2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Convergence of $\mathcal{H}_\infty$-Optimal Actuator Locations

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***Abstract*— In control of vibrations, diffusion and many other problems governed by partial differential equations, there is freedom in the choice of actuator location. The actuator location should be chosen to optimize performance objectives. In this paper, we consider $\mathcal{H}_\infty$ performance with state-feedback. It is shown that the corresponding optimal actuator problem is well-posed. In practice, approximations are used to determine the optimal actuator location. The optimal performance and the corresponding actuator location of the approximating sequence should converge to the exact optimal performance and location. Conditions for this convergence in the case of $\mathcal{H}_\infty$-control are provided. The results are illustrated with an example.**

I. Introduction

In many control systems governed by partial differential equations, the location of actuators can be chosen. The actuator locations should be selected in order to optimize the performance criterion of interest. Since the 1980's, many cost functions for actuator placement have been used; see for example the survey paper [4]. In this work we are interested in finding locations that minimize the response of the controlled system to disturbances, that is locations that minimize the $\mathcal{H}_\infty$- norm of the controlled system.

Although the state-space for the full partial differential equation model is infinite-dimensional, approximations are used in controller design and thus in selection of the actuator location. The theory that guarantees optimality of the cost and existence of the optimal actuator location for these models has not been developed in its entirety. Furthermore, the issues associated with the usage of approximations in determining optimal actuator locations have not been extensively investigated. In [5], it was shown that using the first n modes to find the actuator location that maximizes the decay rate of the solution to the wave equation yields the worst location for the $(n+1)$th mode. Conditions that guarantee optimality of the actuator location with a linear quadratic cost are developed in [10].

In this work the problem of using approximations to determine optimal actuator location for $\mathcal{H}_\infty$-control with state-feedback is considered. Optimal disturbance attenuation as a function of actuator location is used as the cost function. Criterion for optimality using the original model are obtained. Convergence of the cost obtained using approximations to the exact cost for a fixed actuator location was proven in [6] using conditions similar to those required for convergence of linear quadratic controllers. Here, continuity of the optimal attenuation with respect to actuator locations calculated using approximations is proved. This leads to the main result of this paper: conditions under which $\mathcal{H}_\infty$- optimal actuator locations calculated using approximations converge to the exact optimal locations. Conditions derived in this paper are applicable towards calculation of $\mathcal{H}_\infty$ optimal actuator location in many problems modeled by diffusion, vibration and noise control applications. Vibration control of a simply supported beam is provided as an illustration.

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II. Framework

Consider the system described on a Hilbert space $\mathcal{Z}$ by

$$\frac{dz}{dt} = Az(t) + Bu(t) + Dv(t), \quad z(0) = z_0 \in \mathcal{Z} \qquad \text{(II.1)}$$

where A is a linear, closed and densely defined operator with domain $\mathbf{D}(A)$ generating a C_0 semigroup, $S(t)$, on $\mathcal{Z}$; $B \in \mathcal{L}(U, \mathcal{Z})$, $D \in \mathcal{L}(W, \mathcal{Z})$. We assume that U and W are separable Hilbert spaces. The signal $u(\cdot) \in L_2(0, \infty; U)$ is the control input and $v(\cdot) \in L_2(0, \infty; W)$ is the exogenous disturbance. Write $\mathcal{U} = L_2(0, \infty; U)$ and $\mathcal{W} = L_2(0, \infty; W)$ to denote the space of all admissible controls and disturbances respectively. For a separable Hilbert space Y, let $C \in \mathcal{L}(\mathcal{Z}, Y)$ denote the measurement operator. With control cost, $R \in \mathcal{L}(U, U)$, where R is coercive, define

$$y(t) = \begin{bmatrix} Cz(t) \\ R^{\frac{1}{2}} u(t) \end{bmatrix}, \qquad \text{(II.2)}$$

and the index

$$\rho(u, v; z_0) = \|y\|^2_{L_2(0,\infty;Y)} = \int_0^\infty \|Cz(t)\|^2 + \|R^{\frac{1}{2}} u(t)\|^2 dt. \qquad \text{(II.3)}$$

Systems of the form (II.1)-(II.2) will often be abbreviated $(A, [B \;\; D], C)$. The system (II.1)-(II.2) is a special form of the generalised plant configuration, known as the full information problem.

Definition 2.1: The C_0-semigroup $S(t)$ is stable if there exists constants M and $\alpha > 0$ such that $\|S(t)\| \leq Me^{-\alpha t}$ for all $t \geq 0$.

Definition 2.2: The pair (A, B) is stabilizable if there exists a bounded linear operator $K : \mathcal{Z} \to U$ such that $A - BK$ generates a stable semigroup.

Definition 2.3: The pair (A, C) is detectable if there exists a bounded linear operator $F : Y \to \mathcal{Z}$ such that $A - FC$ generates a stable semigroup.

The notation $\mathcal{H}_\infty$ indicates the Hardy space of functions $G(s)$ which are analytic in the right-half plane $Re(s) > 0$ and for which

$$\sup_\omega \lim_{x \downarrow 0} |G(x + j\omega)| < \infty. \qquad \text{(II.4)}$$

The norm of a function in $\mathcal{H}_\infty$ is

$$\|G\|_\infty = \sup_\omega \lim_{x\downarrow 0} |G(x+j\omega)|. \tag{II.5}$$

For any operator-valued function $G(s) \in \mathcal{L}(H_1, H_2)$, where H_1 and H_2 are Hilbert spaces, the space for which

$$\|G\|_\infty = \sup_\omega \lim_{x\downarrow 0} \|G(x+j\omega)\| < \infty \tag{II.6}$$

is indicated by $\mathcal{H}_\infty(H_1, H_2)$. The norm of a function in $\mathcal{H}_\infty(H_1, H_2)$ is given in (II.6) .By the Paley-Weiner Theorem, a system with input in U and outputs in Y is L_2-stable if and only if the system transfer function $G \in \mathcal{H}_\infty(U, Y)$.

Let G be the transfer function of the system (II.1)-(II.2):

$$G(s) := \begin{bmatrix} C \\ 0 \end{bmatrix} \mathrm{R}(s; A)[B \ D] + \begin{bmatrix} 0 & 0 \\ R^{\frac{1}{2}} & 0 \end{bmatrix} \tag{II.7}$$

where $\mathrm{R}(s;A)$ is the resolvent operator of A. With state feedback control $u(t) = -Kz(t)$, the closed loop transfer function from the disturbance v to the output y is

$$G_{yv}(K) = \begin{bmatrix} C \\ R^{\frac{1}{2}} K \end{bmatrix} \mathrm{R}(s; A - BK)D.$$

Definition 2.4: The $\mathcal{H}_\infty$-control problem is to construct a state feedback control $u(t) = -Kz(t), K \in \mathcal{L}(\mathcal{Z}, U)$, for the system (II.1) - (II.2) for a given $\gamma > 0$, such that the closed loop transfer function $G_{yv}(K) \in \mathcal{H}_\infty(W, Y)$ and

$$\|G_{yv}(K)\|_\infty < \gamma. \tag{II.8}$$

Definition 2.5: If there is a $\eta > 0$ such that for each disturbance $v \in \mathcal{W}$, there exists a control $u \in \mathcal{U}$ with

$$\rho(u, v; 0) \leq (\gamma^2 - \eta) \parallel v \parallel_{\mathcal{W}}^2, \tag{II.9}$$

then the system (II.1) with (II.2) is said to be stabilizable with attenuation γ.

Definition 2.6: The state feedback $K \in \mathcal{L}(\mathcal{Z}, U)$ is said to be γ - admissible if it is stabilizing and the linear feedback $u(t) = -Kz(t)$ is such that the attenuation bound (II.9) is achieved.

Theorem 2.7: [1], [7] Assume that (A, B) is stabilizable and (A, C) is detectable. For $\gamma > 0$ the following are equivalent:
(1) there exists a γ-admissible state feedback;
(2) the system is stabilizable with disturbance attenuation γ;
(3) there exists a non-negative, self-adjoint operator Σ on $\mathcal{Z}$ satisfying the $\mathcal{H}_\infty$-Riccati operator equation,

$$(A^*\Sigma + \Sigma A - \Sigma B R^{-1} B^* \Sigma + \frac{1}{\gamma^2} \Sigma D D^* \Sigma + C^* C)z = 0 \tag{II.10}$$

for all $z \in \mathbf{D}(A)$, and $A - BR^{-1}B^*\Sigma + \frac{1}{\gamma^2} DD^*\Sigma$ generates an exponentially stable semigroup on $\mathcal{Z}$. Moreover, in this case a γ-admissible state feedback is given by $K = R^{-1}B^*\Sigma$.
In other words, if a system can be stabilized with disturbance attenuation γ, the system can be stabilized with the same attenuation using constant state feedback.

Definition 2.8: The optimal $\mathcal{H}_\infty$-control problem is to calculate

$$\hat{\gamma} = \inf \gamma \tag{II.11}$$

over all $K \in \mathcal{L}(\mathcal{Z}, U)$ such that $G_{yv}(K) \in \mathcal{H}_\infty(W, Y)$ and $\|G_{yv}(K)\|_\infty < \gamma$. The infimum $\hat{\gamma}$ is called the optimal disturbance attenuation for the system (II.1) with (II.2).

III. Approximation Theory

In practice, the operator equation (II.10) cannot be solved and the control is calculated using an approximation. Let $\mathcal{Z}^N$ be a family of finite-dimensional subspaces of $\mathcal{Z}$ and P^N the orthogonal projection of $\mathcal{Z}$ onto $\mathcal{Z}^N$. The space $\mathcal{Z}^N$ is equipped with the norm inherited from $\mathcal{Z}$. Consider a sequence of operators $A^N \in \mathcal{L}(\mathcal{Z}^N, \mathcal{Z}^N)$, $B^N \in \mathcal{L}(U, \mathcal{Z}^N)$, $D^N \in \mathcal{L}(W, \mathcal{Z}^N)$, and $C^N = C|_{\mathcal{Z}^N}$. We will make the following standard assumptions on the approximation scheme.

(A1) For all $z \in \mathcal{Z}$,

$$\lim_{N\to\infty} \|e^{A^N t} P^N z - S(t)z\| = 0, \tag{III.1}$$

$$\lim_{N\to\infty} \|(e^{A^N t})^* P^N z - S(t)^* z\| = 0. \tag{III.2}$$

uniformly in t on bounded intervals.

(A2)(i) The family of pairs (A^N, B^N) is uniformly exponentially stabilizable, that is, there exists a uniformly bounded sequence of operators $K^N \in \mathcal{L}(\mathcal{Z}^N, U)$ such that

$$\|e^{(A^N - B^N K^N)t} P^N\| \leq M_1 e^{-\alpha_1 t}, \tag{III.3}$$

for some positive constants $M_1 \geq 1$ and α_1.

(A2)(ii) The family of pairs (A^N, C^N) is uniformly exponentially detectable, that is, there exists a uniformly bounded sequence of operators $F^N \in \mathcal{L}(Y, \mathcal{Z}^N)$ such that

$$\|e^{(A^N - F^N C^N)t} P^N\| \leq M_2 e^{-\alpha_2 t}, \tag{III.4}$$

for some positive constants $M_2 \geq 1$ and α_2.

(A3) The approximating sequence of input and disturbance operators converge in norm

$$\lim_{N\to\infty} \|B^N - B\| = 0, \tag{III.5}$$

$$\lim_{N\to\infty} \|D^N - D\| = 0. \tag{III.6}$$

Remarks 3.1: (1) Assumption (A1) implies that $P^N z \to z$ for $z \in \mathcal{Z}$. Assumption (A1)(i) is required for convergence of initial conditions. Assumption (A1)(i) is often satisfied by ensuring that the conditions of the Trotter-Kato Theorem hold, see for instance [11, Chap. 3, Thm. 4.2]. The convergence (A1)(ii) of the adjoint semigroup sequence is required for the strong convergence of the approximating linear-quadratic Riccati operators. A counter-example may be found in [2].
(2) If the original problem is exponentially stabilizable(detectable) and the eigenfunctions of A form an orthonormal basis for $\mathcal{Z}$, then an approximation scheme formed using the first n eigenfunctions is uniformly stabilizable(detectable); that is assumption (A2) is satisfied. In practice, other approximation methods such as finite-elements are typically used. Many such approximations, such

as linear splines for the diffusion equation and cubic splines for damped beam vibrations are uniformly stabilizable (detectable), provided that the original system is stabilizable (detectable) [8, Thm. 5.2, Thm. 5.3], [9].

(3) Since the approximating spaces $\mathcal{Z}^N$ are finite-dimensional, B^N and D^N are finite rank operators. Thus, assumption (A3) holds if and only if the operators B and D are compact.

Theorem 3.2: Assume that (A, B) is stabilizable and (A, C) is detectable, and (A1)-(A3) hold. If the original system is stabilizable with attenuation γ, then so are the approximating systems for sufficiently large N. For such N, the Riccati equation

$$(A^N)^*\Sigma^N + \Sigma^N A^N - \Sigma^N B^N R^{-1}(B^N)^*\Sigma^N + ...$$
$$\frac{1}{\gamma^2}\Sigma^N D^N (D^N)^*\Sigma^N + (C^N)^* C^N = 0$$

has a non-negative, self-adjoint solution Σ^N and $\Sigma^N P^N z \to \Sigma z$ strongly in $\mathcal{Z}$ as $N \to \infty$. Moreover, $K^N = R^{-1}(B^N)^*\Sigma^N$ converges to $K = R^{-1}B^*\Sigma$ in norm. For N sufficiently large, K^N is γ-admissible for the original system.

Proof: This follows from [6, Thm. 2.5, Cor. 2.6] with the extension of $B^N = P^N B$ and $D^N = P^N D$ to more general approximations. ■

Let $\{\hat{\gamma}^N\}$ indicate the optimal disturbance attenuation for the approximating problems.

Theorem 3.3: Assume that (A1)-(A3) hold, (A, B) is stabilizable, and (A, C) is detectable. Then

$$\lim_{N\to\infty} \hat{\gamma}^N = \hat{\gamma}. \qquad \text{(III.7)}$$

Proof: This follows from Theorem 3.2 and [6, Thm. 2.8] with the extension of $B^N = P^N B$ and $D^N = P^N D$ to more general approximations. ■

IV. Optimal Actuator Location

Consider a situation where there are m actuators whose location could be varied over some compact set $\Omega \subset \mathcal{R}^q$. Parametrize the actuator locations by r and denote the dependence of the corresponding input operator with respect to the actuator location by $B(r)$. Note that r is a vector of length m with components in Ω so that r varies over a space denoted by Ω^m. For each location r, we have an optimal $\mathcal{H}_\infty$ control problem. Let $\hat{\gamma}(r)$ denote the $\mathcal{H}_\infty$ performance of (II.1)-(II.2) with actuators at the location r.

Definition 4.1: The optimal performance μ is

$$\mu = \inf_{r\in\Omega^m} \hat{\gamma}(r). \qquad \text{(IV.1)}$$

We now show continuity of the $\mathcal{H}_\infty$ performance $\hat{\gamma}(r)$ with respect to actuator location under the following assumptions:

(C1) The family of input operators $B(r) \in \mathcal{L}(U, \mathcal{Z})$, $r \in \Omega^m$ are continuous functions of r in the operator norm, that is for any $r_0 \in \Omega^m$,

$$\lim_{r\to r_0} \| B(r) - B(r_0) \| = 0. \qquad \text{(IV.2)}$$

(C2) The family of pairs $(A, B(r))$, $r \in \Omega^m$ are stabilizable and the pair (A, C) is detectable.

(C3) The input operators $B(r)$ and the disturbance operator D are compact.

Lemma 4.2: Let $(A, [B(r)\ D], C)$ be a family of systems such that assumptions (C1)-(C3) are satisfied. Assume that the system at r_0 is stabilizable with attenuation $\gamma(r_0)$ and $K(r_0) \in \mathcal{L}(\mathcal{Z}, U)$ is $\gamma(r_0)$-admissible. For every $\epsilon > 0$ there is $\delta > 0$ such that for all $\|r - r_0\| < \delta$ the systems $(A, [B(r)\ D], C)$ are stabilizable with attenuation $\gamma(r_0) + \epsilon$. Furthermore, a sequence of state feedback operators $K(r) \in \mathcal{L}(\mathcal{Z}, U)$ can be chosen that are $(\gamma(r_0) + \epsilon)$-admissible at r and also $K(r)$ is continuous at r_0.

Proof: Consider a sequence $\{r\}$ that converges to r_0. Choose some K so that $A - B(r_0)K$ generates an exponentially stable semigroup $S_{K,r_0}(t)$ with bound $Me^{-\alpha t}$, where $\alpha > 0$. Let δ be such that $A - B(r)K$ generates an exponentially stable semigroup with bound $Me^{-\frac{\alpha}{2}t}$ for all $\|B(r) - B(r_0)\| < \delta$. There is $\epsilon > 0$ such that for all $\|r - r_0\| < \epsilon$, $\|B(r) - B(r_0)\| < \delta$. We thus have a sequence of uniformly exponentially stabilizable systems $(A, B(r))$. Now, the assumptions (A1)-(A3) are satisfied by the sequence of systems $(A, [B(r)\ D], C)$ where B^N is replaced by $B(r)$. A proof similar to that of Theorem 2.5 in [6] yields that the system at r is stabilizable with attenuation $\gamma(r_0) + \epsilon$. Also, the $\mathcal{H}_\infty$-Riccati operator $\Sigma(r)$ converges strongly to $\Sigma(r_0)$. Furthermore, a sequence $K(r) \in \mathcal{L}(\mathcal{Z}, U)$ can be chosen that are $(\gamma(r_0) + \epsilon)$-admissible at r and also $K(r)$ converges uniformly to $K(r_0)$ in norm. ■

Theorem 4.3: Consider a family of control systems $(A, [B(r)\ D], C)$ such that the assumptions (C1) - (C3) are satisfied. Then

$$\lim_{r\to r_0} \hat{\gamma}(r) = \hat{\gamma}(r_0), \qquad \text{(IV.3)}$$

where $\hat{\gamma}(r_0)$ is the optimal disturbance attenuation at r_0.

Proof: Consider any sequence $\{r_n\}$ that converges to r_0. Since the optimal disturbance attenuation at r_0 is $\hat{\gamma}(r_0)$, for every $\epsilon > 0$ the system at r_0 is stabilizable with an attenuation $\hat{\gamma}(r_0) + \epsilon$ with some state feedback $\hat{K}(r_0) \in \mathcal{L}(\mathcal{Z}, U)$.

It follows from Lemma 4.2 that there exists N such that for $n > N$ the system at r_n is stabilizable with attenuation $\gamma(r_n)$ where

$$\gamma(r_n) \ \leq\ \hat{\gamma}(r_0)\ +\ 2\epsilon,$$

Hence, the optimal disturbance attenuation at the location r_n, $\hat{\gamma}(r_n)$, satisfies

$$\hat{\gamma}(r_n) \ \leq\ \hat{\gamma}(r_0) + 2\epsilon.$$

Since ϵ is arbitrary, it follows that

$$\limsup_{n\to\infty}\ \hat{\gamma}(r_n) \leq \hat{\gamma}(r_0). \qquad \text{(IV.4)}$$

Because of (IV.4), it is sufficient to show that

$$\liminf_{n\to\infty}\ \hat{\gamma}(r_n) \geq \hat{\gamma}(r_0). \qquad \text{(IV.5)}$$

Assume that this statement is false. Then there is an $\delta > 0$ such that for all n there is $p > n$ with $\hat{\gamma}(r_p) < \hat{\gamma}(r_0) - \delta$. In this way we can construct a subsequence $\{r_p\}$ of the sequence $\{r_n\}$ with $\hat{\gamma}(r_p) < \hat{\gamma}(r_0) - \delta$ Thus, the system at r_p is stabilizable with attenuation $\hat{\gamma}(r_0) - \frac{\delta}{2}$ with some state feedback $K(r_p)$ and

$$\rho_{r_p}(-K(r_p)z, v; 0) \leq (\hat{\gamma}(r_0) - \frac{\delta}{2})^2 \|v\|^2_{\mathcal{W}} \tag{IV.6}$$

for $v \in \mathcal{W}$. Since r_p converges to r_0, a state feedback $K(r_0)$ can be chosen such that $\|K(r_p) - K(r_0)\| \to 0$ (Lemma 4.2). Therefore, the problem at r_0 is stabilizable with attenuation $\hat{\gamma}(r_0) - \frac{\delta}{2}$. This contradicts the optimality of $\hat{\gamma}(r_0)$ and thus (IV.5) holds. Hence (IV.3) holds. ■

At each actuator location r, (II.11) is posed on a infinite-dimensional space and in general can only be solved by replacing the original system by a finite-dimensional approximation that satisfies the assumptions discussed earlier. For the sequence of approximating problems $(A^N, [B^N(r)\ D^N], C^N)$, the optimal performance μ^N and the optimal location $\hat{r}^N$ are defined similarly as μ, $\hat{r}$ for the original problem.

Corollary 4.4: There exists an optimal actuator location $\hat{r}$ such that

$$\mu = \inf_{r \in \Omega^m} \hat{\gamma}(r) = \hat{\gamma}(\hat{r}),$$

and similarly for each N there exists $\hat{r}^N$ such that

$$\mu^N = \inf_{r \in \Omega^m} \hat{\gamma}^N(r) = \hat{\gamma}^N(\hat{r}^N).$$

Proof: From Theorem 4.3, $\hat{\gamma}(r)$ is a continuous function with respect to the actuator location r, and since r varies over a compact set Ω^m, $\inf_{r \in \Omega^m} \hat{\gamma}(r)$ exists and there exists some location that satisfies the infimum. Call it $\hat{r} \in \Omega^m$. A similar argument holds for the approximating problem. ■

V. Computation of Optimal Actuator Location

We now have the problem of determining the optimal $\mathcal{H}_\infty$ performance, over all possible actuator locations.

Theorem 5.1: Let $(A, [B(r)\ D], C)$ be a family of control systems depending on actuator location such that assumptions (C1) - (C3) are satisfied. Choose some approximation scheme such that assumptions (A1) - (A3) are satisfied for each $(A^N, [B^N(r)\ D^N], C^N)$ where $B^N = P^N B$. Letting $\hat{r}$ be an optimal actuator location for $(A, [B(r)\ D], C)$ with optimal cost μ and defining similarly $\hat{r}^N$, μ^N, it follows that

$$\mu = \lim_{N \to \infty} \mu^N,$$

and there exists a subsequence $\{\hat{r}^M\}$ of $\{\hat{r}^N\}$ such that

$$\mu = \lim_{M \to \infty} \hat{\gamma}(\hat{r}^M).$$

Proof: A similar proof for the convergence of linear quadratic optimal actuator locations may be found in [10].

$$\begin{aligned} \mu^N &= \inf_{r \in \Omega^m} \hat{\gamma}^N(r) \\ &\leq \hat{\gamma}^N(\hat{r}) \\ &= \hat{\gamma}^N(\hat{r}) - \hat{\gamma}(\hat{r}) + \hat{\gamma}(\hat{r}) \\ &= \hat{\gamma}^N(\hat{r}) - \hat{\gamma}(\hat{r}) + \mu. \end{aligned}$$

Since $\lim_{N \to \infty} \hat{\gamma}^N(\hat{r}) = \hat{\gamma}(\hat{r})$ (Theorem 3.3),

$$\limsup \mu^N \leq \mu. \tag{V.1}$$

It remains only to show that

$$\liminf \mu^N \geq \mu. \tag{V.2}$$

To this end, choose a subsequence $\mu^M \to \liminf \mu^N$, with corresponding actuator locations $\hat{r}^M$. Since $\{\hat{r}^M\} \subset \Omega^m$, it has a convergent subsequence, also denoted by $\hat{r}^M$, with limit $\underline{r}$. Now,

$$\begin{aligned} &\|B^M(\hat{r}^M) - B(\underline{r})\| \\ &\quad = \|P^M B(\hat{r}^M) - B(\underline{r})\| \\ &\quad \leq \|P^M B(\hat{r}^M) - P^M B(\underline{r})\| + \|P^M B(\underline{r}) - B(\underline{r})\| \\ &\quad \leq \|P^M\| \|B(\hat{r}^M) - B(\underline{r})\| + \|P^M B(\underline{r}) - B(\underline{r})\|. \end{aligned}$$

Thus, $\| B^M(\hat{r}^M) - B(\underline{r}) \| \to 0$. By assumption (A2)(i), there is a uniformly bounded sequence $K^M_{\underline{r}} \in \mathcal{L}(\mathcal{Z}, U)$ such that $A^M - B^M(\underline{r}) K^M_{\underline{r}}$ generate semigroups bounded by $M_1 e^{-\omega_1 t}$ for some $M_1 \geq 1, \omega_1 > 0$. For some $\epsilon < \omega_1$, choose N large enough such that $\| B^M(\hat{r}^M) - B^M(\underline{r}) \| < \frac{\epsilon}{M_1 \|K^M_{\underline{r}}\|}$ for $M > N$. Then for all $M > N$, $A^M - B^M(\hat{r}^M) K^M_{\underline{r}}$ generates an exponentially stable C_0-semigroup with bound $M_1 e^{(-\omega_1 + \epsilon)t}$. Applying then Theorem 3.3 to the sequence $(A^M, [B^M(\hat{r}^M)\ D^M], C^M)$, it follows that $\hat{\gamma}^M(\hat{r}^M) \to \hat{\gamma}(\underline{r})$. Thus,

$$\begin{aligned} \liminf \mu^N &= \lim_{M \to \infty} \mu^M \\ &= \lim_{M \to \infty} \hat{\gamma}^M(\hat{r}^M) \\ &= \hat{\gamma}(\underline{r}) \\ &\geq \mu. \end{aligned} \tag{V.3}$$

Thus, $\liminf \mu^N \geq \mu$ and so $\lim \mu^N = \mu$ as required.

Since $\mu = \lim \mu^N = \liminf \mu^N$, (V.3) implies that

$$\begin{aligned} \mu &= \liminf \mu^N \\ &= \hat{\gamma}(\underline{r}) \\ &= \lim_{M \to \infty} \hat{\gamma}(\hat{r}^M). \end{aligned} \tag{V.4}$$

where the latter equality follows from continuity of $\mathcal{H}_\infty$ performance with respect to actuator location (Theorem 4.3). Thus, as was to be shown, a sequence of approximating actuator locations yield performance arbitrarily close to optimal. ■

Note that unlike the case of linear-quadratic optimal control where the actuator location is chosen to minimize $\|\Pi\|$ or traceΠ where Π is the solution to the LQ algebraic Riccati equation, it is not required that the measurement operator C be compact. This is illustrated by the example below.

VI. EXAMPLE

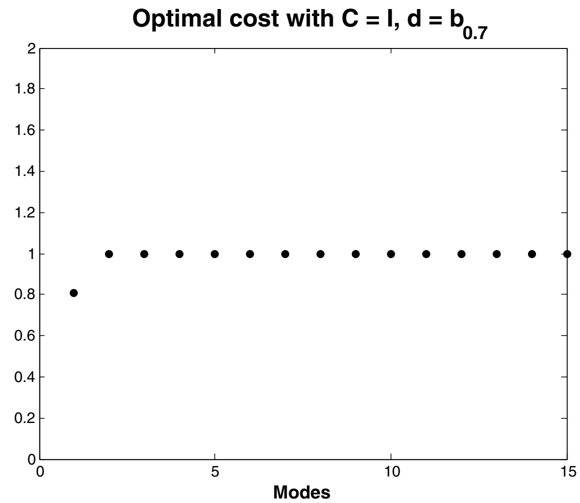

Fig. 1. Performance $\hat{\gamma}$ at optimal location for different approximations of the viscously damped beam with $C = I, R = 1, d = b_{0.7}$

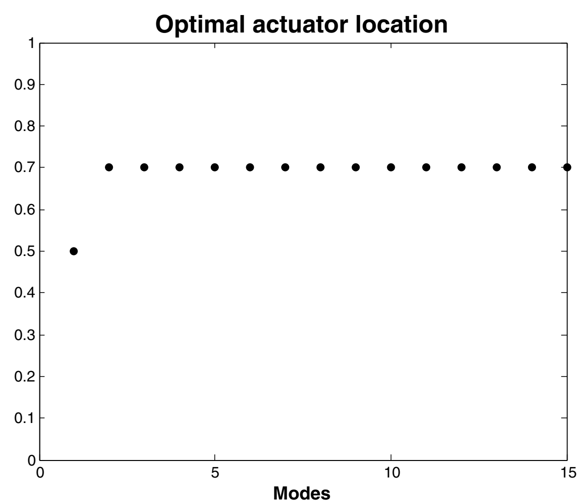

Fig. 2. Optimal actuator location for different approximations of the viscously damped beam with $C = I, R = 1, d = b_{0.7}$

Consider a simply supported Euler-Bernoulli beam and let $w(x,t)$ denote the deflection of the beam from its rigid body motion at time t and position x. The deflection is controlled by applying a force $u(t)$ around the point r with width Δ. The exogenous disturbance $v(t)$ induces a distributed load $d(x)v(t)$ where $d(x) \in \mathcal{C}(0,1)$. If we normalize the variables and include viscous damping with parameter ξ, we obtain the partial differential equation

$$\frac{\partial^2 w}{\partial t^2}+\xi\frac{\partial w}{\partial t}+\frac{\partial^4 w}{\partial x^4} = b_r(x)u(t)+d(x)v(t), \quad t \geq 0, 0 < x < 1, \tag{VI.1}$$

where, letting Δ indicate the width of the actuator and r its

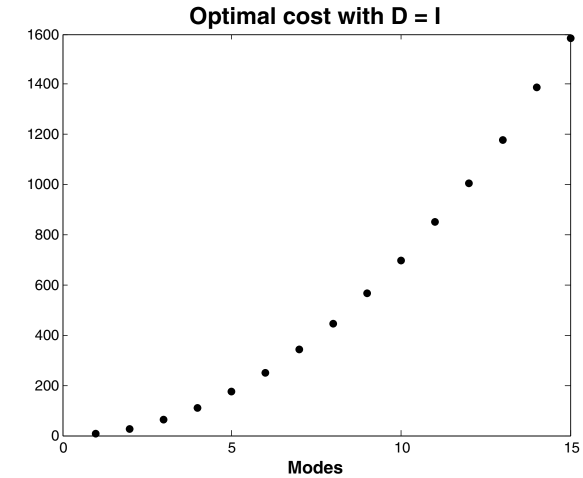

Fig. 3. Optimal performance $\hat{\gamma}(\hat{r})$ for different approximations of the viscously damped beam with $C = I, R = 1, D = I$

location,

$$b_r(x) = \begin{cases} \frac{1}{\Delta}, & |r-x| < \frac{\Delta}{2} \\ 0, & |r-x| \geq \frac{\Delta}{2}. \end{cases}$$

The boundary conditions are

$$\begin{aligned} w(0,t) = 0, \quad w''(0,t) = 0, \\ w(1,t) = 0, \quad w''(1,t) = 0. \end{aligned} \tag{VI.2}$$

In the computer simulations, the parameters were set to $\xi = 0.1, \Delta = 0.001$. Let

$$H_s(0,1) = \{w \in \mathcal{H}^2(0,1), w(0) = 0, w(1) = 0\} \tag{VI.3}$$

and define the state-space $\mathcal{Z} = H_s(0,1) \times L_2(0,1)$ with state $z(t) = (w(\cdot,t), \frac{\partial}{\partial t}w(\cdot,t))$. A state-space formulation of the above partial differential equation problem is

$$\frac{d}{dt}z(t) = Az(t) + B(r)u(t) + Dv(t), \tag{VI.4}$$

where

$$A = \begin{bmatrix} 0 & I \\ -\frac{d^4}{dx^4} & -\xi I \end{bmatrix}, B(r) = \begin{bmatrix} 0 \\ b_r(\cdot) \end{bmatrix}, D = \begin{bmatrix} 0 \\ d(\cdot) \end{bmatrix}, \tag{VI.5}$$

with domain

$$\mathbf{D}(A) = \{(\phi,\psi) \in H_s(0,1) \times H_s(0,1) \text{ with } \phi'' \in H_s(0,1)\}.$$

It is well-known that A with domain $\mathbf{D}(A)$ generates an exponentially stable semigroup on $\mathcal{Z}$ [3]. Since there is only one control, choose control weight $R = 1$. An obvious choice of measurement is $C = I$. Consider the disturbance $d = b_{0.7}$ centered at $r = 0.7$ with width $\Delta = 0.001$. The operators $B(r)$ and D are finite rank. Therefore, the corresponding $\mathcal{H}_\infty$-control with full-information problem satisfies the assumptions of Theorem 4.3. Hence, the cost $\hat{\gamma}(r)$ depends continuously on the actuator location and there exists an optimal actuator location. Since a closed form solution to the partial differential equation problem is not available, the optimal actuator location must be calculated using an approximation. Let $\phi_i(x)$ indicate the eigenfunctions of $\frac{\partial^4 w}{\partial x^4}$ with boundary conditions (VI.2). For any positive integer N, define Z^N to be the span of $\phi_i, i = 1...N$. Choose $\mathcal{Z}^N = Z^N \times Z^N$ and define P^N to be the projection onto $\mathcal{Z}^N$. Define A^N to be the Galerkin approximation to A, $B^N := P^N B$ and $D^N := P^N D$. This approximation scheme satisfies all the assumptions of Theorem 5.1 [8] and hence, we obtain convergence of the approximating optimal performance and the actuator locations. This is illustrated in Figures 1 and 2.

Consider the same example as before, except that now, we minimize the effect of worst disturbance on the entire state and choose $D = I$. Now D is not a compact operator. As shown in Figures 3 and 4, neither optimal cost nor the optimal actuator location converges.

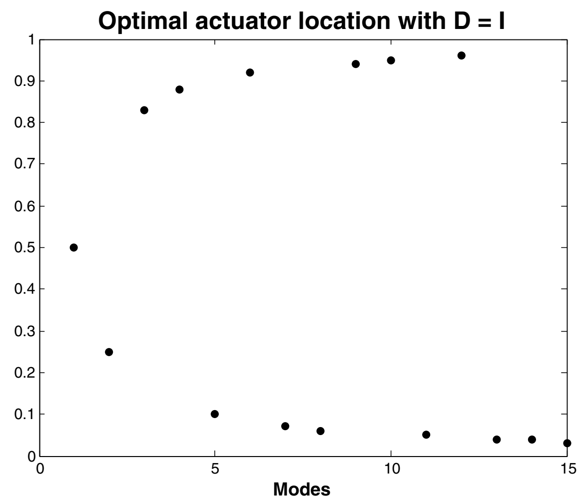

Fig. 4. Optimal actuator location for different approximations of the viscously damped beam with $C = I, R = 1, D = I$

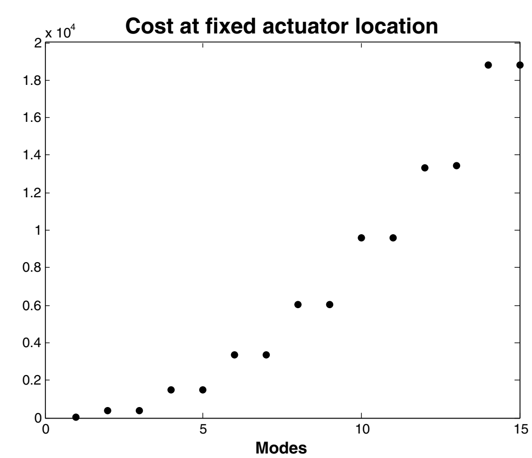

Fig. 5. Performance $\hat{\gamma}$ at actuator location $r = 0.5$ for different approximations of the viscously damped beam with $C = I, R = 1, D = I$

In fact, the optimal attenuation does not converge even at a fixed actuator location, say for example at $r = 0.5$, as the approximation size increases. This is illustrated in Figure 5.

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