moreover, the convexity of P_2 implies the following inequality:

$$\frac{\partial P_2}{\partial \theta_2}(\check{\theta}_2)\,(\theta_2 - \check{\theta}_2) < 0 \quad \forall \check{\theta}_2 \in \mathbb{R}^{\sigma_2}\backslash\Theta_2$$

To incorporate the modifier to the estimates dynamics, we define

$$P_{2,r}(\check{\theta}_2) := \begin{cases} \frac{\exp\left(\frac{1}{1-P_2(\check{\theta}_2)}\right)}{\left(\rho_{2,o}-P_2(\check{\theta}_2)\right)^3} \left(\frac{\partial P_2}{\partial\theta_2}(\check{\theta}_2)\right)' & \forall\theta_2 \in \Theta_{2,o}\backslash\Theta_2 \\ 0_{\sigma_2\times 1} & \forall\theta_2 \in \Theta_2 \end{cases}$$

and introduce $l_{2,2} = [-(P_{2,r}(\check{\theta}_2)' \ \mathbf{0}_{1\times n_2}]'$. The dynamics of $\check{\xi}_2$ is then given as follows,

$$\dot{\check{\xi}}_2 = -\bar{\Sigma}_2 \begin{bmatrix} (P_{2,r}(\check{\theta}_2))' & \mathbf{0}_{1\times n_2} \end{bmatrix}' + \bar{A}_2\check{\xi}_2 + \bar{B}_2 u + \bar{A}_{2,y} y_1 + \bar{\check{D}}_2\check{w}_2 - \bar{\Sigma}_2\bar{Q}_2(\hat{\xi}_2 - \check{\xi}_2) + \zeta_2^2(\gamma^2\bar{\Sigma}_2\bar{C}_2' + \bar{L}_2) \cdot (y_2 - b_{2,p0}u - \bar{C}_2\check{\xi}_2); \quad \check{\xi}_2(0) = \begin{bmatrix} \check{\theta}_{2,0}' & \check{x}_{2,0}' \end{bmatrix}'$$

where $\bar{L}_2$ is defined as $\bar{L}_2 = [\mathbf{0}_{1\times\sigma_2} \ L_2']'$.

To analyze the stability of the close-loop system easily, we implement the dynamics of Φ_2 as the following pre-filtering systems for y_1, y_2, u and $\check{w}_2$.

$$\begin{aligned}
A_{2,f} &= A_2 - \zeta_2^2 L_2 C_2 - \zeta_2^2 \Pi_2 C_2' C_2 \\
\dot{\eta}_2 &= A_{2,f}\eta_2 + p_{2,n_2} y_2; \quad \eta_2(0) = \eta_{2,0} \\
\dot{\lambda}_2 &= A_{2,f}\lambda_2 + p_{2,n_2} u; \quad \lambda_2(0) = \lambda_{2,0} \\
\dot{\lambda}_{2,o} &= A_{2,f}\lambda_{2,o}; \quad \lambda_{2,o}(0) = p_{2,n_2} \\
\dot{\eta}_{2,\check{w},2} &= A_{2,f}\eta_{2,\check{w},1} + p_{2,n_2}\check{w}_{2,1}; \eta_{2,\check{w},1}(0) = \eta_{2,\check{w}20} \\
&\vdots \\
\dot{\eta}_{2,\check{w},\check{q}_2} &= A_{2,f}\eta_{2,\check{w},\check{q}_2} + p_{2,n_2}\check{w}_{2,\check{q}_2}; \eta_{2,\check{w},\check{q}_2}(0) = \eta_{2,\check{w},\check{q}_2 0} \\
\dot{\eta}_{2,y} &= A_{2,f}\eta_{2,y} + p_{2,n_2} y_1; \quad \eta_{2,y}(0) = \eta_{2y,0} \\
\Phi_2 &= \Phi_{2,u} + \Phi_{2,y} \\
\Phi_{2,y} &= \begin{bmatrix} A_{2,f}^{n_2-1}\eta_2 & \cdots & A_{2,f}\eta_2 & \eta_2 \end{bmatrix} M_{2,f}^{-1}\bar{A}_{2,21} \\
&= \begin{bmatrix} T_{2,1}'\eta_i & \cdots & T_{2,n_2}'\eta_2 \end{bmatrix}' \\
\dot{\Phi}_{2,u} &= A_{2,f}\Phi_{2,u} + (\bar{A}_{2,22} - \zeta_2^2 L_2\bar{C}_{2,0} - \Pi_2\zeta^2 C_2'\bar{C}_{2,0})u + \sum_{j=1}^{\check{q}_2} \bar{A}_{j,23\,j}\check{w}_{i,j} + \bar{A}_{2,24} y_1; \Phi_{2,u}(0) = \Phi_{2,u0}
\end{aligned}$$

where $M_{2,f} := \begin{bmatrix} A_{2,f}^{n_2-1} p_{2,n_2} & \cdots & A_{2,f} p_{2,n_2} & p_{2,n_2} \end{bmatrix}$; p_{2,n_2} is a n_2-dimensional vector such that the pair $(A_{2,f}, p_{2,n_2})$ is controllable.

This completes the estimation design of $\mathbf{S_2}$.

Associated with the estimation design of $\mathbf{S_1}$ in [12] and the above identifier and estimator of subsystem $\mathbf{S}_2$, we introduce the value function $W_i : \mathbb{R}^{n_i+\sigma_i} \times \mathbb{R}^{n_i+\sigma_i} \times \mathcal{S}_{+(n_i+\sigma_i)} \to \mathbb{R}$, $i = 1, 2$ as

$$W_i(\xi_i, \check{\xi}_i, \bar{\Sigma}_i) = |\theta_i - \check{\theta}_i|^2_{\Sigma_i^{-1}} + \gamma^2 |x_i - \check{x}_i - \Phi_i(\theta_i - \check{\theta}_i)|^2_{\Pi_i^{-1}}$$

whose time derivative is as follows

$$\begin{aligned}
\dot{W}_1 \;=\;& -|x_{1,1} - y_d|^2 - \gamma^4 |x_1 - \hat{x}_1 - \Phi_1(\theta_1 - \hat{\theta}_1)|^2_{\Pi_1^{-1}\Delta_1\Pi_1^{-1}} \\
& -\epsilon_1 \gamma^2 \zeta_1^2 |\theta_1 - \hat{\theta}_1|^2_{\Phi_1' C_1' C_1 \Phi_1} + |C_1 \check{x}_1 - y_d|^2 + |\xi_{1,c}|^2_{\bar{Q}_1} \\
& -\gamma^2 \zeta_1^2 |y_1 - C_1 \check{x}_1|^2 + \gamma^2 |w_1|^2 - \gamma^2 |w_1 - w_{1,*}|^2 \\
& +2(\theta_1 - \check{\theta}_1)' P_{1,r}(\check{\theta}_1) + \epsilon_1 |\theta_1 - \hat{\theta}_1|^2_{\Phi_1' C_1' C_1 \Phi_1} \qquad (12) \\
\dot{W}_2 \;=\;& -\gamma^4 |x_2 - \hat{x}_2 - \Phi_2(\theta_2 - \hat{\theta}_2)|^2_{\Pi_2^{-1}\Delta_2\Pi_2^{-1}} + |\xi_{2,c}|^2_{\bar{Q}_2} \\
& -\gamma^2 \zeta_2^2 |y_2 - b_{2,p0} u - C_2 \check{x}_2 - \bar{C}_{2,0} \check{\theta}_2 u|^2 + \gamma^2 |w_2|^2 \\
& -\epsilon_2 \gamma^2 \zeta_2^2 |\theta_2 - \hat{\theta}_2|^2_{\Phi_2' C_2' C_2 \Phi_2} - \gamma^2 |w_2 - w_{2,*}|^2 \\
& +2(\theta_2 - \check{\theta}_2)' P_{2,r}(\check{\theta}_2) \qquad (13)
\end{aligned}$$

where $\xi_{i,c} = \check{\xi}_i - \hat{\xi}_i$, and $w_{i,*}$ is the worst-case disturbance, given by $w_{i,*} : \mathbb{R} \times \mathbb{R}^{n_i+\sigma_i} \times \mathbb{R}^{n_i+\sigma_i} \times \mathcal{S}_{+(n_i+\sigma_i)} \longrightarrow \mathbb{R}$

$$\begin{aligned}
w_{1,*}(\xi_1, \check{\xi}_1, \bar{\Sigma}_1, w_1) =\;& \zeta_1^2 E_1'(y_1 - \bar{C}_1 \xi_1) + \gamma^{-2}(I_{q_1} - \zeta_1^2 E_1' E_1) \\
& \cdot \bar{D}_1' \bar{\Sigma}_1^{-1} (\xi_1 - \check{\xi}_1) \\
w_{2,*}(\xi_2, \check{\xi}_2, \bar{\Sigma}_2, w_2) =\;& \zeta_2^2 E_2'(y_2 - b_{2,p0} u - \bar{C}_2 \xi_2) \\
& +\gamma^{-2}(I_{q_2} - \zeta_2^2 E_2' E_2) \bar{D}_2' \bar{\Sigma}_2^{-1} (\xi_2 - \check{\xi}_2)
\end{aligned}$$

We note that (12) and (13) hold when $\Sigma_i > 0$ and $\theta_i \in \Theta_{i,0}$, and the last term in $\dot{W}_i$ is nonpositive, zero on the set Θ_i and approaches $-\infty$ as $\check{\theta}_i$ approaches the boundary of the set $\Theta_{i,o}$, which guarantees the boundedness of $\check{\theta}_i$, $i = 1, 2$. This completes the identification design step.

B. Control Design

In this section, we describe the controller design for the uncertain system under consideration. Note that, we ignored some terms in the cost function (5) in the identification step, since they are constant when y_1, y_2, $\check{w}_1$ and $\check{w}_2$ are given. In the control design step, we will include such terms. Then, based on the cost function (5), the controller design is to guarantee that the following supremum is less than or equal to zero for all measurement waveforms,

$$\begin{aligned}
& \sup_{w_1 \in \mathcal{W}_1, w_2 \in \mathcal{W}_2} J_{\gamma t_f} \\
& \leq \sup_{\omega_m \in \mathcal{W}_m} \Bigg\{ \int_0^{t_f} \Big(|C_1 \check{x}_1 - y_d|^2 + |\xi_{1,c}|^2_{\bar{Q}_1} + |\xi_{2,c}|^2_{\bar{Q}_2} \\
& \quad + \check{l}_1 + \check{l}_2 - \gamma^2 |\check{w}_{1,a}|^2 - \gamma^2 \zeta_1^2 |y_1 - C_1 \check{x}_1|^2 \\
& \quad - \gamma^2 \zeta_2^2 |y_2 - b_{2,p0} u - \bar{C}_{2,0} \check{\theta}_2 u - C_2 \check{x}_2|^2 \Big) \mathrm{d}\tau \Bigg\} \qquad (14)
\end{aligned}$$

where function $\check{l}_1(\tau, y_{1[0,\tau]}, Y_{d[0,\tau]}, \check{w}_1)$ is part of the weighting function $l_1(\tau, \theta_1, x_1, y_{1[0,\tau]}, Y_{d[0,\tau]}, \check{w}_1)$, and $\check{l}_2(\tau, y_{2[0,\tau]}, Y_{d[0,\tau]}, \check{w}_2)$ is part of the weighting function $l_2(\tau, \theta_2, x_2, y_{2[0,\tau]}, Y_{d[0,\tau]}, \check{w}_2)$ to be designed, which are constants in the identifier design step and are therefore neglected.

By equation (14), we observe that the cost function is expressed in term of the states of the estimator we derived, whose dynamics are driven by the measurement y_1, y_2, $\check{w}_1$, $\check{w}_2$, the reference trajectory y_d, the input u, and the worst-case estimate for the expanded state vector $\hat{\xi}_1$ and $\hat{\xi}_2$, which are signals we can either measure or construct. This is then a nonlinear H^∞-optimal control problem under full information measurements. We can equivalently deal with the following transformed variables instead of considering y_1, y_2, $\check{w}_1$ and $\check{w}_2$ as the maximizing variable,

$$v = \begin{bmatrix} \zeta_1 (y_1 - C_1 \check{x}_1) \\ \check{w}_{1,a} \\ \hline \zeta_2 (y_2 - b_{2,p0} u - \bar{C}_{2,0} \check{\theta}_2 - C_2 \check{x}_2) \end{bmatrix} = \begin{bmatrix} v_1 \\ \hline v_2 \end{bmatrix}$$

Then our control design objective is to achieve desired attenuation level γ with respect to variables v, and the variables to be designed at this stage are u, $\xi_{1,c}$, and $\xi_{2,c}$.

We observe that Σ_1, Π_1, $s_{1,\sigma}$ and $\check{\theta}_1$ of subsystem $\mathbf{S_1}$ are always bounded by the estimation design, and $\eta_{1,\check{w},1}, \cdots, \eta_{1,\check{w},\check{q}_1}$ and $\lambda_{1,o}$ are bounded since $A_{1,f}$ is Hurwitz. Then, we treat these variables as states of the stable zero dynamics in the control design procedure. We can not stabilize $\Phi_{1,u}$ in conjunction with $\check{x}_1$ in the control design. We will assume they are bounded and prove later they are indeed so under the derived control law. Since there is a nonnegative definite weighting on $\xi_{1,c}$ in the cost function (14), we can not use integrator backstepping to design feedback law for $\xi_{1,c}$ either. Hence, we set $\xi_{1,c} = 0$ in the backstepping procedure. After the completion of the backstepping procedure, we will then optimize the choice of $\xi_{1,c}$ based on the value function obtained.

Note that the structures of A_1 and $A_{1,f}$ in the dynamics are in strict-feedback form, we will use the backstepping methodology, see [13], to design the control input $\grave{u}$ of subsystem $\mathbf{S_1}$.

First, we will stabilize η_1 by introducing single $\eta_{1,d}$, which is of the following dynamics with initial condition $\eta_{1,d}(0) = \eta_{1,d0}$, $\dot{\eta}_{1,d} = A_{1,f} \eta_{1,d} + p_{1,n_1} y_d$, and is the reference trajectory for η_1 to track. Treating $\check{x}_{1,1}$ as the virtual control input, and choosing value function $V_{1,0} := |\eta_1 - \eta_{1,d}|^2_{Z_1}$, where Z_1 is the solution to the following algebraic Riccati equation,

$$A_{1,f}' Z_1 + Z_1 A_{1,f} + \frac{1}{\gamma^2 \zeta_1^2} Z_1 p_{1,n_1} p_{1,n_1}' Z_1 + Y_1 = 0$$

and Y_1 is a positive-definite matrix, we complete the step 0 with the virtual control law $\alpha_{1,0} = y_d$, which will guarantee the $\dot{V}_{1,0} \leq 0$ under $\check{x}_{1,1} = \alpha_{1,0}$.

At step 1, we introduce $z_{1,1} := \check{x}_{1,1} - y_d$, and choose value function $V_{1,1} = V_{1,0} + \frac{1}{2} z_{1,1}^2$. Treating $\check{x}_{1,2}$ as the virtual control input, we end the step 1 with the virtual control law $\alpha_{1,1}$, which guarantees $\dot{V}_{1,1} \leq 0$ under $\check{x}_{1,2} = \alpha_{1,1}$. Repeating the backstepping procedure until step r_1, the control input $\grave{u}$ will appear in the dynamic of $\dot{z}_{1,r_1}$. Using the similar procedure as previous steps, we can derive the robust adaptive controller α_{1,r_1} such that $\dot{V}_{1,r_1} \leq 0$ under $\grave{u} = \alpha_{1,r_1}$ to guarantee the dissipation inequality with supply rate,

$$-|x_{11} - y_d|^2 - |\eta_1 - \eta_{1,d}|^2_{Y_1} - \frac{1}{2} z_{11}^2 - \sum_{i=1}^{r_1} \beta_{1,i} z_{1i}^2 + \gamma^2 |v_1|^2$$

Please see [12] for the design detail.

In the design step of $\mathbf{S_2}$, we can equivalently deal with the transformed variable, v_2. In view of $\grave{u} = y_2$, a clear choice for control input u and the worst-case estimate $\xi_{2,c}$, which guarantees that the right-hand-side of (14) is nonpositive, is

$$\begin{aligned}
u \;:=\;& \bar{\mu}(\omega_m) = \frac{\alpha_{1,r_1} - \frac{\check{b}_{1,0}^2 z_{1,r_1}}{4\gamma^2 \zeta_2^2} - C_2 \check{x}_2}{\bar{C}_{2,0} \check{\theta}_2 + b_{2,p0}} \qquad (15) \\
\hat{\xi}_2 \;=\;& \check{\xi}_2 \qquad (16)
\end{aligned}$$

where $\check{b}_{1,0} = \begin{bmatrix} 0_{1\times(r_1-1)} & 1 & 0_{1\times(n_1-r_1)} \end{bmatrix} (B_1 + A_{1,22} \check{\theta}_1)$.

For the closed-loop adaptive nonlinear system, we have the following value function, $U = W_1 + W_2 + V_{1,r_1}$, and its time derivative is given by

$$\dot{U} \le -|x_{1,1} - y_d|^2 - \frac{1}{4}|\varsigma_{1,r_1}|^2_{\bar{Q}_1} + \left|\xi_{1,c} + \frac{1}{2}\varsigma_{1,r_1}\right|^2_{\bar{Q}_1}$$
$$+\sum_{i=1}^{2}\gamma^2\left(|w_i|^2 - |w_i - w_{i,*}|^2\right) + \gamma^2|\breve{w}_{1,a}|^2 - \gamma^2|\breve{w}_{1,a} - \breve{w}_{1,*}|^2$$

where ς_{1,r_1} is function obtained after step r_1, $\breve{w}_{1,*}$ is the worst case disturbance with respect to the value function U, and both of them are defined in [12] and [11].

Then the optimal choice for the variable $\xi_{1,c}$ is, $\xi_{1,c*} = -\frac{1}{2}\varsigma_{1,r_1}$, which yields that the closed-loop system is dissipative with storage function U and supply rate:

$$-|x_{1,1} - y_d|^2 + \gamma^2|w_1|^2 + \gamma^2|w_2|^2 + \gamma^2|\breve{w}_{1,a}|^2$$

The optimal choice of $\xi_{1,c*}$ is generally very complicated. We could simply choose $\xi_{1,c} = 0$, i.e., $\hat{\xi}_1 = \check{\xi}_1$. Since it will result in a simplified identifier structure, this suboptimal choice of $\hat{\xi}_1$ may be preferable over the optimal one. Then, we summarize the choice of $\hat{\xi}_1$ and $\hat{\xi}_2$ as follows,

$$\hat{\xi}_1 = \check{\xi}_1 - \frac{\epsilon_{1,c}}{2}\varsigma_{1,r_1}; \quad \epsilon_{1,c} = \in [0,1] \qquad (17)$$
$$\hat{\xi}_2 = \check{\xi}_2 \qquad (18)$$

Next, we state the strong robustness property of the closed-loop system in the following theorem.

Theorem 1: Consider the robust adaptive control problem formulated in Section III, the robust adaptive controller μ in (15) with the choise of $\xi_{1,c}$ as (17) and $\xi_{2,c}$ as (18), achieves the following strong robustness properties for the closed-loop system.

1) For uncertainties $\omega_1 \in \mathcal{W}_1$ and $\omega_2 \in \mathcal{W}_2$, the controller μ achieves disturbance attenuation level γ with respect to w_1 and w_2, arbitrary disturbance attenuation level $\tilde{\gamma}$ with respect to $\breve{w}_{1,a}$, and disturbance attenuation level zero with respect to $\breve{w}_{1,b}$, $\breve{w}_{1,c}$, and $\breve{w}_2$.
2) For subsystem $\mathbf{S_1}$ and $\mathbf{S_2}$, the controller μ guarantees the boundedness of all closed-loop state variables for any bounded uncertainty $\omega_1 \in \mathcal{W}_1$ and $\omega_2 \in \mathcal{W}_2$.
3) For uncertainties $\omega_1 \in \mathcal{W}_1$ and $\omega_2 \in \mathcal{W}_2$ with $w_{1,[0,\infty)} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, $w_{2,[0,\infty)} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, $\breve{w}_{1,a[0,\infty)} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, $\breve{w}_{1,b[0,\infty)} \in \mathcal{L}_\infty$, $\breve{w}_{1,c[0,\infty)} \in \mathcal{L}_\infty$, $\breve{w}_{2[0,\infty)} \in \mathcal{L}_\infty$ and $Y_{d[0,\infty)} \in \mathcal{L}_\infty$, the output of the subsystem $\mathbf{S_1}$, $x_{1,1}$, asymptotically tracks the reference trajectory, y_d, *i.e.*,
$$\lim_{t\to\infty}(x_{1,1}(t) - y_d(t)) = 0$$
4) The ultimate lower bound on the achievable performance level, γ, is only depended on the subsystem $\mathbf{S_1}$, i.e., $\gamma \ge (E_1E_1')^{\frac{1}{2}}$ or $\gamma > (E_1E_1')^{\frac{1}{2}}$.

Proof We can prove the theorem by a line of reasoning that is similar to that of [12] and [14]. The detailed proof is skipped due to page limitation.

V. Conclusions

In this paper, we present the worst case based adaptive control design for the linear system with plant and actuation uncertainties. We formulate the actuation and plant of the linear system as two subsystems sequentially interconnected with additional feedback, and we assume that the plant subsystem satisfies the same assumptions as [12] and actuation subsystem satisfies the assumptions as [11]. We formulate the robust adaptive control problem as a nonlinear H^∞ control problem under imperfect state measurements, and then apply *cost-to-come function* analysis to obtain the finite dimensional estimators of two subsystems independently due to the sequentially interconnected structure. The controller of plant subsystem can be obtained by utilizing the integrator backstepping methodology recursively, and the controller of actuation subsystem can be obtained directly in one step. The controller then guarantees the total stability of the closed-loop system with bounded exogenous disturbances and achieves asymptotic tracking of the reference trajectory when the disturbance is of finite energy and uniformly bounded. The controller also achieves a desired disturbance attenuation level with respect to the exogenous disturbance inputs, where ultimate lower bound for the achievable attenuation performance level is only related to the noise intensity in the measurement channel of the plant subsystem, and zero or arbitrary positive distance attenuation level with respect to the measured disturbances. It further leads to a stronger asymptotic tracking property for the measured disturbances that the controller can achieve disturbance attenuation level zero with respect to, namely, the asymptotic tracking objective is achieved when the above measured disturbances are only bounded, without requiring it to be of finite energy.

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