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Stabilization of Rigid Formations with Direction-Only Constraints

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Abstract— **Direction-based formation shape control for a collection of autonomous agents involves the design of distributed control laws that ensure the formation moves so that certain relative bearing constraints achieve, and maintain, some desired value. This paper looks at the design of a distributed control scheme to solve the direction-based formation shape control problem. A gradient control law is proposed based on the notion of bearing-only constrained graph rigidity and parallel drawings. This work provides an interesting and novel contrast to much of the existing work in formation control where distance-only constraints are typically maintained. A stability analysis is sketched and a number of illustrative examples are also given.**

I. Introduction

The general distributed formation control problem involves a group of agents which are tasked with maintaining a prescribed geometrical formation described in terms of relative distance and/or angular constraints. There are two common aspects of each formation control scheme that precede the controller design. Firstly, the sensing technology and sensing graph should be formed. The sensing technology describes what kind of measurements are taken and the sensing graph describes for each agent what aspects of what other agents in the formation are measurements taken. The sensing technology for formation control usually consists of either bearing measurements [1]–[3] or distance measurements. Typically both kinds of measurements are taken which amounts to a relative position measurement [4]–[13]. Secondly, albeit not independently, the control graph and the controlled parameters are defined [9]. It is typical for the topology of the control and sensing graph to be equivalent meaning that agents control some geometrical relationship to those agents concerning which some measurements are taken. However, it is typical that the control constraints be either distance or bearing-only constraints and not both, while the sensing technology often results in the relative position of certain, so-called neighbour, agents.

The control graph together with the particular controlled parameters determines what desired formation shapes/scales are feasible along with their uniqueness. Obviously, defining a complete distance constraint graph between a group of agents will suffice to define a unique formation shape. However, defining a certain (well-chosen) subset of these distance constraints can often (generically) define a unique formation shape; e.g. see the notion of graph rigidity as it applies to formation control in [4]–[6], [9], [10], [13]. Directed constraints can also be considered, where some agents are tasked at maintaining certain distances to other agents while the converse is not true; e.g. see [11], [14]. Relative angular constraints can also be considered [15], [16].

Given a sensing and control architecture, one then seeks to design the control laws that, actively, and in a distributed fashion, seek to control the desired parameters using the locally sensed information at each agent. There now exists a large literature on formation control, e.g. see the related work in [4]–[13], but the problem remains interesting due to the various problem formulations, the distributed nature of the problem itself and the existence of undesired equilibria in much of the existing systems [12], [13], [17].

In this work, the shape of a formation is controlled by actively, and in a distributed fashion, controlling a certain set of inter-agent bearings using relative position measurements. Specifically, our contribution is the design and analysis of a novel distributed controller for an arbitrary number of agents with direction-only constraints to a subset of neighbour agents. A stability and convergence analysis is undertaken in line with [10]. Since the agents only seek to control their relative bearings (to this subset of agents), the formation control problem considered here and the control law outlined differs from much of the existing work on formation control.

This work provides an appealing basis for further work on direction-only constrained formation control. It is also a stepping-stone to work on formation control with mixed bearing and distance constraints; e.g. one could look to combine this work with [10].

II. Rigidity Theory with Direction Constraints

Consider n agents indexed by $\mathcal{V} = \{1, 2, \dots, n\}$ and with positions $\mathbf{p}_i \in \mathbb{R}^2$. Agent i can measure the bearing and range (or relative position) to agent j iff $j \in \mathcal{N}_i \Leftrightarrow i \in \mathcal{N}_j$ where $\mathcal{N}_i$ is the set of neighbours of i. The sets $\mathcal{V} = \{1, 2, \dots, n\}$ and $\mathcal{N}_i$, $\forall i \in \mathcal{V}$ define a graph that represents the interactions, e.g. measurements, constraints etc, between the agents. Denote this graph by $\mathcal{G}(\mathcal{V}, \mathcal{E})$ where $\mathcal{E}$ is the set of m links (i, j) where (i, j) exists iff $j \in \mathcal{N}_i \Leftrightarrow i \in \mathcal{N}_j$.

Definition 1 (Formal Point Formation)**.** *A point formation $\mathcal{F}_p(\mathcal{G})$ is defined by a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ and a map $p : \mathcal{V} \to \mathbb{R}^2$ which takes agent i in $\mathcal{V}$ to its respective position $\mathbf{p}_i$ in $\mathbb{R}^2$.*

The bearing to agent $j \in \mathcal{N}_i$ at agent i is denoted by ϕ_{ij}. The set $\mathcal{E}$ defines the set of measurements taken by the agents in $\mathcal{V}$. The set of bearing measurements $\mathcal{B}$ is

$$\mathcal{B} = \{\phi_{ij}, \phi_{ji} \in [0, 2\pi) : (i, j) \in \mathcal{E}\} \quad (1)$$

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where $\phi_{ij} \equiv (\pi + \phi_{ji}) \bmod (2\pi)$ and $|\mathcal{B}| = 2m$.

Assumption 1 (Global Coordinate System)**.** *It is assumed that each agent i, $\forall i$, measures the bearing ϕ_{ij} to agent $j \in \mathcal{N}_i$ with respect to a global direction in $\mathbb{R}^2$.*

Define also a set of range measurements

$$\mathcal{D} = \{\|\mathbf{p}_i - \mathbf{p}_j\| \in \mathbb{R}^+ : (i,j) \in \mathcal{E}\} \tag{2}$$

where $|\mathcal{D}| = m$. Let $d_{ij} = \|\mathbf{p}_i - \mathbf{p}_j\| = d_{ji}$ denote the range.

Definition 2 (Equivalent Formations)**.** *Two formations $\mathcal{F}_q$ and $\mathcal{F}_p$ are said to be equivalent if their underlying graphs are identical and the set of bearing measurements $\mathcal{B}$ in one of the formations is equal to the set in the other.*

Consider two formations $\mathcal{F}_p$ and $\mathcal{F}_q$ defined by the same graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ and respective mappings $p : \mathcal{V} \to \mathbb{R}^2$ and $q : \mathcal{V} \to \mathbb{R}^2$. For each $(i,j) \in \mathcal{E}$ consider the constraint

$$(\mathbf{p}_i - \mathbf{p}_j)^{\perp} \cdot (\mathbf{q}_i - \mathbf{q}_j) = 0 \tag{3}$$

where the operator $(\cdot)^{\perp}$ rotates a plane vector by $\pi/2$ counterclockwise. Then it follows that $\mathcal{F}_p$ and $\mathcal{F}_q$ are parallel drawings [15], [16] of each other in the sense that for each $(i,j) \in \mathcal{E}$ the vectors $(\mathbf{p}_i - \mathbf{p}_j)$ and $(\mathbf{q}_i - \mathbf{q}_j)$ are parallel. The system of equations (3) for all $(i,j) \in \mathcal{E}$ is a system of $|\mathcal{E}| = m$ homogenous linear equations in the $\mathbf{q}_i$ and $\mathbf{q}_j$ when the $\mathbf{p}_i$ and $\mathbf{p}_j$ are treated as known parameters.

Definition 3 (Parallel Point Formations)**.** *Assume $\mathcal{F}_p$ is given and $\mathcal{F}_q$ is defined on the same underlying graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ as $\mathcal{F}_p$. Then $\mathcal{F}_q$ is said to be a parallel point formation with respect to $\mathcal{F}_p$ if and only if (3) is satisfied for all $(i,j) \in \mathcal{E}$. A parallel point formation $\mathcal{F}_q$ is trivial with respect to $\mathcal{F}_p$ if it is equivalent to $\mathcal{F}_p$ and if $\mathcal{F}_q$ can be obtained from $\mathcal{F}_p$ via a translation then a dilation[1] (or vice-versa) on $\mathbb{R}^2$. All other parallel point formations are non-trivial.*

Consider a formation trajectory defined by a time-varying $\mathbf{q}_i(t)$ for all $i \in \mathcal{V}$ such that $\mathcal{F}_{q(t)}$ is defined by the same $\mathcal{G}(\mathcal{V},\mathcal{E})$ and the time-varying map $q(t) : \mathcal{V} \to \mathbb{R}^2$. Then $\mathcal{F}_{q(t)}$ is a parallel point formation to $\mathcal{F}_p$ if

$$(\mathbf{p}_i - \mathbf{p}_j)^{\perp} \cdot (\mathbf{q}_i(t) - \mathbf{q}_j(t)) = 0, \quad (i,j) \in \mathcal{E}, \quad t \geq 0 \tag{4}$$

Conversely, a solution to the resulting linear system of equations defines a parallel point formation $\mathcal{F}_{q(t)}$. Differentiating (4) with respect to time we have

$$(\mathbf{p}_i - \mathbf{p}_j)^{\perp} \cdot (\dot{\mathbf{q}}_i(t) - \dot{\mathbf{q}}_j(t)) = 0, \quad (i,j) \in \mathcal{E}, \quad t \geq 0 \tag{5}$$

which can be written in matrix form as

$$\mathbf{R}(\mathbf{p})\dot{\mathbf{q}} = 0 \tag{6}$$

where $\mathbf{p} = [\mathbf{p}_1^\top \ \mathbf{p}_2^\top \ \dots \ \mathbf{p}_n^\top]^\top$ and similarly for $\mathbf{q}$. $\mathbf{R}(\mathbf{p}) \in \mathbb{R}^{m \times 2n}$ is called the rigidity matrix for formations with bearing constraints (or the bearing-constrained rigidity matrix) [15], [16].

[1]For a formation $\mathcal{F}_p$, a dilation changes the size but not the shape or orientation in $\mathbb{R}^2$ of the formation. That is, for each pair $\mathbf{p}_i, \mathbf{p}_j \in \mathbb{R}^2$, $i,j \in \mathcal{V}$ a dilation of the object $\mathcal{F}_p$ preserves the bearing ϕ_{ij} and thus ϕ_{ji} but scales all $d_{ij} = d_{ji}$ by the same positive constant.

Definition 4 (Parallel Rigid Formations)**.** *A point formation $\mathcal{F}_p$ is said to be a parallel rigid formation if all parallel point formations of $\mathcal{F}_p$ are trivial with respect to $\mathcal{F}_p$.*

A formation that is parallel rigid is one in which the bearing between agents i and j is uniquely defined regardless of whether or not $(i,j) \in \mathcal{E}$ and thus is a formation in which the shape and orientation, albeit not the scale, is uniquely defined in $\mathbb{R}^2$. The novelty of this characterization is that it allows one to consider only the graphical topology of the formation and by appropriately choosing the agent interactions, e.g. the links $(i,j) \in \mathcal{E}$, one can define a unique formation shape with a minimal number of bearing measurements. Later we design a formation shape control law where the desired shape is specified by certain bearing constraints, defining a rigid formation, and the agents attempt to steer their measured bearings to the desired bearing constraints and achieve the desired shape and orientation.

Example 1. *Consider four agents indexed by 1, 2, 3, and 4. An example of a rigid network is illustrated in Figure 1. Conditions for testing and confirming parallel rigidity are given subsequently.*

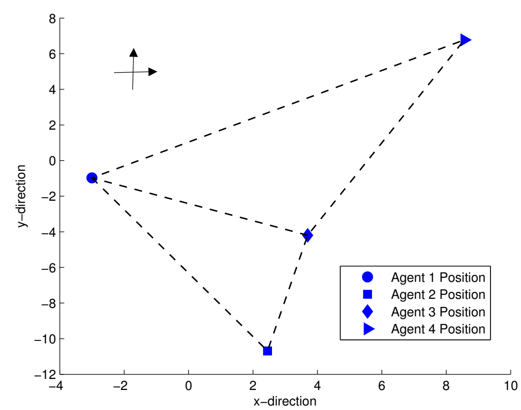


Fig. 1. A parallel rigid formation defined by the interaction graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ and a random embedding of the four agents on the plane. In this case $\mathcal{V} = \{1,2,3,4\}$ and $\mathcal{E} = \{(1,2),(1,3),(1,4),(2,3),(2,4)\}$.

The graph associated with the formation has 5 edges and there are thus a total of 10 bearing constraints (albeit only 5 independent constraints). The edges, arranged in lexicographical order, are $\{(1,2),(1,3),(1,4),(2,3),(2,4)\}$. The bearing-constrained rigidity matrix for the formation is given by (7). The bearing-constrained rigidity matrix is a 5×8 matrix in this example. The rows correspond to the edges in the graph associated with the formation and the columns correspond to the agents.

We highlight both a graph theoretical and linear algebra test for bearing-only network rigidity.

Theorem 1. *A formation $\mathcal{F}_p$ of n agents is parallel rigid if* $\operatorname{rank}(\mathbf{R}(\mathbf{p})) = 2n - 3$.

Refer to (6) and note the condition $\operatorname{rank}(\mathbf{R}(\mathbf{p})) = 2n - 3$ implies the kernel of $\mathbf{R}(\mathbf{p})$ is of dimension 3. It is easily

$$
\begin{array}{c}
\\
edge\ (1,2)\\
edge\ (1,3)\\
edge\ (1,4)\\
edge\ (2,3)\\
edge\ (2,4)
\end{array}
\begin{array}{c}
\begin{array}{cccc} agent\ 1 & agent\ 2 & agent\ 3 & agent\ 4 \end{array}\\
\left[\begin{array}{c|c|c|c}
((\mathbf{p}_1-\mathbf{p}_2)^{\perp})^{\top} & ((\mathbf{p}_2-\mathbf{p}_1)^{\perp})^{\top} & \mathbf{0} & \mathbf{0}\\
((\mathbf{p}_1-\mathbf{p}_3)^{\perp})^{\top} & \mathbf{0} & ((\mathbf{p}_3-\mathbf{p}_1)^{\perp})^{\top} & \mathbf{0}\\
((\mathbf{p}_1-\mathbf{p}_4)^{\perp})^{\top} & \mathbf{0} & \mathbf{0} & ((\mathbf{p}_4-\mathbf{p}_1)^{\perp})^{\top}\\
\mathbf{0} & ((\mathbf{p}_2-\mathbf{p}_3)^{\perp})^{\top} & ((\mathbf{p}_3-\mathbf{p}_2)^{\perp})^{\top} & \mathbf{0}\\
\mathbf{0} & ((\mathbf{p}_2-\mathbf{p}_4)^{\perp})^{\top} & \mathbf{0} & ((\mathbf{p}_4-\mathbf{p}_2)^{\perp})^{\top}
\end{array}\right]
\end{array}
= \mathbf{R}(\mathbf{p}) \tag{7}
$$

shown that this is the lowest dimension the kernel can take on and it corresponds to the fact that the trajectories of the formation at $\mathbf{q}$ in (6) are free up to translations (accounting for two linearly independent solutions $\dot{\mathbf{q}}$ to (6)) and dilations (accounting for the third solution linearly independent $\dot{\mathbf{q}}$ to (6)) even when $\mathcal{F}_q$ is parallel rigid.

The condition $\mathrm{rank}(\mathbf{R}(\mathbf{p})) = 2n-3$ is almost necessary in the sense that if $\mathcal{F}_p$ is parallel rigid and $\mathrm{rank}(\mathbf{R}(\mathbf{p})) < 2n-3$ then there exists a $\mathbf{p}'$ arbitrarily close to $\mathbf{p}$ such that $\mathrm{rank}(\mathbf{R}(\mathbf{p}')) = 2n-3$.

Definition 5 (Generic Formations)**.** *A formation is said to be in generic position $\mathbf{p}$ in $\mathbb{R}^{2n}$ if the set of its coordinates are not algebraically dependent; e.g. see [9] for more details.*

Theorem 2. *Consider two formations $\mathcal{F}_p$ and $\mathcal{F}_q$ in generic positions defined on the same underlying graph $\mathcal{G}(\mathcal{V},\mathcal{E})$. Then $\mathcal{F}_p$ is parallel rigid if and only if $\mathcal{F}_q$ is parallel rigid.*

This theorem underpins the following definition.

Definition 6 (Generically Parallel Rigid Graph)**.** *When $\mathcal{F}_p$ is parallel rigid for all generic points $\mathbf{p}$ then we say the graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ associated with$\mathcal{F}_p$ is generically parallel rigid.*

We often refer also to the formation $\mathcal{F}_p$ whose graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ is generically parallel rigid as a generically parallel rigid formation. There is a combinatorial test.

Theorem 3 ([15], [16])**.** *A $\mathcal{G}(\mathcal{V},\mathcal{E})$ is generically parallel rigid if and only if there is a subset $\mathcal{E}' \subseteq \mathcal{E}$ satisfying: (1) $|\mathcal{E}'| = 2n-3$; and (2) for all $\mathcal{E}'' \subseteq \mathcal{E}'$, $\mathcal{E}'' \neq \emptyset$ then $|\mathcal{E}''| \leq 2|\mathcal{V}(\mathcal{E}'')| - 3$ where $\mathcal{V}(\mathcal{E}'')$ is the set of vertices that are end-vertices of the edges in $\mathcal{E}''$.*

Despite the fact that the linear algebraic test for parallel rigidity may fail when a formation is not in a generic position we make the following claim.

Claim 1. *A generically parallel rigid formation $\mathcal{F}_p$ at $\mathbf{p} \in \mathbb{R}^{2n}$ is also a parallel rigid formation as per Definition 4.*

That is, every parallel point formation of a particular generically parallel rigid $\mathcal{F}_p$ located at $\mathbf{p} \in \mathbb{R}^{2n}$ is trivial with respect to $\mathcal{F}_p$.

III. The Rigid Formation Control Problem with Bearing Constraints

Consider $\mathcal{F}_z$ in $\mathbb{R}^2$ with an associated graph $\mathcal{G}(\mathcal{V},\mathcal{E})$. Agent i's position is $\mathbf{z}_i \in \mathbb{R}^2$ and $\mathbf{z} = [\mathbf{z}_1^\top\ \mathbf{z}_2^\top\ \ldots\ \mathbf{z}_n^\top]^\top$.

Assumption 2. *The formation $\mathcal{F}_z$ is generically parallel rigid and $\mathbf{z}_i \neq \mathbf{z}_j$ at $t=0$, $\forall i,j \in \mathcal{V}$.*

This assumption does not imply $\mathrm{rank}(\mathbf{R}(\mathbf{z})) = 2n-3$ since $\mathbf{z}$ at $t=0$ may not be a generic point.

The operator $\mathrm{column}(\mathcal{B})$ stacks the bearing measurements into a column vector that exhibits a lexicographical order such that ϕ_{ij} is above ϕ_{il} if $j<l$ and ϕ_{ij} is above ϕ_{kl} if $i<k$. Similarly, define $\mathbf{d}(\mathbf{z}) = \mathrm{column}(\mathcal{D}(\mathbf{z}))$.

Now define the subset $\mathcal{A}(\mathbf{z}) \subset \mathcal{B}(\mathbf{z})$ such that

$$\mathcal{A}(\mathbf{z}) = \{\phi_{ij} \in [0,2\pi) : (i,j) \in \mathcal{E},\ i<j\} \tag{8}$$

where $|\mathcal{A}| = m$ and then define $\alpha(\mathbf{z}) = \mathrm{column}(\mathcal{A}(\mathbf{z}))$.

Let $t \in [0,\infty)$ denote time. The motion of agent i is governed by

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{z}_i = \dot{\mathbf{z}}_i = \mathbf{u}_i \tag{9}$$

where $\mathbf{u}_i$ is a control vector to be determined. The combined motion of the formation is $\dot{\mathbf{z}} = \mathbf{u}$.

Now define a set of desired bearing values

$$\mathcal{B}_c = \{\phi^c_{ij}, \phi^c_{ji} \in [0,2\pi) : (i,j) \in \mathcal{E}\} \tag{10}$$

where $\phi^c_{ij} \equiv (\pi + \phi^c_{ji}) \bmod (2\pi)$ and $|\mathcal{B}_c| = 2m$. It suffices to define only a sub-set of desired bearing values

$$\mathcal{A}_c = \{\phi^c_{ij} \in [0,2\pi) : (i,j) \in \mathcal{E},\ i<j\} \tag{11}$$

where $|\mathcal{A}_c| = m$. Let $\alpha_c = \mathrm{column}(\mathcal{A}_c)$.

If agent i knows ϕ_{ij} then it knows ϕ_{ji} since $\phi_{ij} \equiv (\pi + \phi_{ji}) \bmod (2\pi)$. Similarly, $\phi^c_{ij} \equiv (\pi + \phi^c_{ji}) \bmod (2\pi)$.

Definition 7 (Realizable Bearing Sets)**.** *Assume a formation $\mathcal{F}_p$ is given. Then a set $\mathcal{B}'$ of bearings are realizable if and only if each $\phi_{ij} \in \mathcal{B}'$ can exist between the respective $\mathbf{p}_i$ and $\mathbf{p}_j$ simultaneously.*

Assumption 3. *The set of desired bearing values $\mathcal{B}_c$, and consequently $\mathcal{A}_c$, that define the desired formation shape and orientation is realizable. Moreover, there is a value ϕ^c_{ij} for each $(i,j) \in \mathcal{E}$ and due to Assumption 2 the desired formation is generically parallel rigid.*

In this work we seek to control only the bearings (between certain agents) and thus only the desired bearing set $\mathcal{B}_c$ (or $\mathcal{A}_c$) is given and must be realizable.

Note that $\alpha(\mathbf{z})$ is determined by the bearing measurements and is a function of $\mathbf{z}$ whereas α_c is a vector of desired bearing constraints and is constant.

Now define an error vector as

$$\mathbf{e} = \alpha(\mathbf{z}) - \alpha_c = \mathrm{column}(\mathcal{A}(\mathbf{z})) - \mathrm{column}(\mathcal{A}_c) \tag{12}$$

and note $\mathbf{e} \to 0$ implies $(\mathrm{column}(\mathcal{B}) - \mathrm{column}(\mathcal{B}_c)) \to 0$. Thus, $\mathbf{e}$ is set to become our control error.

Problem 1. *The formation control problem given only bearing constraints is to design a control input* $\mathbf{u}_i$*,* $\forall i \in \mathcal{V}$*, as a function of at most* ϕ_{ij}*,* d_{ij} *and* ϕ_{ij}^c*, for all* $j \in \mathcal{N}_i$*, such that* $\mathbf{e} = (\alpha(\mathbf{z}) - \alpha_c) \to 0$*.*

Before outlining the control law proposed to solve Problem 1 we note that the Jacobian of $\mathbf{e} \in \mathbb{R}^m$ evaluated at a point $\mathbf{p} \in \mathbb{R}^{2n}$ is given by

$$\begin{aligned}\mathbf{J_e}(\mathbf{p}) &= \nabla\mathbf{e} \\ &= \frac{\partial}{\partial \mathbf{z}}(\alpha(\mathbf{z}) - \alpha_c)\Big|_{\mathbf{z}=\mathbf{p}} = \frac{\partial}{\partial \mathbf{z}}\alpha(\mathbf{z})\Big|_{\mathbf{z}=\mathbf{p}} \quad (13)\end{aligned}$$

where $\mathbf{J_e}(\mathbf{p}) \in \mathbb{R}^{m\times 2n}$.

Let $\mathrm{diag}(\mathcal{D}(\mathbf{z}))$ take the actual distances, in $\mathcal{D}$, into the diagonal components of an $m\times m$ matrix with a lexicographical ordering such that d_{ij} is above d_{il} if $j < l$ and d_{ij} is above d_{kl} if $i < k$ etc. Let $\mathbf{D} = \mathrm{diag}(\mathcal{D}(\mathbf{z}))$.

The ℓ^{th} element of a m-vector $\mathbf{x}$ is $(\mathbf{x} : \ell)$. We then have

$$\begin{aligned}\mathbf{J_e}(\mathbf{p}) &= \frac{\partial}{\partial \mathbf{z}}\alpha(\mathbf{z})\Big|_{\mathbf{z}=\mathbf{p}} \\ &= \begin{bmatrix} \frac{\partial(\alpha(\mathbf{z}):1)}{\partial \mathbf{z_1}}\Big|_{\mathbf{z}_1=\mathbf{p}_1} & \cdots & \frac{\partial(\alpha(\mathbf{z}):1)}{\partial \mathbf{z_n}}\Big|_{\mathbf{z}_n=\mathbf{p}_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial(\alpha(\mathbf{z}):m)}{\partial \mathbf{z_1}}\Big|_{\mathbf{z}_1=\mathbf{p}_1} & \cdots & \frac{\partial(\alpha(\mathbf{z}):m)}{\partial \mathbf{z_n}}\Big|_{\mathbf{z}_n=\mathbf{p}_n} \end{bmatrix} \\ &= -\mathbf{D}^{-2}\mathbf{R}(\mathbf{p}) \quad (14)\end{aligned}$$

where $\mathbf{R}(\mathbf{p})$ is the bearing-constrained rigidity matrix for the formation $\mathcal{F}_z|_{z=p}$.

Example 2. *Consider four agents indexed by* 1*,* 2*,* 3*, and* 4 *and the rigid network illustrated in Figure 1 of Example 1. Again, the edges, arranged in lexicographical order, are* $\{(1,2),(1,3),(1,4),(2,3),(2,4)\}$*. There are a total of* 10 *bearing measurements and constraints (albeit only* 5 *independent measurements/constraints). We write*

$$\alpha(\mathbf{z}) = [\phi_{12}\ \phi_{13}\ \phi_{14}\ \phi_{23}\ \phi_{24}]^\top \quad (15)$$

where each measured ϕ_{ij} *is a function of* $\mathbf{z}_i$ *and* $\mathbf{z}_j$*. The bearing-constrained rigidity matrix for the formation is given by (7). The bearing-constrained rigidity matrix is a* 5×8 *matrix in this example. The Jacobian* $\mathbf{J_e}(\mathbf{p})$ *of the error vector* $\mathbf{e}$ *is given by (16) and is of the same dimension as (7). The rows correspond to the edges in the graph associated with the formation and the columns correspond to the agents. We note again that an agent* i *that knows* ϕ_{ij} *also knows* ϕ_{ji} *and vice versa. Thus, given the measurements* ϕ_{ij} *and* d_{ij} *at agent* i *for* $j \in \mathcal{N}_i$ *it follows that the rows of* $\mathbf{J_e}(\mathbf{p})$ *corresponding to an edge incident on* i *are known locally at agent* i *and the two columns corresponding to the agent itself are also known locally.*

For a matrix $\mathbf{X} \in \mathbb{R}^{n\times m}$ define $\ker(\mathbf{X})$ to be the kernel of $\mathbf{X}$ such that $\ker(\mathbf{X}) = \{\mathbf{x} \in \mathbb{R}^m : \mathbf{X}\mathbf{x} = 0\}$.

Lemma 1. *Suppose that* $d_{ij} \neq 0$ *for* $(i,j) \in \mathcal{E}$ *and consider a generically parallel rigid formation* $\mathcal{F}_p$*. Then* $\ker(\mathbf{R}(\mathbf{p})) = \ker(\mathbf{J_e}(\mathbf{p}))$ *and* $\dim(\ker(\mathbf{R}^\top(\mathbf{p}))) = \dim(\ker(\mathbf{J}_\mathbf{e}^\top(\mathbf{p})))$*.*

The proof is immediate from (14). In particular, the sparsity pattern of both $\mathbf{R}(\mathbf{p})$ and $\mathbf{J_e}(\mathbf{p})$ is identical for an arbitrary formation $\mathcal{F}_p$.

A. The Proposed Control Law

The control law proposed is a gradient-type control law, associated with the function $\frac{1}{2}\mathbf{e}^\top\mathbf{e}$, and can be written as

$$\begin{aligned}\mathbf{u} &\triangleq -(\nabla\mathbf{e})^\top\mathbf{e} \\ &= -\mathbf{J_e}(\mathbf{z})^\top\mathbf{e} \\ &= \mathbf{R}^\top(\mathbf{z})\mathbf{D}^{-2}\mathbf{e} \quad (17)\end{aligned}$$

from (14), and there results

$$\begin{aligned}\dot{\mathbf{z}} &= \mathbf{u} = -\mathbf{J_e}(\mathbf{z})^\top\mathbf{e} \\ &= \mathbf{R}^\top(\mathbf{z})\mathbf{D}^{-2}\mathbf{e} \quad (18)\end{aligned}$$

More specifically, the control law for an individual agent is

$$\begin{aligned}\dot{\mathbf{z}}_i &= \mathbf{u}_i \\ &= \sum_{j\in\mathcal{N}_i}\frac{1}{d_{ij}}\begin{bmatrix}\cos\phi_{ij} \\ -\sin\phi_{ij}\end{bmatrix}(\phi_{ij} - \phi_{ij}^c) \quad (19)\end{aligned}$$

which amounts to a superposition of $|\mathcal{N}_i|$ vectors pointing perpendicular to the respective $|\mathcal{N}_i|$ links in the formation $\mathcal{F}_z$ leaving agent i and where each vector is scaled by the length of the link in the formation and an appropriate error term (which may be negatively signed). The form (19) can be easily intuited using Example 2.

The controller proposed in this work is similar in principle to the controller proposed in [10] for formation control with range-only shape constraints. Indeed, there is a strong connection between this work and [4], [9]–[11], [13] due to the relationship between the rigidity matrix with range constraints and the rigidity matrix with bearing constraints.

Lemma 2. *The trajectory of* $\mathbf{z}$ *over* $t \in [0,\infty)$ *is such that* $\mathbf{z}_i \neq \mathbf{z}_j$*, for all* t *and for all neighbours* i, j *such that* $i \in \mathcal{N}_j \Leftrightarrow j \in \mathcal{N}_i$*.*

The proof this lemma is based on the fact that as agent i approaches agent j it follows that

$$\dot{\mathbf{z}}_i \to \frac{1}{d_{ij}}\begin{bmatrix}\cos\phi_{ij} \\ -\sin\phi_{ij}\end{bmatrix}(\phi_{ij} - \phi_{ij}^c) \quad (20)$$

with a similar expression for $\dot{\mathbf{z}}_j$. Now these differential equations imply that both agents are heading on a line perpendicular to the line connecting the agents.

The existence and uniqueness of the coupled system of differential equations (19) is consequently guaranteed using standard arguments [10], [18].

The next controller property concerns the formation's centre-of-mass and follows a similar result in [10] for formation control with range-only shape constraints.

Lemma 3. *Let* $\overline{\mathbf{z}} = \frac{1}{|\mathcal{V}|}\sum_{i\in\mathcal{V}}\mathbf{z}_i$*. Then* $\dot{\overline{\mathbf{z}}} = 0$*.*

We must omit the proof for reasons of space. The next lemma concerns the controller and its invariance to the global coordinate system chosen. We omit the proof for brevity.

$$
\begin{array}{cc}
 & \begin{array}{cccc} \textit{agent } 1 & \textit{agent } 2 & \textit{agent } 3 & \textit{agent } 4 \end{array} \\
\begin{array}{c} \textit{edge } (1,2) \\ \textit{edge } (1,3) \\ \textit{edge } (1,4) \\ \textit{edge } (2,3) \\ \textit{edge } (2,4) \end{array} &
\left[\begin{array}{cc|cc|cc|cc}
-\frac{\cos\phi_{12}}{d_{12}} & \frac{\sin\phi_{12}}{d_{12}} & \frac{\cos\phi_{12}}{d_{12}} & -\frac{\sin\phi_{12}}{d_{12}} & 0 & 0 & 0 & 0 \\
-\frac{\cos\phi_{13}}{d_{13}} & \frac{\sin\phi_{13}}{d_{13}} & 0 & 0 & \frac{\cos\phi_{13}}{d_{13}} & -\frac{\sin\phi_{13}}{d_{13}} & 0 & 0 \\
-\frac{\cos\phi_{14}}{d_{14}} & \frac{\sin\phi_{14}}{d_{14}} & 0 & 0 & 0 & 0 & \frac{\cos\phi_{14}}{d_{14}} & -\frac{\sin\phi_{14}}{d_{14}} \\
0 & 0 & -\frac{\cos\phi_{23}}{d_{23}} & \frac{\sin\phi_{23}}{d_{23}} & \frac{\cos\phi_{23}}{d_{23}} & -\frac{\sin\phi_{23}}{d_{23}} & 0 & 0 \\
0 & 0 & -\frac{\cos\phi_{24}}{d_{24}} & \frac{\sin\phi_{24}}{d_{24}} & 0 & 0 & \frac{\cos\phi_{24}}{d_{24}} & -\frac{\sin\phi_{24}}{d_{24}}
\end{array}\right]
\end{array}
= \mathbf{J_e}(\mathbf{p}) \quad (16)
$$

Lemma 4. *For all $\mathbf{w} \in \mathbb{R}^2$ it follows that $\mathbf{J_e}(\mathbf{z}) = \mathbf{J_e}(\mathbf{z} + (\mathbf{1} \otimes \mathbf{w}))$ where $\mathbf{1}$ is an n-dimensional column vector of all 1's. Moreover, for every orthogonal matrix $\mathbf{X} \in \mathbb{R}^{2\times 2}$ it follows that $\mathbf{J_e}(\mathbf{z})(\mathbf{I}_n \otimes \mathbf{X})^\top = \mathbf{J_e}((\mathbf{I_n} \otimes \mathbf{X})\mathbf{z})$ where $\mathbf{I}_n$ is a $n \times n$ identity matrix.*

IV. Stability Results

A. Minimally and Generically Parallel Rigid Formation

We know that a generically parallel rigid formation $\mathcal{F}_z$ is one that can be characterized entirely by the associated graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ defining the formation interactions; e.g. see Theorem 1. It is also the case, from e.g. Theorem 1, that a necessary condition for the formation to be (generically) parallel rigid is that $|\mathcal{E}| = m \geq 2|\mathcal{V}| - 3 = 2n - 3$.

Definition 8 (Minimally Parallel Rigid)**.** *A formation $\mathcal{F}_z$ with $|\mathcal{E}| = m = 2|\mathcal{V}| - 3 = 2n - 3$ at $\mathbf{z} \in \mathbb{R}^{2n}$ is called a minimally parallel rigid formation if $\operatorname{rank}(\mathbf{R}(\mathbf{z})) = 2n - 3$. A formation $\mathcal{F}_z$ with $|\mathcal{E}| = m = 2|\mathcal{V}| - 3 = 2n - 3$ is called a minimally and generically parallel rigid formation if and only if it is generically parallel rigid and $|\mathcal{E}| = m = 2|\mathcal{V}| - 3 = 2n - 3$.*

Lemma 5. *Suppose $\mathcal{F}_z$ is a minimally and generically parallel rigid formation. For the existence of each distinct (i.e. different, but not necessarily disjoint) set*

$$\mathcal{C}_{i,j,k} = \{i, j, k \in \mathcal{V} : \phi_{ij} \equiv \phi_{ik} \operatorname{mod}(\pi),\ \phi_{ij}, \phi_{ik}, \phi_{jk} \in \mathcal{A}\} \tag{21}$$

where $|\mathcal{C}_{i,j,k}| = 3$ then $\operatorname{rank}(\mathbf{R}(\mathbf{z}))$ drops by 1.

Each $\mathcal{C}_{i,j,k}$ is a set of three agents that are collinear in $\mathbb{R}^2$ and which form a cycle in $\mathcal{G}$. This rank condition has been established previously for the Jacobian $\mathbf{J_e}(\mathbf{z})$ in [19] and $\operatorname{rank}(\mathbf{R}(\mathbf{z})) = \operatorname{rank}(\mathbf{J_e}(\mathbf{z}))$.

The stability analysis in this subsection concerns minimally and generically parallel rigid formations $\mathcal{F}_z$ and the resulting differential system (18). Consider the set

$$\mathcal{Z}^* = \{\mathbf{z} \in \mathbb{R}^{2n} : \alpha(\mathbf{z}) - \alpha_c = \mathbf{e} = \mathbf{0}\} \tag{22}$$

of equilibrium points corresponding to the formation $\mathcal{F}_z$ reaching the desired shape and orientation defined by α_c. Each formation that lives in $\mathcal{Z}^*$ is generically parallel rigid due to Assumptions 2 and 3.

Definition 9 (Connected Space)**.** *A topological space $\mathcal{X}$ is said to be disconnected if there exists two open sets $\mathcal{U} \neq \emptyset$ and $\mathcal{W} \neq \emptyset$ such that $\mathcal{U} \cap \mathcal{W} = \emptyset$ and $\mathcal{X} = \mathcal{U} \cup \mathcal{W}$. If $\mathcal{X}$ is not disconnected than it is said to be connected.*

The maximal connected subsets of a nonempty topological space are called the connected components of the space. The components of any topological space $\mathcal{X}$ are disjoint, nonempty, and their union is $\mathcal{X}$.

Lemma 6. *The set $\mathcal{Z}^*$ is connected and each $\mathbf{z}' \in \mathcal{Z}^*$ can be obtained from $\mathbf{z} \in \mathcal{Z}^*$ by a translation and then a dilation (or vice versa).*

The proof of the preceding lemma will appear in an extended version of this work but is easily obtained.

Unfortunately, the set $\mathcal{Z}^*$ is not the only equilibrium set for the differential system (19) and minimally parallel rigid formations under Assumption 2. Consider the set

$$\mathcal{Z}_* = \{\mathbf{z} \in \mathbb{R}^{2n} : \mathbf{R}^\top(\mathbf{z})\mathbf{D}^{-2}\mathbf{e} = \mathbf{0}\} \tag{23}$$

and note that it is trivial to conclude that $\dot{\mathbf{z}} = \mathbf{0}$ if and only if $\mathbf{z} \in \mathcal{Z}_*$. A question remains as to when $\mathcal{Z}^* \equiv \mathcal{Z}_*$.

Theorem 4. *Suppose Assumption 2 holds and the formation $\mathcal{F}_z$ is minimally and generically parallel rigid. Assume that $\operatorname{rank}(\mathbf{R}(\mathbf{z})) = m = 2n - 3$ for all $t \in [0, \infty)$. Then $\dot{\mathbf{z}} = \mathbf{0}$ if and only if $\mathbf{z} \in \mathcal{Z}^*$.*

Proof: The *if* part of the theorem is obvious from (18). If $\operatorname{rank}(\mathbf{R}(\mathbf{z})) = m = 2n - 3$ then $\mathbf{R}(\mathbf{z})$ has full (column) rank and the kernel of $\mathbf{R}^\top(\mathbf{z})$ is trivial. Thus, $\mathcal{Z}^* \equiv \mathcal{Z}_*$. ☐

We know from Lemma 5 that $\operatorname{rank}(\mathbf{R}(\mathbf{z}))$ drops by 1 for the existence of each set (21). Thus, any undesirable equilibria in $\{\mathcal{Z}_* \setminus \mathcal{Z}^*\}$ must coincide with the existence of some (possible multiple) sets (21). We conjecture, in the spirit of [10], that any equilibria in $\{\mathcal{Z}_* \setminus \mathcal{Z}^*\}$ are non-attractive. Note that in distance-constraint-based formation control using the rigidity matrix [10], every initial collinear formation will remain collinear. The situation here is less troublesome in this respect.

Analysis concerning the state space and the equilibrium sets $\{\mathcal{Z}_*$ and $\mathcal{Z}^*\}$ is the topic of further work and will appear in an extended version of this work.

Theorem 5. *Suppose Assumption 2 holds and the formation $\mathcal{F}_z$ is minimally and generically parallel rigid. Then $\mathcal{Z}^*$ is locally asymptotically stable and there exists a neighbourhood $\mathcal{U}$ of $\mathcal{Z}^*$ such that for all $\mathbf{z}(0) \in \mathcal{U}$ there exists a point $\mathbf{z}^* \in \mathcal{Z}^*$ such that $\lim_{t\to\infty}\mathbf{z} = \mathbf{z}^*$.*

For reasons of space, the proof will appear elsewhere. Moreover, the theorem's validity is not surprising given the gradient-like nature of the system and the structural similarity between the differential system considered here and that considered in [10].

B. Generically Parallel Rigid Formation

The stability analysis in this subsection concerns (arbitrary) generically parallel rigid formations $\mathcal{F}_z$ and the resulting differential system (18). The only equilibrium set considered in this subsection is the desired one (22).

Theorem 6. *Suppose Assumption 2 holds and the formation $\mathcal{F}_z$ is generically parallel rigid. Then $\mathcal{Z}^*$ is locally asymptotically stable and there is a neighbourhood $\mathcal{U}$ of $\mathcal{Z}^*$ such that $\forall \mathbf{z}(0) \in \mathcal{U}$ there exists a $\mathbf{z}^* \in \mathcal{Z}^*$ such that $\lim_{t\to\infty} \mathbf{z} = \mathbf{z}^*$.*

The proof of the preceding theorem will appear elsewhere.

V. Illustrative Examples

In this section we demonstrate the algorithm developed for distributed formation control with direction constraints.

A. Four Agent Control

Suppose there are four agents indexed by $\mathcal{V} = \{1,2,3,4\}$ and the interaction topology is defined by the links $\mathcal{E} = \{(1,2),(1,3),(1,4),(2,3),(2,4)\}$. The control error for this formation $\mathcal{F}_z$ has the form

$$\begin{aligned}\mathbf{e} &= [\phi_{12}\ \phi_{13}\ \phi_{14}\ \phi_{23}\ \phi_{24}]^\top - [\phi_{12}^c\ \phi_{13}^c\ \phi_{14}^c\ \phi_{23}^c\ \phi_{24}^c]^\top \\ &= \alpha(\mathbf{z}) - \alpha_c \qquad (24)\end{aligned}$$

Thus, the formation is defined as in Example 1 and 2 and the bearing-constrained rigidity matrix and error Jacobian take the form of (7) and (16) respectively.

1) Random Four-Agent Formation to a Square Formation: The first example illustrates how the formation converges to a square given a random initial placement of the agents. The desired formation in this case is characterized by

$$\alpha_c = [\pi/4\ \pi/2\ 0\ \pi\ 3\pi/2]^\top \qquad (25)$$

and the formation motion is illustrated in Figure 2 along with the convergence of $(\phi_{ij} - \phi_{ij}^c) \to 0$ for $(i,j) \in \mathcal{E}$, $i<j$.

2) Random Four-Agent Formation to a Non-Square Formation: This example illustrates how the formation converges to a non-square shape given a random initial placement of the agents. The desired formation in this case is characterized by

$$\alpha_c = [\pi/4\ \pi/2\ 0\ 3\pi/4\ 7\pi/4]^\top \qquad (26)$$

and the formation motion is illustrated in Figure 3 along with the convergence of $(\phi_{ij} - \phi_{ij}^c) \to 0$ for $(i,j) \in \mathcal{E}$, $i<j$.

In this case, the desired formation has three of the agents collinear, i.e. agents 2, 3 and 4 are collinear in the desired formation. We see that in this example convergence also occurs given both a desired three-agent collinear condition and a random initial placement.

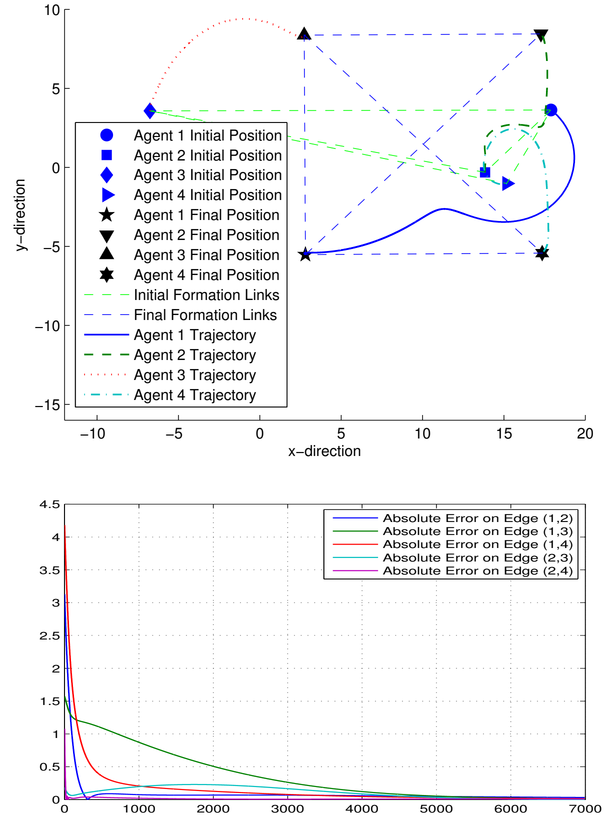


Fig. 2. The motion of a formation consisting of four mobile agents starting in a random initial configuration and given a desired square constraint.

3) Collinear Four-Agent Formation to a Square Formation: This example illustrates how the formation converges to a square shape given that the initial formation is collinear. The desired formation in this case is characterized by

$$\alpha_c = [\pi/4\ \pi/2\ 0\ \pi\ 3\pi/2]^\top \qquad (27)$$

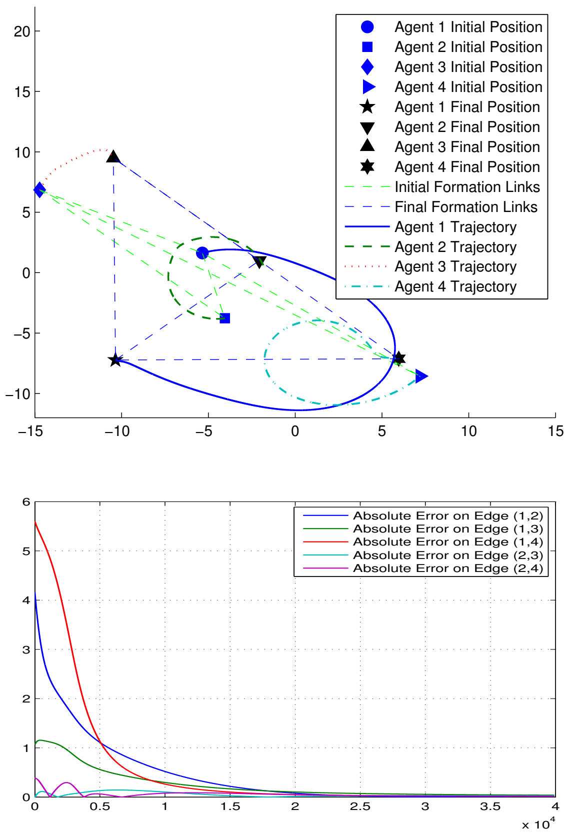
and the formation motion is illustrated in Figure 4 along with the convergence of $(\phi_{ij} - \phi_{ij}^c) \to 0$ for $(i,j) \in \mathcal{E}$, $i<j$.

In this case, the desired formation is a square but the initial formation has all four agents collinear. We thus have an example where collinearity is not an equilibrium (due to the chosen α_c) and the formation converges to the desired shape from collinearity.

Note that in distance-constraint based formation control using the idea of the rigidity matrix [10], any initial collinear formation will remain collinear.

VI. Conclusion

This paper looks at the design of a distributed control scheme to solve the direction-based formation shape control problem. In particular, a gradient control law is proposed based on the notion of bearing-only constrained graph rigidity theory and parallel drawings. An outline stability analysis is provided.



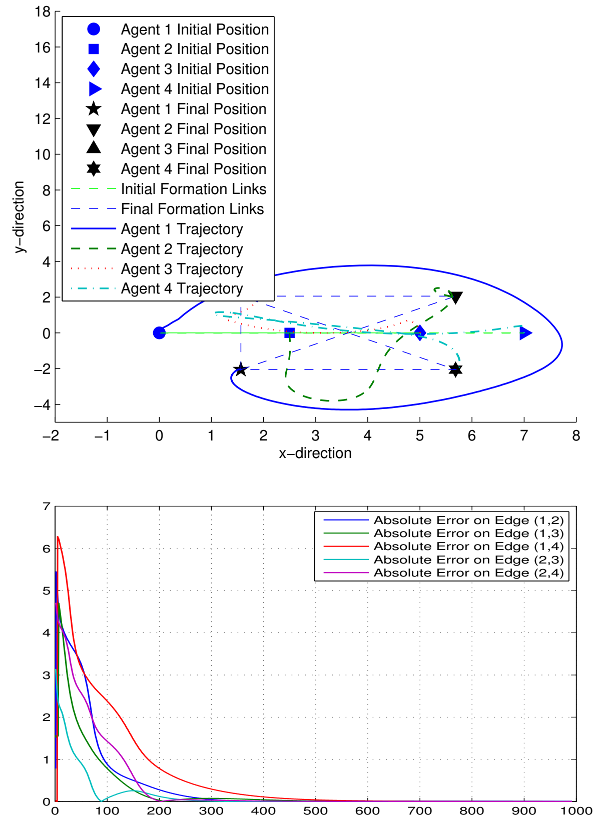


Fig. 3. The motion of a formation consisting of four mobile agents starting in a random initial configuration and given a non-square desired formation where three of the agents are collinear.

Fig. 4. The motion of a formation consisting of four mobile agents starting in a collinear formation and given a desired square constraint.

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