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A Lyapunov approach in incremental stability

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Abstract— **The notion of incremental stability was proposed by several researchers as a strong property of dynamical and control systems. Incremental stability describes the convergence of trajectories with respect to themselves, rather than with respect to an equilibrium point or a particular trajectory. Similarly to stability, Lyapunov functions play an important role in the study of incremental stability. In this paper, we propose new notions of incremental Lyapunov functions which are coordinate independent and provide the description of incremental stability in terms of the proposed Lyapunov functions. Moreover, we develop a backstepping design approach providing a recursive way of constructing controllers, enforcing incremental stability, as well as incremental Lyapunov functions. The effectiveness of the proposed method is illustrated by synthesizing a controller rendering a single-machine infinite-bus electrical power system incrementally stable.**

I. Introduction

Incremental stability is a the requirement that all trajectories of a dynamical system converge to each other, rather than to an equilibrium point or a particular trajectory. While it is well-known that for linear systems such a property is equivalent to stability, it can be a much stronger property for nonlinear systems. The study of incremental stability goes back to the work of Zames in the 60's [1]; see [2] for a historical discussion and a broad list of applications of incremental stability.

Similarly to stability, Lyapunov functions play an important role in the study of incremental stability. Angeli [3] proposed the notions of incremental Lyapunov function and incremental input-to-state Lyapunov function, and used these notions to prove charactrizations of incremental global asymptotic stability (δ-GAS) and incremental input-to-state stability (δ-ISS). Both proposed notions of Lyapunov functions in [3] are not coordinate independent, in general. In this paper, we propose new notions of incremental Lyapunov functions and incremental input-to-state Lyapunov functions that are coordinate invariant. Moreover, we use these new notions of Lyapunov functions to describe notions of incremental stability, proposed in [2].

Since the proposed notions of Lyapunov functions in this paper are coordinate invariant, they potentiate the development of synthesis tools for incremental stability. As an example, we develop a backstepping design method for incremental stability for strict-feedback[1] form systems. The proposed approach was inspired by the incremental backstepping approach provided in [2]. While the approach in [2], provides a recursive way of constructing contraction metrics, the proposed approach in this paper provide a recursive way of constructing incremental Lyapunov functions, identified as a key property for the construction of finite abstractions in [5], [6], [7]. See [8], for a broad list of applications of incremental Lyapunov functions. Like the original backstepping method, the proposed approach in this paper provides a recursive way of constructing controllers as well as incremental Lyapunov functions. Our design approach is illustrated by designing a controller rendering a single-machine infinite-bus electrical power system incrementally stable.

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II. Control Systems and Stability Notions

A. Notation

The symbols $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}_0^+$ denote the set of real, positive, and nonnegative real numbers, respectively. The symbol I_n denotes the identity matrix in $\mathbb{R}^{n\times n}$. Given a vector $x \in \mathbb{R}^n$, we denote by x_i the i–th element of x, and by $\|x\|$ the Euclidean norm of x; we recall that $\|x\| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$. Given a measurable function $f : \mathbb{R}_0^+ \to \mathbb{R}^n$, the (essential) supremum of f is denoted by $\|f\|_\infty$; we recall that $\|f\|_\infty := \text{(ess)sup}\{\|f(t)\|, t \geq 0\}$. f is essentially bounded if $\|f\|_\infty < \infty$. For a given time $\tau \in \mathbb{R}^+$, define f_τ so that $f_\tau(t) = f(t)$, for any $t \in [0, \tau)$, and $f(t) = 0$ elsewhere; f is said to be locally essentially bounded if for any $\tau \in \mathbb{R}^+$, f_τ is essentially bounded. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called radially unbounded if $f(x) \to \infty$ as $\|x\| \to \infty$. A continuous function $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$; function γ is said to belong to class $\mathcal{K}_\infty$ if $\gamma \in \mathcal{K}$ and $\gamma(r) \to \infty$ as $r \to \infty$. A continuous function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the map $\beta(r, s)$ belongs to class $\mathcal{K}_\infty$ with respect to r and, for each fixed nonzero r, the map $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$. If $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is a global diffeomorphism, and if $X : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous map, we denote by ϕ_*X the map defined by $(\phi_*X)(y) = \frac{\partial\phi}{\partial x}\big|_{x=\phi^{-1}(y)} X \circ \phi^{-1}(y)$. A function $\mathbf{d} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$ is a metric on $\mathbb{R}^n$ if for any $x, y, z \in \mathbb{R}^n$, the following three conditions are satisfied: i) $\mathbf{d}(x, y) = 0$ if and only if $x = y$; ii) $\mathbf{d}(x, y) = \mathbf{d}(y, x)$; and iii) $\mathbf{d}(x, z) \leq \mathbf{d}(x, y) + \mathbf{d}(y, z)$.

[1]See equation (III.8) or [4] for a definition.

B. Control systems

The class of control systems that we consider in this paper is formalized in the following definition.

Definition 2.1: A *control system* Σ is a quadruple $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f)$, where:

- $\mathbb{R}^n$ is the state space;
- $\mathsf{U} \subseteq \mathbb{R}^m$ is the input set;
- $\mathcal{U}$ is the set of all measurable, locally essentially bounded functions of time from intervals of the form $]a, b[\subseteq \mathbb{R}$ to U with $a < 0$ and $b > 0$;
- $f : \mathbb{R}^n \times \mathsf{U} \to \mathbb{R}^n$ is a continuous map satisfying the following Lipschitz assumption: for every compact set $Q \subset \mathbb{R}^n$, there exists a constant $Z \in \mathbb{R}^+$ such that $\|f(x,u) - f(y,u)\| \leq Z\|x - y\|$ for all $x, y \in Q$ and all $u \in \mathsf{U}$.

A curve $\xi :]a, b[\to \mathbb{R}^n$ is said to be a *trajectory* of Σ if there exists $\upsilon \in \mathcal{U}$ satisfying $\dot{\xi}(t) = f(\xi(t), \upsilon(t))$, for almost all $t \in]a, b[$. We also write $\xi_{x\upsilon}(t)$ to denote the point reached at time t under the input υ from initial condition $x = \xi_{x\upsilon}(0)$; this point is uniquely determined, since the assumptions on f ensure existence and uniqueness of trajectories [9]. A control system Σ is said to be forward complete if every trajectory is defined on an interval of the form $]a, \infty[$. Sufficient and necessary conditions for a system to be forward complete can be found in [10].

C. Stability notions

Here, we recall the notions of incremental global asymptotic stability ($\delta_\exists$-GAS) and incremental input-to-state stability ($\delta_\exists$-ISS), presented in [2].

Definition 2.2 ([2]): A control system Σ is incrementally globally asymptotically stable ($\delta_\exists$-GAS) if it is forward complete and there exist a metric $\mathbf{d}$ and a $\mathcal{KL}$ function β such that for any $t \in \mathbb{R}_0^+$, any $x, x' \in \mathbb{R}^n$ and any $\upsilon \in \mathcal{U}$ the following condition is satisfied:

$$\mathbf{d}\left(\xi_{x\upsilon}(t), \xi_{x'\upsilon}(t)\right) \leq \beta\left(\mathbf{d}\left(x, x'\right), t\right). \tag{II.1}$$

As defined in [3], δ-GAS requires the metric $\mathbf{d}$ to be the Euclidean metric. However, Definition 2.2 only requires the existence of a metric. We note that while δ-GAS is not generally invariant under changes of coordinates, $\delta_\exists$-GAS is. When the origin is an equilibrium point for Σ and the map $\psi : \mathbb{R}^n \to \mathbb{R}_0^+$, defined by $\psi(x) = \mathbf{d}(x, 0)$, is radially unbounded, both $\delta_\exists$-GAS and δ-GAS imply global asymptotic stability.

Definition 2.3 ([2]): A control system Σ is incrementally input-to-state stable ($\delta_\exists$-ISS) if it is forward complete and there exist a metric $\mathbf{d}$, a $\mathcal{KL}$ function β, and a $\mathcal{K}_\infty$ function γ such that for any $t \in \mathbb{R}_0^+$, any $x, x' \in \mathbb{R}^n$, and any υ, $\upsilon' \in \mathcal{U}$ the following condition is satisfied:

$$\mathbf{d}\left(\xi_{x\upsilon}(t), \xi_{x'\upsilon'}(t)\right) \leq \beta\left(\mathbf{d}\left(x, x'\right), t\right) + \gamma\left(\|\upsilon - \upsilon'\|_\infty\right). \tag{II.2}$$

By observing (II.1) and (II.2), it is readily seen that $\delta_\exists$-ISS implies $\delta_\exists$-GAS while the converse is not true in general. Moreover, whenever the metric $\mathbf{d}$ is the Euclidean metric, $\delta_\exists$-ISS becomes δ-ISS as defined in [3]. We note that while δ-ISS is not generally invariant under changes of coordinates, $\delta_\exists$-ISS is. When the origin is an equilibrium point for Σ and the map $\psi : \mathbb{R}^n \to \mathbb{R}_0^+$, defined by $\psi(x) = \mathbf{d}(x, 0)$, is radially unbounded, both $\delta_\exists$-ISS and δ-ISS imply input-to-state stability.

D. Descriptions of incremental stability

This section contains the description of $\delta_\exists$-GAS and $\delta_\exists$-ISS in terms of existence of incremental Lyapunov functions. We start by introducing the following definition which was inspired by the notions of incremental global asymptotic stability (δ-GAS) Lyapunov function and incremental input-to-state stability (δ-ISS) Lyapunov function presented in [3].

Definition 2.4: Consider a control system Σ and a smooth function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$. Function V is called a $\delta_\exists$-GAS Lyapunov function for Σ, if there exist a metric $\mathbf{d}$, $\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\overline{\alpha}$, and $\kappa \in \mathbb{R}^+$ such that:

(i) for any $x, x' \in \mathbb{R}^n$
$\underline{\alpha}(\mathbf{d}(x, x')) \leq V(x, x') \leq \overline{\alpha}(\mathbf{d}(x, x'))$;

(ii) for any $x, x' \in \mathbb{R}^n$ and any $u \in \mathsf{U}$
$\frac{\partial V}{\partial x} f(x, u) + \frac{\partial V}{\partial x'} f(x', u) \leq -\kappa V(x, x')$.

Function V is called a $\delta_\exists$-ISS Lyapunov function for Σ, if there exist a metric $\mathbf{d}$, $\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\overline{\alpha}$, σ, and $\kappa \in \mathbb{R}^+$ satisfying conditions (i) and:

(iii) for any $x, x' \in \mathbb{R}^n$ and for any $u, u' \in \mathsf{U}$
$\frac{\partial V}{\partial x} f(x, u) + \frac{\partial V}{\partial x'} f(x', u') \leq -\kappa V(x, x') + \sigma(\|u - u'\|)$.

While δ-GAS and δ-ISS Lyapunov functions, as defined in [3], require the metric $\mathbf{d}$ to be the Euclidean metric, Definition 2.4 only requires the existence of a metric. We note that while δ-GAS and δ-ISS Lyapunov functions are not invariant under changes of coordinates in general, $\delta_\exists$-GAS and $\delta_\exists$-ISS Lyapunov functions are.

In the next lemma, we show that $\delta_\exists$-GAS and $\delta_\exists$-ISS Lyapunov functions, defined in Definition 2.4, are invariant under changes of coordinates.

Lemma 2.5: Let $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f)$ be a control system and let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a global diffeomorphism. If the function V is a $\delta_\exists$-GAS (resp. $\delta_\exists$-ISS) Lyapunov function for Σ, then the function $V\left(\phi^{-1}, \phi^{-1}\right)$ is a $\delta_\exists$-GAS (resp. $\delta_\exists$-ISS) Lyapunov function for $\Sigma' = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, \phi_* f)$.

Proof: For simplifying the proof, we abuse the notation and use $V \circ \phi^{-1}$ to denote $V\left(\phi^{-1}, \phi^{-1}\right)$. Inequalities (i) in Definition 2.4, transforms under ϕ to:

$$\underline{\alpha}\left(\mathbf{d}\left(\phi^{-1}(y), \phi^{-1}(y')\right)\right) \leq V\left(\phi^{-1}(y), \phi^{-1}(y')\right) \leq \overline{\alpha}\left(\mathbf{d}\left(\phi^{-1}(y), \phi^{-1}(y')\right)\right). \tag{II.3}$$

Therefore, function $V \circ \phi^{-1}$ satisfies the inequalities (i) in Definition 2.4 by the metric $\mathbf{d}'(y, y') = \mathbf{d}\left(\phi^{-1}(y), \phi^{-1}(y')\right)$. Let us now show that condition (ii) in Definition 2.4 holds for $V \circ \phi^{-1}$. Using

$\frac{\partial \phi^{-1}}{\partial y}\frac{\partial \phi}{\partial x}\left(\phi^{-1}(y)\right) = I_n$, we obtain:

$$\begin{aligned} &\frac{\partial\left(V\circ\phi^{-1}\right)}{\partial y}(\phi_* f)(y,u) + \frac{\partial\left(V\circ\phi^{-1}\right)}{\partial y'}(\phi_* f)(y',u) \\ &= \frac{\partial V}{\partial x}\Big|_{x=\phi^{-1}(y)}\frac{\partial \phi^{-1}}{\partial y}\left(\phi_* f\right)(y,u) \qquad \text{(II.4)} \\ &+\frac{\partial V}{\partial x'}\Big|_{x'=\phi^{-1}(y')}\frac{\partial \phi^{-1}}{\partial y}\left(\phi_* f\right)(y',u) \\ &= \frac{\partial V}{\partial x}\Big|_{x=\phi^{-1}(y)} f\left(\phi^{-1}(y),u)\right) \\ &+\frac{\partial V}{\partial x'}\Big|_{x'=\phi^{-1}(y')} f\left(\phi^{-1}(y'),u)\right) \\ &\leq -\kappa V\left(\phi^{-1}(y),\phi^{-1}(y')\right), \end{aligned}$$

which completes the proof. Similarly, it can be shown that $V\circ\phi^{-1}$ satisfies the condition (iii) in Definition 2.4 for Σ' if V satisfies it for Σ. ■

The following theorem describes $\delta_\exists$-ISS (resp. $\delta_\exists$-GAS) in terms of existence of a $\delta_\exists$-ISS (resp. $\delta_\exists$-GAS) Lyapunov function.

Theorem 2.6: A forward complete control system Σ is $\delta_\exists$-ISS (resp. $\delta_\exists$-GAS) if it admits a $\delta_\exists$-ISS (resp. $\delta_\exists$-GAS) Lyapunov function.

Proof: The proof is inspired by the proof of Theorem 5.2 in [11]. By using property (i) in Definition 2.4, we obtain:

$$\mathbf{d}\left(\xi_{x\upsilon}(t),\xi_{x'\upsilon'}(t)\right) \leq \underline{\alpha}^{-1}\left(V\left(\xi_{x\upsilon}(t),\xi_{x'\upsilon'}(t)\right)\right), \qquad \text{(II.5)}$$

for any $t\in\mathbb{R}_0^+$. By using property (iii) and the comparison lemma [12], one gets:

$$\begin{aligned} V(\xi_{x\upsilon}(t),\xi_{x'\upsilon'}(t)) &\leq e^{-\kappa t}V(\xi_{x\upsilon}(0),\xi_{x'\upsilon'}(0)) \qquad \text{(II.6)} \\ &+e^{-\kappa t} * \sigma(\|\upsilon(t)-\upsilon'(t)\|), \end{aligned}$$

for any $t\in\mathbb{R}_0^+$, where $*$ denotes the convolution integral[2]. By combining inequalities (II.5) and (II.6), one gets:

$$\begin{aligned} &\mathbf{d}\left(\xi_{x\upsilon}(t),\xi_{x'\upsilon'}(t)\right) \leq \\ &\underline{\alpha}^{-1}\left(e^{-\kappa t}V(x,x') + e^{-\kappa t} * \sigma(\|\upsilon(t)-\upsilon'(t)\|)\right) \leq \\ &\underline{\alpha}^{-1}\left(e^{-\kappa t}V(x,x') + \frac{1-e^{-\kappa t}}{\kappa}\sigma(\|\upsilon-\upsilon'\|_\infty)\right) \leq \\ &\underline{\alpha}^{-1}\left(e^{-\kappa t}V(x,x') + \frac{1}{\kappa}\sigma(\|\upsilon-\upsilon'\|_\infty)\right) = \gamma(\rho,\phi), \end{aligned}$$

where $\gamma(\rho,\phi) = \underline{\alpha}^{-1}(\rho+\phi)$, $\rho = e^{-\kappa t}V(x,x')$, and $\phi = \frac{1}{\kappa}\sigma\left(\|\upsilon-\upsilon'\|_\infty\right)$. Since γ is nondecreasing in each variable, we have:

$$\begin{aligned} \mathbf{d}\left(\xi_{x\upsilon}(t),\xi_{x'\upsilon'}(t)\right) &\leq h\left(e^{-\kappa t}V(x,x')\right) \\ &+h\left(\frac{1}{\kappa}\sigma\left(\|\upsilon-\upsilon'\|_\infty\right)\right), \end{aligned}$$

where $h(r) = \gamma(r,r) = \underline{\alpha}^{-1}(2r)$ and $h:\mathbb{R}_0^+\rightarrow\mathbb{R}_0^+$ is a $\mathcal{K}_\infty$ function. Moreover, using $V(x,x')\leq\overline{\alpha}(\mathbf{d}\left(x,x'\right))$, one obtains:

$$\begin{aligned} \mathbf{d}(\xi_{x\upsilon}(t),\xi_{x'\upsilon'}(t)) &\leq \underline{\alpha}^{-1}\left(2e^{-\kappa t}\overline{\alpha}(\mathbf{d}(x,x'))\right) \\ &+\underline{\alpha}^{-1}\left(\frac{2}{\kappa}\sigma(\|\upsilon-\upsilon'\|_\infty)\right). \end{aligned}$$

[2] $e^{-\kappa t} * \sigma(\|\upsilon(t)-\upsilon'(t)\|) = \int_0^t e^{-\kappa(t-\tau)}\sigma(\|\upsilon(\tau)-\upsilon'(\tau)\|)d\tau$.

Therefore, by defining functions β and γ as

$$\begin{aligned} \beta(\mathbf{d}(x,x'),t) &= \underline{\alpha}^{-1}\left(2e^{-\kappa t}\overline{\alpha}\left(\mathbf{d}(x,x')\right)\right) \\ \gamma(\|\upsilon-\upsilon'\|_\infty) &= \underline{\alpha}^{-1}\left(\frac{2}{\kappa}\sigma(\|\upsilon-\upsilon'\|_\infty)\right), \end{aligned}$$

the condition (II.2) is satisfied. Hence, the system Σ is $\delta_\exists$-ISS. The proof also works for $\delta_\exists$-GAS case by simply forcing $\upsilon=\upsilon'$. ■

In the next section, we propose a backstepping design procedure, providing a recursive way of constructing controllers as well as incremental Lyapunov functions, to render control systems incrementally stable .

III. Backstepping Design Procedure

The method described here was inspired by the incremental backstepping approach provided in [2]. While the approach proposed in [2] provides a recursive way of constructing contraction metrics, the proposed approach in this paper provides a recursive way of constructing incremental Lyapunov functions, identified as a key tool for the construction of finite abstractions of nonlinear control systems, proposed in [5], [6], [7]. See [8] for a list of applications of incremental Lyapunov functions. Consider the class of control systems $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f)$ with f of the parametric-strict-feedback form [4]:

$$\begin{aligned} f_1(x,u) &= h_1(x_1) + b_1 x_2, \\ f_2(x,u) &= h_2(x_1,x_2) + b_2 x_3, \\ &\vdots \qquad \text{(III.1)} \\ f_{n-1}(x,u) &= h_{n-1}(x_1,\cdots,x_{n-1}) + b_{n-1}x_n, \\ f_n(x,u) &= h_n(x) + g(x)u, \end{aligned}$$

where $x\in\mathbb{R}^n$ is the state and $u\in\mathsf{U}\subseteq\mathbb{R}$ is the control input. The functions $h_i:\mathbb{R}^i\rightarrow\mathbb{R}$, for $i=1,\ldots,n$, and $g:\mathbb{R}^n\rightarrow\mathbb{R}$ are smooth, $g(x)\neq 0$ over the domain of interest, and $b_i\in\mathbb{R}$, for $i=1,\ldots,n$, are nonzero constants.

To clarify later results more, we should note that we use notations $\widehat{u}$ and $\widehat{u}'$ to denote points inside U, and their script version $\widehat{\upsilon}$ and $\widehat{\upsilon}'$ to denote input curves inside $\mathcal{U}$.

We can now state one of the results, describing a backstepping controller for the control system (III.1).

Theorem 3.1: For any control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f)$ with f of the form (III.1), for any $\lambda\in\mathbb{R}^+$, and any $\widehat{u}\in\mathsf{U}$, the state feedback control law:

$$k(x,\widehat{u}) = \frac{1}{g(x)}\Big[k_n(x) - h_n(x)\Big] + \frac{1}{g(x)}\widehat{u}, \qquad \text{(III.2)}$$

where

$$\begin{aligned} k_l(x) &= -b_{l-1}\left(x_{l-1}-\phi_{l-2}(x)\right) - \lambda\left(x_l - \phi_{l-1}(x)\right) \\ &+\frac{\partial\phi_{l-1}}{\partial x}f(x,k(x)),\ \text{for } l=1,\cdots,n, \qquad \text{(III.3)} \\ \phi_l(x) &= \frac{1}{b_l}\Big[k_l(x) - h_l(x)\Big],\ \text{for } l=1,\cdots,n-1, \\ \phi_{-1}(x) &= \phi_0(x) = 0\ \forall x\in\mathbb{R}^n,\ b_0=0,\ \text{and } x_0=0, \end{aligned}$$

renders the control system Σ $\delta_\exists$-ISS with respect to the input $\widehat{\upsilon}$ and the function

$$\widehat{V}(x,x')=\sqrt{\sum_{l=0}^{n-1}\left[(x_{l+1}-\phi_l(x))-\left(x'_{l+1}-\phi_l(x')\right)\right]^2},$$

is a $\delta_\exists$-ISS Lyapunov function for Σ.

Proof: The proposed control law (III.2) transforms a control system $\Sigma=(\mathbb{R}^n,\mathsf{U},\mathcal{U},f)$ with f of the form (III.1) into:

$$\begin{array}{rcl} f_1(x,k(x,\widehat{u})) &=& h_1(x_1)+b_1x_2,\\ f_2(x,k(x,\widehat{u})) &=& h_2(x_1,x_2)+b_2x_3,\\ &\vdots& \\ f_{n-1}(x,k(x,\widehat{u})) &=& h_{n-1}(x_1,\cdots,x_{n-1})+b_{n-1}x_n,\\ f_n(x,k(x,\widehat{u})) &=& k_n(x)+\widehat{u}. \end{array} \quad \text{(III.4)}$$

The coordinate transformation $z=\psi(x)$, where $\psi:\mathbb{R}^n\rightarrow\mathbb{R}^n$ is a global diffeomorphism, defined by:

$$z=\psi(x)=\begin{bmatrix} x_1\\ x_2-\phi_1(x)\\ x_3-\phi_2(x)\\ \vdots\\ x_n-\phi_{n-1}(x)\end{bmatrix}, \quad \text{(III.5)}$$

transforms the control system Σ with f of the form (III.4) into the control system $\Sigma'=(\mathbb{R}^n,\mathsf{U},\mathcal{U},f')$, where $f'=\psi_*f$ has the following form:

$$f'(z,\widehat{u})=Az+B\widehat{u}, \quad \text{(III.6)}$$

where

$$A=\begin{bmatrix} -\lambda & b_1 & 0 & 0 & \cdots & 0\\ -b_1 & -\lambda & b_2 & 0 & \cdots & 0\\ 0 & -b_2 & -\lambda & b_3 & \cdots & 0\\ \vdots & & & & \ddots & \vdots\\ 0 & & \cdots & 0 & -b_{n-1} & -\lambda \end{bmatrix}, B=\begin{bmatrix}0\\ \vdots\\ 0\\ 1\end{bmatrix}.$$

It can be easily checked that the function

$$V(z,z')=\sqrt{(z-z')^T(z-z')},$$

satisfies

$$\frac{\partial V}{\partial z}(Az+B\widehat{u})+\frac{\partial V}{\partial z'}(Az'+B\widehat{u}')\leq -\lambda V(z,z')+|\widehat{u}-\widehat{u}'|. \quad \text{(III.7)}$$

Hence the function V satisfies conditions (i) and (iii) in Definition 2.4 implying that it is a $\delta_\exists$-ISS Lyapunov function for Σ'. Using Theorem 2.6, we obtain that Σ' is $\delta_\exists$-ISS with respect to the input $\widehat{\upsilon}$. Using Lemma 2.5, we conclude that the function:

$$\widehat{V}(x,x')=V(\psi(x),\psi(x'))=\sqrt{(\psi(x)-\psi(x'))^T(\psi(x)-\psi(x'))}=\sqrt{\sum_{l=0}^{n-1}\Big[(x_{l+1}-\phi_l(x))-\left(x'_{l+1}-\phi_l(x')\right)\Big]^2},$$

is a $\delta_\exists$-ISS Lyapunov function for Σ. Therefore, using Theorem 2.6, we obtain that Σ is $\delta_\exists$-ISS with respect to the input $\widehat{\upsilon}$. The $\delta_\exists$-ISS condition (II.2), as shown in Theorem 2.6, is given by:

$$\mathbf{d}\left(\xi_{x\widehat{\upsilon}}(t),\xi_{x'\widehat{\upsilon}'}(t)\right)\leq 2e^{-\lambda t}\mathbf{d}(x,x')+\frac{2}{\lambda}|\widehat{\upsilon}-\widehat{\upsilon}'|_\infty,$$

where $\mathbf{d}(x,x')=\|\psi(x)-\psi(x')\|$, for any $x,x'\in\mathbb{R}^n$. ■

Remark 3.2: It can be readily seen that the state feedback control law (III.2) renders the control system Σ $\delta_\exists$-GAS and the function

$$\widehat{V}(x,x')=\sqrt{\sum_{l=0}^{n-1}\Big[(x_{l+1}-\phi_l(x))-\left(x'_{l+1}-\phi_l(x')\right)\Big]^2},$$

is a $\delta_\exists$-GAS Lyapunov function for Σ.

Now, we extend the result in Theorem 3.1 to the class of control systems $\Sigma=(\mathbb{R}^n,\mathsf{U},\mathcal{U},f)$ with f of the strict-feedback form [4]:

$$\begin{array}{rcl} f_1(x,u) &=& h_1(x_1)+g_1(x_1)x_2,\\ f_2(x,u) &=& h_2(x_1,x_2)+g_2(x_1,x_2)x_3,\\ &\vdots& \\ f_{n-1}(x,u) &=& h_{n-1}(x_1,\cdots,x_{n-1})\\ && +g_{n-1}(x_1,\cdots,x_{n-1})x_n,\\ f_n(x,u) &=& h_n(x)+g_n(x)u, \end{array} \quad \text{(III.8)}$$

where $x\in\mathbb{R}^n$ is the state and $u\in\mathsf{U}\subseteq\mathbb{R}$ is the control input. The functions $h_i:\mathbb{R}^i\rightarrow\mathbb{R}$, and $g_i:\mathbb{R}^i\rightarrow\mathbb{R}$, for $i=1,\ldots,n$, are smooth, and $g_i(x_1,\cdots,x_i)\neq 0$ over the domain of interest.

Theorem 3.3: Let $\Sigma=(\mathbb{R}^n,\mathsf{U},\mathcal{U},f)$ be a control system where f is of the form (III.8). Consider the state feedback control law $u=k(\varphi(x),\widehat{u})$, where k is the controller, defined in (III.2) for the control system $\Sigma'=(\mathbb{R}^n,\mathsf{U},\mathcal{U},\varphi_*f)$, where $\varphi:\mathbb{R}^n\rightarrow\mathbb{R}^n$ is a global diffeomorphism, defined by:

$$\varphi(x)=\begin{bmatrix} x_1\\ g_1(x_1)x_2\\ g_1(x_1)g_2(x_1,x_2)x_3\\ \vdots\\ \prod_{i=1}^{n-1}g_i(x_1,\cdots,x_i)x_n\end{bmatrix}. \quad \text{(III.9)}$$

The proposed control law renders control system Σ $\delta_\exists$-ISS with respect to the input $\widehat{\upsilon}$ and the function

$$\widetilde{V}(x,x')=\sqrt{(\psi\circ\varphi(x)-\psi\circ\varphi(x'))^T(\psi\circ\varphi(x)-\psi\circ\varphi(x'))},$$

where ψ was defined in (III.5), is a $\delta_\exists$-ISS Lyapunov function for Σ.

Proof: As showed in [2], the coordinate transformation $y=\varphi(x)$ transforms the control system $\Sigma=(\mathbb{R}^n,\mathsf{U},\mathcal{U},f)$ with f of the form (III.8) to the control system $\Sigma'=(\mathbb{R}^n,\mathsf{U},\mathcal{U},f')$, where $f'=\varphi_*f$ has the following form:

$$\begin{aligned} f_1'(y,u) &= h_1'(y_1) + y_2, \\ f_2'(y,u) &= h_2'(y_1,y_2) + y_3, \\ &\vdots \\ f_{n-1}'(y,u) &= h_{n-1}'(y_1,\cdots,y_{n-1}) + y_n, \\ f_n'(y,u) &= h_n'(y) + g'(y)u, \end{aligned} \tag{III.10}$$

where $h_i' : \mathbb{R}^i \to \mathbb{R}$, for $i = 1, \cdots, n$, are smooth functions, $g' = \prod_{i=1}^{i=n} g_i$, and $y \in \mathbb{R}^n$ is the state of Σ'. As proved in Theorem 3.1, the state feedback control law $k(y, \widehat{u})$, defined in (III.2), makes the function

$$\widehat{V}(y,y') = \sqrt{(\psi(y) - \psi(y'))^T (\psi(y) - \psi(y'))}, \tag{III.11}$$

a $\delta_\exists$-ISS Lyapunov function, for the control system Σ'. As proved in Lemma 2.5, the function

$$\widetilde{V}(x,x') = \widehat{V}(\varphi(x),\varphi(x')) = \sqrt{(\psi\circ\varphi(x) - \psi\circ\varphi(x'))^T (\psi\circ\varphi(x) - \psi\circ\varphi(x'))},$$

is a $\delta_\exists$-ISS Lyapunov function , for the control system Σ, equipped with the state feedback control law $k(\varphi(x), \widehat{u})$. Therefore, the state feedback control law $k(\varphi(x), \widehat{u})$ makes the control system Σ $\delta_\exists$-ISS with respect to the input $\widehat{\upsilon}$. The $\delta_\exists$-ISS condition (II.2), as shown in Theorem 2.6, is given by:

$$\mathbf{d}\left(\xi_{x\widehat{\upsilon}}(t), \xi_{x'\widehat{\upsilon}'}(t)\right) \le 2e^{-\lambda t}\mathbf{d}(x,x') + \frac{2}{\lambda}|\widehat{\upsilon} - \widehat{\upsilon}'|_\infty,$$

where $\mathbf{d}(x,x') = \|\psi\circ\varphi(x) - \psi\circ\varphi(x')\|$, for any $x, x' \in \mathbb{R}^n$. ■

Remark 3.4: It can be readily seen that the state feedback control law $k(\varphi(x), \widehat{u})$, where k is the controller, defined in (III.2) for the control system $\Sigma' = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, \varphi_* f)$, renders the control system Σ $\delta_\exists$-GAS and the function

$$\widetilde{V}(x,x') = \sqrt{(\psi\circ\varphi(x) - \psi\circ\varphi(x'))^T (\psi\circ\varphi(x) - \psi\circ\varphi(x'))},$$

is a $\delta_\exists$-GAS Lyapunov function for Σ.

IV. Backstepping Controller Design for an Electrical Power System

We illustrate the results in this paper on a single-machine infinite-bus electrical power system with static VAR compensator [13]. The control system $\Sigma = \left(\mathbb{R}^3, \mathsf{U}, \mathcal{U}, f\right)$ with f of the form:

$$\begin{aligned} f_1(x,u) &= x_2, \\ f_2(x,u) &= -\frac{\omega_0}{H}E_q'V_s y_{svc0}\sin(x_1+\delta_0) - \frac{D}{H}x_2 \\ &\quad + \frac{\omega_0}{H}P_m - \frac{\omega_0}{H}E_q'V_s\sin(x_1+\delta_0)x_3, \\ f_3(x,u) &= -\frac{1}{T_{svc}}x_3 + \frac{1}{T_{svc}}u, \end{aligned} \tag{IV.1}$$

models a single-machine infinite-bus (SIMB) electrical power system with static VAR compensator (SVC). In the mentioned model, x_1 is the deviation of the generator rotor angle, x_2 is the relative speed of the rotor of the generator, x_3 is the deviation of the susceptance of the overall system, δ_0 is the operating point of the generator rotor angle, ω_0 is the operating point of the speed of the generator rotor, H is the inertia constant, P_m is the mechanical power on the generator shaft, D is the damping coefficient, E_q' is the inner generator voltage, V_s is the infinite bus voltage, y_{svc0} is the operating point of the susceptance of the overall system, T_{svc} is the time constant of SVC regulator, and u is the input of SVC regulator. We assume that $\sin(x_1 + \delta_0)$ is nonzero over the domain of the interest.

The control system (IV.1) is of the form (III.8). The coordinate transformation (III.9), given by:

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^T = \varphi(x) = \begin{bmatrix} x_1 & x_2 & -\frac{\omega_0}{H}E_q'V_s\sin(\delta_0+x_1)x_3 \end{bmatrix}^T, \tag{IV.2}$$

transforms the control system Σ to the control system $\Sigma' = \left(\mathbb{R}^3, \mathsf{U}, \mathcal{U}, f'\right)$ with $f' = \varphi_* f$ of the form:

$$\begin{aligned} f_1'(y,u) &= h_1'(y_1) + y_2 = y_2, \\ f_2'(y,u) &= h_2'(y_1,y_2) + y_3 = -\frac{D}{H}y_2 + \frac{\omega_0}{H}P_m \\ &\quad - \frac{\omega_0}{H}E_q'V_s y_{svc0}\sin(y_1+\delta_0) + y_3, \\ f_3'(y,u) &= h_3'(y) + g'(y)u = y_2\cot(y_1+\delta_0)y_3 \\ &\quad - \frac{1}{T_{svc}}y_3 - \frac{\omega_0}{HT_{svc}}E_q'V_s\sin(y_1+\delta_0)u. \end{aligned} \tag{IV.3}$$

By using the results in Theorem 3.1 for a control system of the form (IV.3) and for $\lambda = 2$, we have:

$$\begin{aligned} \phi_1(y_1) &= -2y_1, \\ \phi_2(y_1,y_2) &= -5y_1 + \frac{\omega_0}{H}E_q'V_s y_{svc0}\sin(y_1+\delta_0) \\ &\quad - \frac{\omega_0}{H}P_m + \left(\frac{D}{H} - 4\right)y_2, \\ k_3(y) &= -12y_1 + \left(\frac{D}{H} - 6\right)y_3 + \left(\frac{D}{H} - 6\right)\frac{\omega_0}{H}P_m \\ &\quad + \left(6 - \frac{D}{H}\right)\frac{\omega_0}{H}E_q'V_s y_{svc0}\sin(y_1+\delta_0) \\ &\quad + \frac{\omega_0}{H}E_q'V_s y_{svc0}\cos(y_1+\delta_0)y_2 \\ &\quad + \left(6\frac{D}{H} - \frac{D^2}{H^2} - 14\right)y_2. \end{aligned}$$

Therefore, the state feedback control law:

$$\begin{aligned} k(y,\widehat{u}) &= -\frac{HT_{svc}}{\omega_0 E_q'V_s\sin(\delta_0+y_1)}\Bigg[-12y_1 \\ &\quad + \left(\frac{D}{H} - 6 + \frac{1}{T_{svc}}\right)y_3 \\ &\quad + \left(6 - \frac{D}{H}\right)\frac{\omega_0}{H}E_q'V_s y_{svc0}\sin(y_1+\delta_0) \\ &\quad + \frac{\omega_0}{H}E_q'V_s y_{svc0}\cos(y_1+\delta_0)y_2 \\ &\quad + \left(6\frac{D}{H} - \frac{D^2}{H^2} - 14\right)y_2 + \left(\frac{D}{H} - 6\right)\frac{\omega_0}{H}P_m \\ &\quad - y_2\cot(y_1+\delta_0)y_3\Bigg] - \frac{HT_{svc}\widehat{u}}{\omega_0 E_q'V_s\sin(\delta_0+y_1)}, \end{aligned} \tag{IV.4}$$

makes the control system Σ' $\delta_\exists$-ISS with respect to the input $\widehat{\upsilon}$. The corresponding $\delta_\exists$-ISS Lyapunov function for

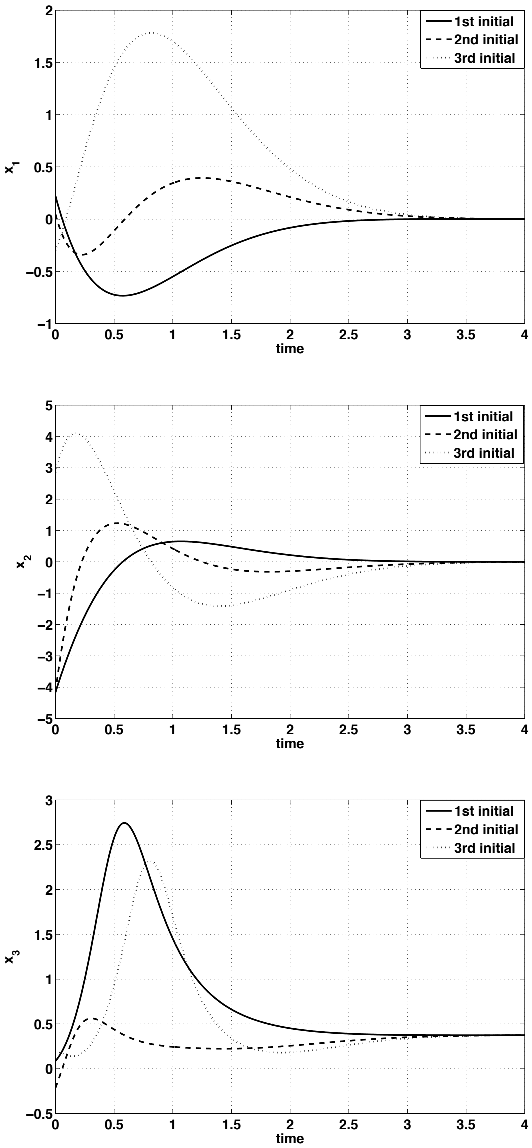

Fig. 1. Evolution of x_1, x_2, and x_3 with initial conditions $(0.2217,\ -4.159,\ 0.086)$, $(0.0471,\ -4.159,\ -0.214)$, and $(-0.3019,\ 2.841,\ 0.186)$, respectively.

the control system (IV.3) is given by:

$$\widehat{V}(y,y') = \Bigg[(y_1-y_1')^2+\big(2(y_1-y_1')+(y_2-y_2')\big)^2 + \bigg[(y_3-y_3')+\left(4-\frac{D}{H}\right)(y_2-y_2') + 5(y_1-y_1')-\frac{\omega_0}{H}E_q'V_s y_{svc0}(\sin(y_1+\delta_0) - \sin(y_1'+\delta_0))\bigg]^2\Bigg]^{\frac{1}{2}}.$$

By using Theorem 3.3, the state feedback control law (IV.4), and the coordinate transformation (IV.3), we obtain the state feedback control law $k(\varphi(x),\widehat{u})$ making Σ $\delta_\exists$-ISS with respect to the input $\widehat{v}$. The corresponding $\delta_\exists$-ISS Lyapunov function for the control system Σ is given by:

$$\widetilde{V}(x,x') = \Bigg[(x_1-x_1')^2+\big(2(x_1-x_1')+(x_2-x_2')\big)^2 + \bigg[-\frac{\omega_0}{H}E_q'V_s\bigg(\sin(\delta_0+x_1)x_3 - \sin(\delta_0+x_1')x_3'\bigg)+\left(4-\frac{D}{H}\right)(x_2-x_2') + 5(x_1-x_1')-\frac{\omega_0}{H}E_q'V_s y_{svc0}(\sin(x_1+\delta_0) - \sin(x_1'+\delta_0))\bigg]^2\Bigg]^{\frac{1}{2}}.$$

We simulate the closed-loop system with $\widehat{u}=0$, and the following parameters: $\omega_0 = 314.159$, $H = 5.9$, $E_q' = 1$, $V_s = 1$, $y_{svc0} = 0.814$, $\delta_0 = 57.3^\circ$, $D = 5$, $P_m = 1$, and $T_{svc} = 0.02$. In Figure 1, we show the closed-loop trajectories stemming from the initial conditions $(0.2217,\ -4.159,\ 0.086)$, $(0.0471,\ -4.159,\ -0.214)$, and $(-0.3019,\ 2.841,\ 0.186)$, respectively.

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