2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Invariant Extended Kalman Filter Design for a Magnetometer-plus-GPS Aided Inertial Navigation System

Martin Barczyk and Alan F. Lynch

***Abstract*— We introduce a magnetometer-plus-GPS aided inertial navigation system for a helicopter UAV. A nonlinear observer is required to estimate the navigation states, typically an Extended Kalman Filter (EKF). A novel approach is the invariant observer, a constructive design method applicable to systems possessing symmetries. We review the theory and design an invariant observer for our example. Using an invariant observer guarantees a simplified form of the nonlinear estimation error dynamics. These are stabilized using an adaptation of the Invariant EKF, a systematic approach to compute the gains of an invariant observer. The resulting design is successfully implemented and validated in experiment and shows an improvement in performance over a conventional EKF.**

I. INTRODUCTION

The University of Alberta's Applied Nonlinear Controls Lab (ANCL) Helicopter UAV project has motivated a number of research directions including aided navigation, e.g. [1] reports on experimental magnetometer integration into an Extend Kalman Filter (EKF)-based design. Aided navigation is a nonlinear observer design problem, and we are interested in using techniques beyond the conventional EKF. The catalogue of these is extensive, c.f. the survey paper [2].

A novel nonlinear design method is the invariant observer [3], [4]. This method provides a systematic approach to deriving the observer equations, and aided navigation applications such as Attitude and Heading Reference Systems (AHRS) and Aided Inertial Navigation Systems (INS) are described by dynamics which naturally possess the required symmetries (defined in Section II-B). The key feature of an invariant observer is that it guarantees a simplified form of the (nonlinear) estimation error dynamics (Theorem 1 in Section II-C), which simplifies the selection of observer gains.

Nonlinear gain selection is not systematic. A natural first-pass approach is to rely on linearization to design the gains, more specifically the EKF method of continuously re-linearizing a dynamic system about its latest estimate. The approach of combining the invariant observer's estimation error dynamics with the EKF was first proposed by [5].

The contribution of this paper is to treat the invariant observer design for a magnetometer-plus-GPS aided INS example, which has not been considered except in [6] for a different set of dynamics and symmetries; to adapt the Invariant EKF design method in [5] to this system as explained in Section III; and to experimentally validate the design and compare it with a conventional EKF in Section IV.

The authors are with the Department of ECE, University of Alberta, Edmonton, AB, T6G 2V4, Canada `martinb@ece.ualberta.ca`; `alan.lynch@ualberta.ca`

II. INVARIANT OBSERVER DESIGN

A. System Dynamics

The aided INS dynamics are described in a local tangent plane as

$$\begin{aligned}\dot{p} &= v\\ \dot{v} &= R(\tilde{f} - b_f - \nu_f) - a\\ \dot{R} &= RS(\tilde{\omega} - b_\omega - \nu_\omega)\\ \dot{b}_f &= \nu_{bf}\\ \dot{b}_\omega &= \nu_{b\omega}\\ \begin{bmatrix}\tilde{y}_p\\ \tilde{y}_m\end{bmatrix} &= \begin{bmatrix}p+\nu_p\\ R^T m + \nu_m\end{bmatrix},\end{aligned} \tag{1}$$

where p, v are the vehicle's position and velocity vectors in the North-East-Down (NED) navigation frame; $R \in SO(3)$ measures the attitude; S is a skew-symmetric matrix such that $S(x)y = x \times y$ where $x, y \in \mathbb{R}^3$; b_f and b_ω are the unknown biases of the measured accelerometer $\tilde{f}$ and rate gyro $\tilde{\omega}$ signals; a and m are the local gravity and magnetic field vectors, which are known constants[1]; $\tilde{y}_p$ and $\tilde{y}_m$ are provided by the on-board GPS receiver and magnetometer, respectively; and ν terms represent Gaussian white-noise vectors with zero mean and diagonal covariance matrices Q_ν, whose values can be identified from logged sensor data [7, Chap. 4]. Remark (1) assumes a random-walk (Wiener) process model for the biases.

We will first design an invariant observer for the nominal (noise-free) version of (1), i.e. with $\nu = 0$. We denote this noise-free system as $\dot{x} = f(x,u)$, $y = h(x,u)$ where $x = [p, v, R, b_f, b_\omega]^T$, $u = [f, \omega, a, m]^T$ and $y = [y_p, y_m]^T$ are the state, input and output vectors, respectively. The noise terms will be brought back for the Invariant EKF design in Section III.

B. System Symmetries

The most general coordinate-free description of the smooth nonlinear dynamics $\dot{x} = f(x,u)$ is the bundle map $F : B \to TX$ where B is the total space of a smooth fiber bundle over the smooth state manifold X with tangent bundle TX, e.g. [8, Sec. 13.5]. This framework is used in [9] to define system symmetries. For invariant observer design problems, we are able to use the simplification $B = X \times U$ where U represents the smooth input manifold, i.e. B is a trivial fiber bundle over X, because the inputs to the observer

[1]The gravity field is $a = [0, 0, -9.81]^T$ [m/s^2]; the magnetic field can be computed from a world magnetic model, e.g. $m = [0.1402, 0.03957, 0.5602]^T$ [G] at flight coordinates 53°25′12”N, 113°23′58”W.

are not functions of its state. Remark this simplification would not apply for invariant feedback control design [10], [11]. Using the trivial bundle assumption, take $\varphi_g : G\times X \to X$, $\psi_g : G \times U \to U$ and $\rho_g : G \times Y \to Y$ to be smooth left Lie group actions acting on the system's state, input and output manifolds, respectively, where G is the Lie group associated to the system. Let $H : X \times U \to Y$ denote the output map. The system is said to be G-invariant and G-equivariant if for all $g \in G$,

$$(\varphi_g)_* F(x,u) = F(\varphi_g(x), \psi_g(u)) \quad \text{and}$$
$$\rho_g H(x,u) = H(\varphi_g(x), \psi_g(u)),$$

respectively, where $(\quad)_*$ denotes the pushforward. In local coordinates, this is equivalent to

$$\frac{d}{dt}\left(\varphi_g(x)\right) = f(\varphi_g(x), \psi_g(u)),$$
$$\rho_g y = h(\varphi_g(x), \psi_g(u)).$$

Finding the symmetry group G and set of actions φ_g, ψ_g making the system G-invariant is non-constructive, but is based on the physics of the problem. For the noise-free version of (1), the Lie group $G = \mathbb{R}^3 \times SO(3) \times \mathbb{R}^3 \times \mathbb{R}^3 \ni (p_0 \quad R_0 \quad b_{f0} \quad b_{\omega 0}) = g$ with group actions

$$\varphi_g \begin{pmatrix} p \\ v \\ R \\ b_f \\ b_\omega \end{pmatrix} = \begin{pmatrix} R_0(p+p_0) \\ R_0 v \\ R_0 R \\ b_f + b_{f0} \\ b_\omega + b_{\omega 0} \end{pmatrix}, \quad \psi_g \begin{pmatrix} f \\ \omega \\ a \\ m \end{pmatrix} = \begin{pmatrix} f + b_{f0} \\ \omega + b_{\omega 0} \\ R_0 a \\ R_0 m \end{pmatrix}$$

can be directly verified to provide G-invariance as well as G-equivariance under the induced action

$$\rho_g \begin{pmatrix} y_p \\ y_m \end{pmatrix} = \begin{pmatrix} R_0(y_p + p_0) \\ y_m \end{pmatrix}.$$

The actions φ_g, ψ_g, ρ_g can be verified to be left actions under group multiplication $g_1 g_2 = (p_0' + p_0'', R_0' R_0'', b_{f0}' + b_{f0}'', b_{\omega 0}' + b_{\omega 0}'')$. Physically, the entries of G represent constant offsets in the position and bias states and constant rotations of the NED frame, and the set of actions φ_g, ψ_g, ρ_g applies these to the state, input and output spaces in such a way that the system's dynamics and output equations are not altered, i.e. invariant.

C. Observer Design

Once the system's symmetries are available, the invariant observer design steps are systematic [3].

1) Invariants of G: Define the group action $\phi_g = \varphi_g \times \psi_g \times \rho_g$ of G acting regularly and freely on the smooth manifold $M = X \times U \times Y$, where $\dim(M) = m$ and $\dim(G) = r$ with $r \leq m$. The function $J : M \to \mathbb{R}$ is defined to be an invariant of G if $J(\phi_g(p)) = J(p), p \in M, \forall g \in G$. The complete set of invariants of G is obtained using the method of normalization [12, p.161]: write the group action ϕ_g in coordinates and partition it into components $(\phi_g^a, \phi_g^b) \in \mathbb{R}^r \times \mathbb{R}^{m-r}$. Choose $c = (c_1, \ldots, c_r)$ in the range of ϕ_g^a, then solve $\phi_g^a(x) = c$ for $g = \gamma(x)$, the moving frame, whose existence is guaranteed at least locally by the Implicit Function Theorem. The complete set of $(m-r)$ invariants of G is then given by $\phi^b_{\gamma(x)}$; any other invariant can be uniquely expressed as an analytic function of these [12, Thm. 8.17].

For our example, we partition ϕ_g as $\phi_g^a = \varphi_g^a = \varphi_g^a(p, R, b_f, b_\omega)$ and $\phi_g^b = \varphi_g^b \times \psi_g \times \rho_g$ and solve $\phi_g^a(x) = c$ for the moving frame:

$$\varphi_g^a \begin{pmatrix} p \\ R \\ b_f \\ b_\omega \end{pmatrix} = \begin{pmatrix} R_0(p+p_0) \\ R_0 R \\ b_f + b_{f0} \\ b_\omega + b_{\omega 0} \end{pmatrix} = \begin{pmatrix} 0 \\ I \\ 0 \\ 0 \end{pmatrix} \Longrightarrow \gamma(x) = \begin{pmatrix} -p \\ R^T \\ -b_f \\ -b_\omega \end{pmatrix}$$

The complete set of invariants is then

$$\phi^b_{\gamma(x)} = \begin{pmatrix} \varphi^b_{\gamma(x)}(v) \\ \psi_{\gamma(x)} \begin{pmatrix} f \\ \omega \\ a \\ m \end{pmatrix} \\ \rho_{\gamma(x)} \begin{pmatrix} y_p \\ y_m \end{pmatrix} \end{pmatrix} = \begin{pmatrix} R^T v \\ f - b_f \\ \omega - b_\omega \\ R^T a \\ R^T m \\ R^T(y_p - p) \\ y_m \end{pmatrix} := \begin{pmatrix} I(x,u) \\ J_h(x,y) \end{pmatrix}$$

2) Invariant Frame: A vector field $w : X \to TX$ is defined to be G-invariant if $(\varphi_g)_* w(x) = w(\varphi_g(x)) \, \forall g \in G$. For $\dim(X) = n$, an invariant frame is defined as an n-tuple of G-invariant vector fields (w_i) which form a global frame for TX over X, i.e. for each $p \in M$, $\big(w_1(p), \ldots, w_n(p)\big)$ forms a basis for the fiber $\pi^{-1}(p) = T_p X$. It can be verified [3, Lem. 1] that an invariant frame is given by

$$w_i(x) = \left(\varphi_{\gamma(x)^{-1}}\right)_* v_i = \frac{d}{d\tau}\Big(\varphi_{\gamma(x)^{-1}}(v_i \tau)\Big)\Big|_{\tau=0}$$

where $(v_i) \in T_e X$ are a set of basis vectors for $T_e X$. The latter form is used to evaluate the pushforward.

In our example, $X = \mathbb{R}^3 \times \mathbb{R}^3 \times SO(3) \times \mathbb{R}^3 \times \mathbb{R}^3$, and we use basis vectors of $T_0\mathbb{R}^3 \ni v_i = e_i$ and $T_I SO(3) \ni v_i = S(e_i)$ where (e_i) denote the standard basis for $\mathbb{R}^3$. We have $\gamma(x)^{-1} = (p, R, b_f, b_\omega)$ and compute

$$\frac{d}{d\tau}\,\varphi_{\gamma(x)^{-1}} \begin{pmatrix} e_i\tau \\ e_i\tau \\ S(e_i)\tau \\ e_i\tau \\ e_i\tau \end{pmatrix}\Bigg|_{\tau=0} = \frac{d}{d\tau} \begin{pmatrix} R(e_i\tau + p) \\ R e_i \tau \\ R S(e_i)\tau \\ e_i\tau + b_f \\ e_i\tau + b_\omega \end{pmatrix}\Bigg|_{\tau=0}$$
$$\Longrightarrow \begin{pmatrix} w_i^p(x) \\ w_i^v(x) \\ w_i^R(x) \\ w_i^{bf} \\ w_i^{b\omega} \end{pmatrix} = \begin{pmatrix} R e_i \\ R e_i \\ R S(e_i) \\ e_i \\ e_i \end{pmatrix}$$

3) Invariant Observer: An invariant observer for the G-invariant, G-equivariant system $\dot{x} = f(x,u)$, $y = h(x,u)$ is $\dot{\hat{x}} = F(\hat{x}, u, y)$ with the following three properties:

- $F(x, u, h(x,u)) = f(x,u)$ (F is a pre-observer)
- $\hat{x} - x \to 0$ as $t \to \infty$ for all $\hat{x}(0)$, or for all $\hat{x}(0)$ close to $x(0)$ (F is globally or locally asymptotic, respectively)
- $\left(\varphi_g\right)_* F(\hat{x}, u, y) = F(\varphi_g(\hat{x}), \psi_g(u), \rho_g(y))$ (F is G-invariant)

By [3, Thm. 1], $F(\hat{x},u,y)$ is an invariant *pre*-observer if and only if

$$F(\hat{x},u,y)=f(\hat{x},u)+\sum_{i=1}^{n}\mathcal{L}_i\left(I(\hat{x},u),E(\hat{x},u,y)\right)w_i(\hat{x}), \tag{2}$$

where $\mathcal{L}_i$ are real-valued functions of the estimated invariants $I(\hat{x},u)$ and the invariant output error

$$E(\hat{x},u,y)=J_h(\hat{x},h(\hat{x},u))-J_h(\hat{x},y),$$

which by construction verifies $E(\hat{x},u,y)=0\iff y=h(\hat{x},u)$ and $E(\varphi_g(\hat{x}),\psi_g(u),\rho_g(y))=E(\hat{x},u,y)$. Obviously, the convergence properties of the observer depend on the choice of $\mathcal{L}_i$, and a general method to compute these is not available; however the stability analysis is simplified due to the key result in Section II-C.4 below.

In our example, the invariant output error $E(\hat{x},u,y)$ is computed as

$$\begin{pmatrix}\hat{R}^T(\hat{p}-\hat{p})\\ \hat{R}^Tm\end{pmatrix}-\begin{pmatrix}\hat{R}^T(y_p-\hat{p})\\ y_m\end{pmatrix}=\begin{pmatrix}\hat{R}^T(\hat{p}-y_p)\\ \hat{R}^Tm-y_m\end{pmatrix}=\begin{pmatrix}E_p\\ E_m\end{pmatrix}$$

and the invariant pre-observer is

$$\begin{aligned}\dot{\hat{p}}&=\hat{v}+\hat{R}\big(L_p^pE_p+L_m^pE_m\big)\\ \dot{\hat{v}}&=\hat{R}(f-\hat{b}_f)-a+\hat{R}\big(L_p^vE_p+L_m^vE_m\big)\\ \dot{\hat{R}}&=\hat{R}S(\omega-\hat{b}_\omega)+\hat{R}S\big[L_p^RE_p+L_m^RE_m\big]\\ \dot{\hat{b}}_f&=L_p^{bf}E_p+L_m^{bf}E_m\\ \dot{\hat{b}}_\omega&=L_p^{b\omega}E_p+L_m^{b\omega}E_m\end{aligned} \tag{3}$$

where each L is a 3×3 gain matrix whose entries are functions of $I(\hat{x},u)$ and $E(\hat{x},u,y)$.

4) Invariant Estimation Error: The stability analysis of the invariant pre-observer is simplified by considering the invariant estimation error

$$\eta(x,\hat{x})=\varphi_{\gamma(x)}(\hat{x})-\varphi_{\gamma(x)}(x) \tag{4}$$

which by construction verifies $\eta(x,\hat{x})=0\iff x=\hat{x}$ and $\eta\left(\varphi_g(x),\varphi_g(\hat{x})\right)=\eta(x,\hat{x})$. The convergence of $\hat{x}$ to x is equivalent to the stability of η dynamics, whose analysis is (potentially greatly) simplified due to the following [3, Thm. 3]:

Theorem 1: For a G-invariant, G-equivariant system $\dot{x}=f(x,u)$, $y=h(x,u)$ with associated invariant observer (2), the dynamics of the invariant estimation error (4) depend on the system only through its estimated invariants:

$$\frac{d}{dt}\eta=\Upsilon\left(\eta,I(\hat{x},u)\right).$$

In our example, the invariant estimation error $\eta=\varphi_{\gamma(x)}(\hat{x})-\varphi_{\gamma(x)}(x)$ is computed as

$$\begin{pmatrix}R^T(\hat{p}-p)\\ R^T\hat{v}\\ R^T\hat{R}\\ \hat{b}_f-b_f\\ \hat{b}_\omega-b_\omega\end{pmatrix}-\begin{pmatrix}R^T(p-p)\\ R^Tv\\ R^TR\\ b_f-b_f\\ b_\omega-b_\omega\end{pmatrix}=\begin{pmatrix}R^T(\hat{p}-p)\\ R^T(\hat{v}-v)\\ R^T\hat{R}-I\\ \hat{b}_f-b_f\\ \hat{b}_\omega-b_\omega\end{pmatrix}=\begin{pmatrix}\eta_p\\ \eta_v\\ \eta_R\\ \eta_{bf}\\ \eta_{b\omega}\end{pmatrix}$$

For convenience, we re-define $\eta_R=R^T\hat{R}$, such that $R=\hat{R}\iff\eta_R=I$ (instead of 0). The estimated invariants are

$$I(\hat{x},u)=\begin{pmatrix}\hat{R}^T\hat{v}\\ f-\hat{b}_f\\ \omega-\hat{b}_\omega\\ \hat{R}^Ta\\ \hat{R}^Tm\end{pmatrix}:=\begin{pmatrix}I_v\\ I_f\\ I_\omega\\ I_a\\ I_m\end{pmatrix}$$

The invariant output error E can be written as

$$\begin{aligned}E_p&=\hat{R}^T(\hat{p}-p)=(\eta_R)^T\eta_p\\ E_m&=\hat{R}^Tm-R^Tm=I_m-\eta_RI_m\end{aligned}$$

and the η dynamics are directly evaluated to be

$$\begin{aligned}\dot{\eta}_p&=-S\big[I_\omega+\eta_{b\omega}\big]\eta_p+\eta_v+\eta_R\big(L_p^pE_p+L_m^pE_m\big)\\ \dot{\eta}_v&=-S\big[I_\omega+\eta_{b\omega}\big]\eta_v+\eta_RI_f\\ &\quad+\eta_R\big(L_p^vE_p+L_m^vE_m\big)-I_f-\eta_{bf}\\ \dot{\eta}_R&=-S\big[I_\omega+\eta_{b\omega}\big]\eta_R+\eta_RS(I_\omega)\\ &\quad+\eta_RS\big[L_p^RE_p+L_m^RE_m\big]\\ \dot{\eta}_{bf}&=L_p^{bf}E_p+L_m^{bf}E_m\\ \dot{\eta}_{b\omega}&=L_p^{b\omega}E_p+L_m^{b\omega}E_m\end{aligned} \tag{5}$$

i.e. $\dot{\eta}$ depends on the system only through $I(\hat{x},u)$ terms, as guaranteed by Theorem 1.

III. INVARIANT EKF

A direct nonlinear design to stabilize (5) by choice of gains L is clearly the most elegant solution. The use of rotation matrices R provides a one-to-one parametrization of the state manifold, making it possible to perform a global stability analysis. Of course this is also very difficult and non-systematic. Instead, we use an EKF-based approach to gain assignment by re-linearizing the invariant estimation error dynamics (5) about the current estimated state $\hat{x}$. The Kalman filter is an optimal observer for systems with process and measurement noise, so the computed gains will provide an invariant observer with increased robustness to sensor noise. The obvious disadvantage is that by relying on linearization, we can guarantee only local stability, whose region of attraction is difficult to quantify, e.g. [13]. Using an EKF to compute the gains of an invariant observer was first proposed in [5] for the subclass of invariant systems where the symmetry group G acts on itself by left or right multiplication [4]. This method was later applied to different examples in [14], [15]. We modify the proposed method to make it applicable to our working example (1).

A. Method Overview

We review the continuous-time Kalman filter, e.g. [16, Chap. 7]. The filter applies to LTV systems

$$\begin{aligned}\dot{x}&=A(t)x+B(t)w\\ y&=C(t)x+D(t)v\end{aligned} \tag{6}$$

where w and v are the process and measurement white noise vectors with covariance matrices Q_ν, R_ν, respectively, and zero correlation: $E\langle wv^T\rangle = 0$. The Kalman filter for (6) is

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + K(y - C\hat{x})\\ K &= PC^T(DR_\nu D^T)^{-1}\\ \dot{P} &= AP + PA^T - PC^T(DR_\nu D^T)^{-1}CP + BQ_\nu B^T,\end{aligned} \tag{7}$$

a linear optimal observer designed to minimize the covariance of the estimation error $e = \hat{x} - x$, i.e. $\min(E\langle e\,e^T\rangle)$. From (7) and (6), the estimation error dynamics are

$$\dot{e} = \dot{\hat{x}} - \dot{x} = (A - KC)e - Bw + KDv. \tag{8}$$

The existing invariant observer $\dot{\hat{x}} = F(\hat{x}, u, y)$ was designed for the noise-free version of (1). Bringing back the noise terms in sensor models $\tilde{u} = u + w$, $\tilde{y} = y + v$ the dynamics of (1) are

$$\dot{x} = f(x, \tilde{u} - w) \tag{9}$$

and the invariant observer is now

$$\dot{\hat{x}} = F(\hat{x}, \tilde{u}, \tilde{y}), \tag{10}$$

i.e. it employs noisy input and output measurements. We recall the invariant estimation error η (4), whose form is unaffected by the addition of w and v, compute $\dot{\eta}$ using (9) and (10), then linearize the result about $\eta = 0$, $w = 0$, $v = 0$ which results in form (8). We can then extract the matrices (A, B, C, D) and use (7) to compute the matrix of observer gains K.

B. Invariant EKF Design

The system dynamics with noise terms were given in (1) and the invariant observer in (3). Taking the latter as $F(\hat{x}, \tilde{u}, \tilde{y})$, such that $\tilde{E}_p = \hat{R}^T(\hat{p} - \tilde{y}_p)$, $\tilde{E}_m = \hat{R}^T m - \tilde{y}_m$, we compute $\dot{\eta}$ to be

$$\begin{aligned}\dot{\eta}_p &= -S\big[\tilde{I}_\omega + \eta_{b\omega}\big]\eta_p + \eta_v\\ &\quad + \eta_R\big(L_p^p\tilde{E}_p + L_m^p\tilde{E}_m\big) - S(\eta_p)\nu_\omega\\ \dot{\eta}_v &= -S\big[\tilde{I}_\omega + \eta_{b\omega}\big]\eta_v + \eta_R\tilde{I}_f\\ &\quad + \eta_R\big(L_p^v\tilde{E}_p + L_m^v\tilde{E}_m\big) - \tilde{I}_f - \eta_{bf} - S(\eta_v)\nu_\omega + \nu_f\\ \dot{\eta}_R &= -S\big[\tilde{I}_\omega + \eta_{b\omega}\big]\eta_R + \eta_R S(\tilde{I}_\omega)\\ &\quad + \eta_R S\big[L_p^R\tilde{E}_p + L_m^R\tilde{E}_m\big] + S(\nu_\omega)\eta_R\\ \dot{\eta}_{bf} &= L_p^{bf}\tilde{E}_p + L_m^{bf}\tilde{E}_m - \nu_{bf}\\ \dot{\eta}_{b\omega} &= L_p^{b\omega}\tilde{E}_p + L_m^{b\omega}\tilde{E}_m - \nu_{b\omega}\end{aligned}$$

where $\tilde{I}_\omega = \tilde{\omega} - \hat{b}_\omega$, $\tilde{I}_f = \tilde{f} - \hat{b}_f$; as expected, the above reduces to (5) for $\nu = 0$. The $\tilde{E}$ terms are expressed as

$$\begin{aligned}\tilde{E}_p &= \hat{R}^T(\hat{p} - p - \nu_p) = (\eta_R)^T\eta_p - \hat{R}^T\nu_p\\ \tilde{E}_m &= \hat{R}^T m - R^T m - \nu_m = I_m - \eta_R I_m - \nu_m\end{aligned}$$

where $I_m = \hat{R}^T m$ as before. We linearize the above about $\bar{\eta} = 0$ ($\bar{\eta}_R = I$) and $\bar{\nu} = 0$ to obtain a system in $\delta\eta = \eta - \bar{\eta}$. We have the output error terms

$$\begin{aligned}\delta\tilde{E}_p &= \tilde{E}_p - \overline{\tilde{E}}_p = \delta\eta_p - \hat{R}^T\nu_p\\ \delta\tilde{E}_m &= \tilde{E}_m - \overline{\tilde{E}}_m = -(\delta\eta_R)I_m - \nu_m\end{aligned}$$

and the linearized invariant estimation error dynamics

$$\begin{aligned}\delta\dot{\eta}_p &= -S(\tilde{I}_\omega)\delta\eta_p + \delta\eta_v + L_p^p\delta\tilde{E}_p + L_m^p\delta\tilde{E}_m\\ \delta\dot{\eta}_v &= -S(\tilde{I}_\omega)\delta\eta_v + \delta\eta_R\tilde{I}_f + L_p^v\delta\tilde{E}_p\\ &\quad + L_m^v\delta\tilde{E}_m - \delta\eta_{bf} + \nu_f\\ \delta\dot{\eta}_R &= \delta\eta_R S(\tilde{I}_\omega) - S(\tilde{I}_\omega)\delta\eta_R - S\big[\delta\eta_{b\omega}\big]\\ &\quad + S\big[L_p^R\delta\tilde{E}_p + L_m^R\delta\tilde{E}_m\big] + S\big[\nu_\omega\big]\\ \delta\dot{\eta}_{bf} &= L_p^{bf}\delta\tilde{E}_p + L_m^{bf}\delta\tilde{E}_m - \nu_{bf}\\ \delta\dot{\eta}_{b\omega} &= L_p^{b\omega}\delta\tilde{E}_p + L_m^{b\omega}\delta\tilde{E}_m - \nu_{b\omega}\end{aligned}$$

We further simplify the above by re-expressing $\delta\eta_R$ as follows: recall the rotational kinematics $\dot{R} = S(\omega)R$ where ω physically represents the angular velocity vector in the ground frame. Defining $\omega = d\gamma/dt$, the kinematics are written as $dR/dt = S(d\gamma/dt)R \Longrightarrow dR = S(d\gamma)R$. Around the linearization point $\bar{\eta}_R = I$, we can write $dR \approx \hat{R} - R$ and $d\gamma = \hat{\gamma} - \gamma$, and so $\hat{R} - R \approx S(\hat{\gamma} - \gamma)R \Longrightarrow \hat{R}R^T - I = S(\hat{\gamma} - \gamma) \Longrightarrow \delta\eta_R = S(\delta\gamma)$. Substituting this last result into the $\delta\dot{\eta}$ expressions and re-arranging into form (8) gives

$$\begin{aligned}\delta\dot{\eta} &= \underbrace{\begin{bmatrix} -S(\tilde{I}_\omega) & I & 0 & 0 & 0\\ 0 & -S(\tilde{I}_\omega) & -S(\tilde{I}_f) & -I & 0\\ 0 & 0 & -S(\tilde{I}_\omega) & 0 & -I\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}}_{A}\delta\eta\\ &\quad - \underbrace{\begin{bmatrix} L_p^p & L_m^p\\ L_p^v & L_m^v\\ L_p^R & L_m^R\\ L_p^{bf} & L_m^{bf}\\ L_p^{b\omega} & L_m^{b\omega}\end{bmatrix}}_{K}\underbrace{\begin{bmatrix} -I & 0 & 0 & 0 & 0\\ 0 & 0 & -S(I_m) & 0 & 0\end{bmatrix}}_{C}\delta\eta\\ &\quad - \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0\\ -I & 0 & 0 & 0\\ 0 & -I & 0 & 0\\ 0 & 0 & I & 0\\ 0 & 0 & 0 & I\end{bmatrix}}_{B}\underbrace{\begin{bmatrix}\nu_f\\ \nu_\omega\\ \nu_{bf}\\ \nu_{b\omega}\end{bmatrix}}_{w}\\ &\quad + \underbrace{\begin{bmatrix} L_p^p & L_m^p\\ L_p^v & L_m^v\\ L_p^R & L_m^R\\ L_p^{bf} & L_m^{bf}\\ L_p^{b\omega} & L_m^{b\omega}\end{bmatrix}}_{K}\underbrace{\begin{bmatrix} -\hat{R}^T & 0\\ 0 & -I\end{bmatrix}}_{D}\underbrace{\begin{bmatrix}\nu_p\\ \nu_m\end{bmatrix}}_{v}\end{aligned} \tag{11}$$

where $\delta\eta = [\delta\eta_p, \delta\eta_v, \delta\gamma, \delta\eta_{bf}, \delta\eta_{b\omega}]^T$.

C. Implementation Details

The invariant observer (3), along with (7) used with the matrices (11) to compute the observer gains L, are numerically implemented using the Modified Euler method. Since the sensor measurements $\tilde{u}$ are available at a higher rate than the aiding measurements $\tilde{y}$, 100 Hz and 10 Hz respectively in our case, we integrate the $f(\hat{x}, \tilde{u})$ part of (3) at the sensor rate to compute (rough) estimates of $\hat{x}$, which are output to the user and also used to build matrices (11) and perform a full integration at the aiding rate. This is precisely the complementary filter approach used in conventional EKF designs [17]. For comparison purposes, we have designed a conventional EKF for (1). The conventional and invariant EKF designs employ the same process noise covariance $Q_\nu = \text{blkdiag}(Q_{\nu_f}, Q_{\nu_\omega}, Q_{\nu_{bf}}, Q_{\nu_{b\omega}})$, measurement noise covariance $R_\nu = \text{blkdiag}(Q_{\nu_p}, Q_{\nu_m})$ and initial estimation error covariance $P(0)$, and use the same sensor data set logged during experiment as their input.

IV. EXPERIMENTAL RESULTS

A. Manual Carry

The first experimental run is performed with the engine off. The helicopter starts aligned with the edge of a rectangular landing pad. It is then picked up by hand and manually carried around the perimeter of the rectangle in two complete circuits, keeping the heading aligned with the direction of travel. The experiment is useful as a preliminary validation of the observer under controlled conditions, e.g. the geometry of the trajectory is known from direct measurements of the landing pad dimensions, and the the sensor noise covariance parameters, identified a priori under engine-off conditions and listed in Table I are directly applicable; note Q_{ν_p} is reported by the GPS receiver.

TABLE I

ENGINE-OFF SENSOR NOISE COVARIANCE MATRICES

diag(Q_{ν_f}) [m^2/s^3]	diag(Q_{ν_ω}) [rad^2/s]	diag($Q_{\nu_{bf}}$) [m^2/s^4]	diag($Q_{\nu_{b\omega}}$) [rad^2/s^2]	diag(Q_{ν_m}) [G^2 s]
0.0079^2	0.0017^2	0.0042^2	0.00029^2	0.00058^2
0.0074^2	0.0017^2	0.0020^2	0.00038^2	0.00051^2
0.0090^2	0.0021^2	0.0016^2	0.00032^2	0.00051^2

An overhead view of estimated positions (p_N, p_E) from the invariant and conventional EKFs is shown in Fig. 1 along with the measured landing pad geometry. The two designs are seen to perform essentially the same, which is confirmed by plotting the deltas between all 15 state estimates in Fig. 2. We can say that under low-noise and well-modeled conditions, the Invariant EKF and conventional EKF designs perform essentially the same.

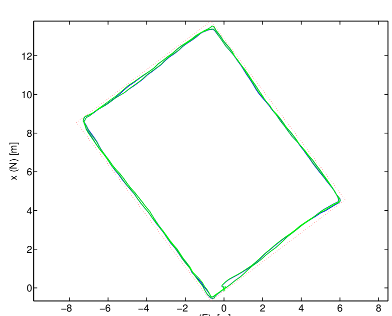

Fig. 1. Manual carry experiment: Overhead view

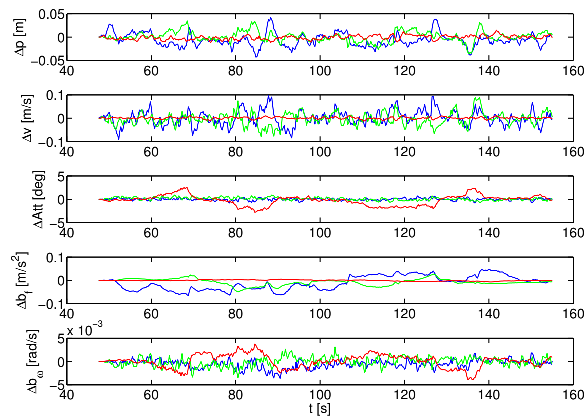

Fig. 2. Manual carry experiment: Invariant vs Conventional EKF estimation differences

B. Hover Flight Test

The more interesting experiment is engine-on flight. The main uncertainty in this setting is noise modeling. The running engine is a source of noise for the IMU sensors, due to increased vibration levels and generated electromagnetic fields. These noise effects also vary with time as a function of throttle input and state of the vehicle, for instance the helicopter passes through a resonant frequency during spool-up. The method used to identify the engine-off noise characteristics requires many hours of log data to obtain reliable results, which is not possible due to the 30 min fuel limit. In practice, we have had good success with scaling the engine-off parameters in Table I as $100\,Q_{\nu_f}$, $10\,Q_{\nu_\omega}$ and $50\,Q_{\nu_m}$ to match the order of magnitude of in-flight signal covariance, and as $0.01\,Q_{\nu_{bf}}$, $0.01\,Q_{\nu_{b\omega}}$ to heuristically increase the time constant of the bias estimates. We also add a small perturbation to the initial covariance of the bias error states, $P(\delta b_f)(0) = 1 \times 10^{-4}$ m/s^2 and $P(\delta b_\omega)(0) = 1 \times 10^{-5}$ rad/s, for better numerical performance during the spool-up resonant frequency effect.

The estimation results for the Invariant EKF design are shown in Fig. 3, where we plot only position and attitude as the signals most relevant to motion visualization. The plots identify the main stages of the experiment: the helicopter's main rotor is spooled up, creating a counter-torque on the body, which is in turn compensated by the on-board heading-hold gyro, standard equipment on R/C helicopters used to reduce pilot workload. Following a take-off period, a hover is established, characterized by a nearly-constant yaw angle ψ provided by the heading-hold gyro. Remark the estimated positions and roll, pitch vary in hover. This is due to the relatively low mass of the helicopter, making it susceptible to unsteady atmospheric effects and requiring constant corrections on the cyclic inputs. The helicopter then lands. The estimates are used to render a 3D animation, which is synchronized with a video taken during the flight experiment. The results show qualitatively good agreement.

The state estimate deltas between Invariant EKF and con-

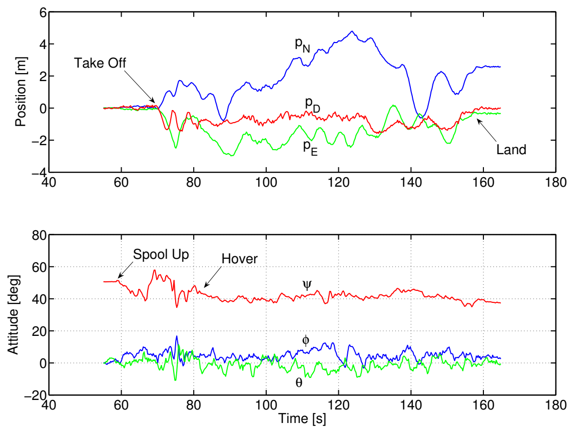

Fig. 3. Hover flight experiment: Invariant EKF

ventional EKF are plotted in Fig. 4. The position estimation is quite similar for both designs, which is expected since position is directly measurable (minus the lever-arm effect), and the receiver provides high-precision (2 cm circular error probable) measurements through carrier-phase differential GPS. The attitude estimation exhibits more differences, particularly in the yaw axis. The overall performance of both designs is sensitive to covariance tuning, so further adjustments may reduce the performance gap. Using a conventional EKF with a Gauss-Markov bias model and a modified set of covariance tunings produces very good performance, c.f. [1]. However, for the present system (1) with the given filter parameters, the Invariant EKF performs better than the conventional EKF design.

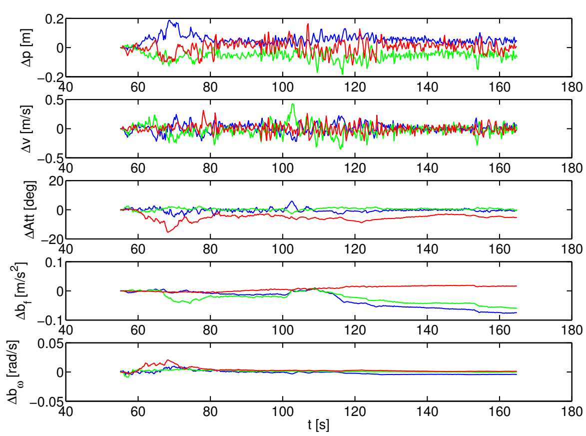

Fig. 4. Hover flight experiment: Invariant vs Conventional EKF estimation differences

V. CONCLUSIONS

We have reviewed the method of invariant observer design [3] and applied it to the problem of magnetometer-plus-GPS aided inertial navigation. The nonlinear observer gains were computed systematically using an adaptation of the Invariant EKF method proposed in [5]. The resulting design was implemented in experiment and compared to a conventional EKF, demonstrating that the invariant version performs as well or better than the conventional design in real-world testing of a helicopter UAV.

VI. ACKNOWLEDGMENTS

We would like to thank Prof. Silvère Bonnabel, Mines ParisTech, France for his time spent answering questions about invariant observer theory. This work was partially supported by the Natural Sciences and Engineering Research Council of Canada (NSERC) under grant number 249681-02.

REFERENCES

[1] M. Barczyk, M. Jost, D. R. Kastelan, A. F. Lynch, and K. D. Listmann, "An experimental validation of magnetometer integration into a GPS-aided helicopter UAV navigation system," in *Proceedings of the 2010 American Control Conference*, Baltimore, MD, July 2010, pp. 4439–4444.

[2] J. L. Crassidis, F. L. Markley, and Y. Cheng, "Survey of nonlinear attitude estimation methods," *Journal Of Guidance, Control, And Dynamics*, vol. 30, no. 1, pp. 12–28, January-February 2007.

[3] S. Bonnabel, P. Martin, and P. Rouchon, "Symmetry-preserving observers," *IEEE Transactions On Automatic Control*, vol. 53, no. 11, pp. 2514–2526, December 2008.

[4] ——, "Non-linear symmetry-preserving observers on Lie groups," *IEEE Transactions On Automatic Control*, vol. 54, no. 7, pp. 1709–1713, July 2009.

[5] S. Bonnabel, "Left-invariant extended Kalman filter and attitude estimation," in *Proceedings of the 46th IEEE Conference on Decision and Control*, New Orleans, LA, December 2007, pp. 1027–1032.

[6] P. Martin and E. Salaün, "Design and implementation of a low-cost aided attitude and heading reference system," in *AIAA Guidance, Navigation and Control Conference and Exhibit*, Honolulu, HI, August 2008, AIAA 2008-7169.

[7] D. R. Kastelan, "Design and implementation of a GPS-aided inertial navigation system for a helicopter UAV," Master's thesis, University of Alberta, 2009.

[8] H. Nijmeijer and A. van der Schaft, *Nonlinear Dynamical Control Systems*. Springer, 1990.

[9] J. W. Grizzle and S. I. Marcus, "The structure of nonlinear control systems possessing symmetries," *IEEE Transactions On Automatic Control*, vol. 30, no. 3, pp. 248–258, March 1985.

[10] P. Rouchon and J. Rudolph, "Invariant tracking and stabilization: problem formulation and examples," in *Stability and Stabilization of Nonlinear Systems*, ser. Lecture Notes in Control and Information Sciences, D. Aeyels, F. Lamnabhi-Lagarrigue, and A. van der Schaft, Eds. London: Springer-Verlag, 1999, vol. 246, ch. 14, pp. 261–273.

[11] P. Martin, P. Rouchon, and J. Rudolph, "Invariant tracking," *ESAIM: Control, Optimisation and Calculus of Variations*, vol. 10, no. 1, pp. 1–13, January 2004.

[12] P. J. Olver, *Classical Invariant Theory*, ser. London Mathematical Society Student Texts. Cambridge University Press, 1999, vol. 44.

[13] Y. Song and J. W. Grizzle, "The Extended Kalman Filter as a local asymptotic observer for discrete-time nonlinear systems," *Journal of Mathematical Systems, Estimation, and Control*, vol. 5, no. 1, pp. 59–78, 1995.

[14] S. Bonnabel, P. Martin, and E. Salaün, "Invariant Extended Kalman Filter: theory and application to a velocity-aided attitude estimation problem," in *Proceedings of the Joint 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference*, Shanghai, P.R. China, December 2009, pp. 1297–1304.

[15] P. Martin and E. Salaün, "Generalized Multiplicative Extended Kalman Filter for aided attitude and heading reference system," in *AIAA Guidance, Navigation, and Control Conference*, Toronto, Canada, August 2010, AIAA 2010-8300.

[16] R. G. Brown and P. Y. Hwang, *Introduction to Random Signals and Applied Kalman Filtering*, 3rd ed. John Wiley & Sons, 1997.

[17] J. A. Farrell, *Aided Navigation: GPS with High Rate Sensors*. McGraw Hill, 2008.