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On Arbitrage Possibilities Via Linear Feedback in an Idealized Brownian Motion Stock Market

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***Abstract*— This paper extends the so-called Simultaneous Long-Short (SLS) linear feedback stock trading analysis given in [2]. Whereas the previous work addresses a class of idealized markets involving continuously differentiable stock prices, this work concentrates on markets governed by Geometric Brownian Motion (GBM). For this class of stock price variations, the main results in this paper address the extent to which a positive trading gain $g(t) > 0$ can be guaranteed. We prove that the SLS feedback controller possesses a remarkable robustness property that guarantees a positive *expected* trading gain $\mathbb{E}[g(t)] > 0$ in *all* idealized GBM markets with non-zero drift. Additionally, the main results of this paper include closed form expressions for both $g(t)$ and its probability density function. Finally, the use of the SLS controller is illustrated via a detailed numerical example involving a large number of simulations.**

I. INTRODUCTION

This paper is part of a growing body of literature, for example, see [1]–[16], intended to address the following question: What might classical control theory have to offer to the area of stock trading? At the simplest level, the controller output is the amount invested $I(t)$ in the trade and basic inputs to the controller might be the stock price $p(t)$ and trading gain or loss $g(t)$. In contrast to classical approaches in finance, for example see [17]–[19], this paper and others in this line of research do not address controller design using a predictive model for the future stock price $p(t)$; e.g., see [1], [2] and [9]. Instead, the controller treats $p(t)$ as an uncontrolled external input and no stochastic model is used. The feedback controller simply processes the history of $p(t)$ to determine the appropriate investment level $I(t)$.

To explore the ultimate potential for control theoretic methods in stock trading, we use the notion of an *idealized market*. The key idea here is to use this market to determine which control algorithms are worthy of the large investment associated with a full-scale back-test. That is, the idealized market serves as the "proving ground" within which theoretical performance certifications are obtained.

The idealized market, more carefully defined in Section II, is characterized by some class of prices $\mathcal{P}$ plus assumptions regarding liquidity, continuous trading and price-taking. For example, in [2], the class $\mathcal{P}$ is the set of non-negative continuously differentiable functions on $[0, T]$. Note that this class of prices is "idealized" in the sense that the possibility of price gaps is ruled out. In contrast, this paper takes $\mathcal{P}$ to be the collection of Geometric Brownian Motions (GBM) with drift μ and volatility σ. Within this context, it is important to note that a performance certification must be robust with respect to $\mathcal{P}$. Hence, feedback gains entering into the controller are not allowed to be selected assuming a priori knowledge of μ and σ.

Within the framework of an idealized GBM market, we consider classical linear time-invariant feedback and explore the ultimate potential of control theory by studying the profit or loss on a trade, henceforth called the cumulative *trading gain* $g(t)$ on $[0, t]$. Subsequently, the following fundamental question is addressed: To what extent can a positive trading gain $g(t) > 0$ be assured? In the previously mentioned case, when the idealized market is defined by continuously differentiable prices, a remarkable result is obtained: By taking the instantaneous investment $I(t)$ to be a combination of two linear feedbacks, one representing a long trade and the other a short trade, for every non-zero price variation, the result $g(t) > 0$ is guaranteed. This is called a *trading gain arbitrage* and the feedback combination above is called the *Simultaneous Long-Short* (SLS) linear feedback control strategy.

The main objective of this paper is to explore the efficacy of this same SLS controller in an idealized GBM market. In this stochastic setting, we can no longer guarantee $g(t) > 0$. However, we prove that the SLS controller in a GBM market possesses an analogous property in which a positive *expected* trading gain $\mathbb{E}[g(t)] > 0$ is guaranteed for every non-zero drift for $t > 0$. Furthermore, we provide closed form expressions for both the trading gain and its probability density function.

II. IDEALIZED GEOMETRIC BROWNIAN MOTION MARKET

As explained in Section 1, we take the point of view that a feedback control trading strategy must provide theoretical certifications of performance in an idealized market prior to a full-scale back-test. Such performance certification results are viewed as a necessary condition for credibility in real markets where things are much less predictable.

In the sequel, at time $t \geq 0$, we use notation $p(t)$ for the *stock price*, $I(t)$ for the *amount invested* with $I(t) < 0$ being a short sale and $g(t)$ for the *cumulative trading gain*

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on $[0, t]$. Since the purpose of this paper is to concentrate exclusively on $g(t)$, the analysis to follow does not include variables such as account value, cash reserves, interest and broker margin and interest accrual.

Continuous Trading: It is assumed that the trader can react instantaneously to observed price variations. Motivation for this assumption is derived in part from the world of high-frequency trading where the ability exists to literally execute thousand of trades per second. It is also noted that this assumption is present in the celebrated Black-Scholes model; e.g., see [17].

Costless Trading: It is assumed that trading occurs with no brokerage commissions, fees or margin costs. In practice, this corresponds to the situation faced by a large trader or investment house.

Perfect Liquidity: It is assumed that the trader faces no gap between a stock's bid and ask prices, and, consistent with the continuous trading assumption, orders are filled instantaneously at the market price $p(t)$.

Trader as a Price-Taker: It is assumed that the trader is not trading sufficiently large blocks of stock so as to have an influence on the price. Note that this assumption would be faulty in the case of a large hedge or mutual fund.

Adequate Resources: It is assumed the the trader has adequate resources so that no transactions are stopped due to a failure to meet collateral requirements. For example, this can be satisfied if the account has a suitably large cash balance or if other securities in the account, not bought on margin, provide adequate collateral.

Stock Price Governed by Geometric Brownian Motion: In this type of idealized market, we assume that prices are governed by the stochastic equation

$$\frac{dp}{p} = \mu dt + \sigma dZ$$

where $\frac{dp}{p}$ is the return of the stock over the next instantaneous time increment dt and $Z(t)$ is a standard Brownian motion. Note that dZ can be viewed as a normal random variable distributed as $\mathcal{N}(0, dt)$. The parameter μ, often called the *drift*, captures the annualized expected return, and the parameter σ, often called the *volatility*, represents the annualized standard deviation associated with the underlying process. This is perhaps the most popular financial model for stock price movements and includes the celebrated Black-Scholes framework [17].

A. Trading Gain Dynamics

Recalling that the instantaneous investment is denoted by $I(t)$, the trading gain is readily calculated over the next incremental interval dt by multiplying the investment by the return of the stock. That is, $dg = \frac{dp}{p} I$. When I is a feedback of the form $I = f(g)$, the increment above becomes

$$dg = \frac{dp}{p} f(g) = (\mu dt + \sigma dZ) f(g)$$

which is the stochastic equation governing $g(t)$. In the section to follow, the Simultaneous Long-Short trading strategy we use is seen to be a combination of two classical linear time-invariant feedbacks. In terms of the notation above, these two feedbacks are of the form $f(g) = I_0 + Kg$ where $I_0 = I(0)$ denotes the initial investment. Moreover, as shown in the next section, the resulting stochastic equation

$$dg = \frac{dp}{p}(I_0 + Kg)$$

is seen to admit a closed form solution.

III. SLS IN AN IDEALIZED GBM MARKET

In this section, our goal is to analyze the performance of the Simultaneous Long-Short feedback controller in an idealized Geometric Brownian Motion market.

A. Simultaneous Long-Short Strategy

We first construct a controller which is a superposition of two linear feedbacks as described above, one being a long trade with $I(t) > 0$ and the other being a short trade with $I(t) < 0$. These trades can be viewed as running simultaneously in parallel. Letting $g_L(t)$ and $g_S(t)$ denote the trading gain from the long and short trades respectively, and taking $I_0 > 0$ and $K > 0$, the corresponding investments are

$$I_L(t) \doteq I_0 + K g_L(t) \quad \text{and} \quad I_S(t) \doteq -I_0 - K g_S(t)$$

leading to overall investment

$$I(t) = I_L(t) + I_S(t) = K(g_L(t) - g_S(t)).$$

The feedback control loop associated with this control law is seen in Figure 1.

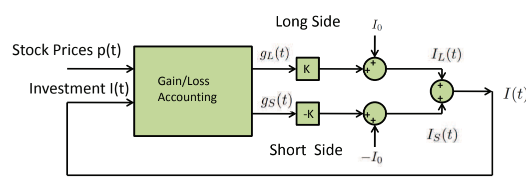

Fig. 1. Feedback Control Loop for SLS Trading

Now, the first main result, a closed form for the trading gain, has two salient features. First, under the SLS feedback controller, $g(t)$ is independent of the drift μ and the price path $p(\cdot)$ over $[0, t]$; only the final value $p(t)$ and volatility σ come into play.

Theorem 3.1: For $t \geq 0$ the SLS feedback controller leads to the trading gain

$$g(t) = \frac{I_0}{K}\left[\left(\frac{p(t)}{p(0)}\right)^K e^{\frac{1}{2}\sigma^2(K-K^2)t} + \left(\frac{p(t)}{p(0)}\right)^{-K} e^{-\frac{1}{2}\sigma^2(K+K^2)t} - 2\right].$$

Proof: We first solve for the trading gain $g_L(t)$ associated with the long strategy. Under GBM, the stochastic equation for the gain is given by

$$dg_L = (I_0 + Kg_L)(\mu dt + \sigma dZ).$$

Making the substitution $f = I_0 + Kg_L$, leads to the equation

$$df = K(\mu dt + \sigma dZ)f,$$

which itself is of the form of GBM. The solution is given in closed form as

$$f(t) = f(0)e^{(\mu K - \frac{1}{2}\sigma^2 K^2)t + \sigma K Z(t)}.$$

By substituting back into $f = I_0 + Kg_L$, solving for g_L, and recognizing that $\frac{p(t)}{p(0)} = e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma Z(t)}$, recalling that $g(0) = 0$, we obtain

$$g_L(t) \quad = \quad \frac{I_0}{K}\left(\left(\frac{p(t)}{p(0)}\right)^K e^{(\frac{1}{2}\sigma^2(K-K^2))t} - 1\right).$$

The gain on the short trade is similarly obtained by replacing I_0 and K with $-I_0$ and $-K$; i.e.,

$$g_S(t) \quad = \quad \frac{I_0}{K}\left(\left(\frac{p(t)}{p(0)}\right)^{-K} e^{-\frac{1}{2}\sigma^2(K+K^2)t} - 1\right).$$

Thus, the gain on the SLS feedback controller is given by the sum of g_L and g_S. Namely,

$$g(t) = \frac{I_0}{K}\left[\left(\frac{p(t)}{p(0)}\right)^K e^{\frac{1}{2}\sigma^2(K-K^2)t} + \left(\frac{p(t)}{p(0)}\right)^{-K} e^{-\frac{1}{2}\sigma^2(K+K^2)t} - 2\right]. \quad \square$$

Figure 2 provides a plot of the gain for various values of the volatility σ. Note that in contrast with the continuously differentiable idealized market in [2], no trading gain arbitrage is possible in the idealized GBM market. In fact, Figure 2 indicates that if the stock price $p(t)$ is within an interval about the initial price $p(0)$, the feedback controller will incur a loss. In fact, beginning with the formula above, the following sections derive the statistics of the gain, quantify its win/loss boundary, and provide an explicit expression for its probability density function.

B. Statistics of the Trading Gain

Since positivity of the trading gain is not guaranteed in this idealized market, it is natural to ask: When will the expected value of the trading gain be positive? Remarkably, the following theorem, the main result of this paper, indicates that the answer to these questions is "always." In the results to follow, we make use of the constants

$$a = e^{\frac{1}{2}\sigma^2(K-K^2)t}; \quad c = e^{-\frac{1}{2}\sigma^2(K+K^2)t}.$$

Theorem 3.2: The expectation and variance of the trading gain resulting from the SLS feedback control are

$$\mathbb{E}[g(t)] = \frac{I_0}{K}\left[e^{\mu K t} + e^{-\mu K t} - 2\right]$$

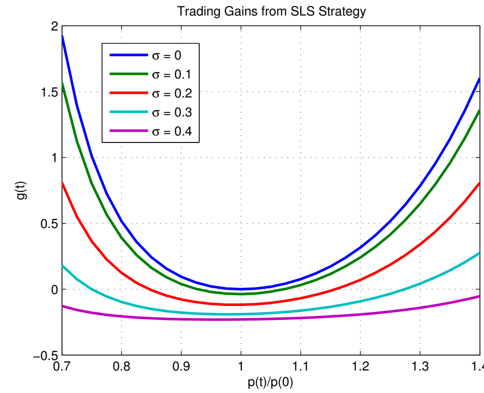

Fig. 2. Trading Gain for Various Values of σ. ($K = 8$, $t = 0.5$, $I_0 = 1$.)

and

$$Var[g(t)] = \frac{I_0^2}{K^2}\left(e^{\sigma^2 K^2 t} - 1\right)\left(e^{2\mu K t} + e^{-2\mu K t} + e^{-\sigma^2 K^2 t}\right).$$

Moreover, for all non-zero drifts μ and $t > 0$,

$$\mathbb{E}[g(t)] > 0.$$

Proof: Recall the fact that the K-th moment of a log normal random variable X with $\log X \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ is

$$\mathbb{E}[X^K] = e^{K\mu_Y + \frac{1}{2}K^2\sigma_Y^2}.$$

Now taking $X = \frac{p(t)}{p(0)}$; $\mu_Y = (\mu - \frac{1}{2}\sigma^2)t$; $\sigma_Y = \sigma^2 t$, the expected value of the trading gain is given by

$$\begin{aligned}\mathbb{E}[g(t)] \quad &= \quad \frac{I_0}{K}\left(a\mathbb{E}(X^K) + c\mathbb{E}(X^{-K}) - 2\right) \\ &= \quad \frac{I_0}{K}\left(ae^{K\mu_Y + \frac{1}{2}K^2\sigma_Y^2} + ce^{-K\mu_Y + \frac{1}{2}K^2\sigma_Y^2}\right)\end{aligned}$$

which, upon substitution for μ_Y, σ_Y, a and c, leads to the formula for $\mathbb{E}[g(t)]$. Now, to complete the proof for expectation, we simply note that $\mathbb{E}[g(t)] > 0$ follows immediately from the fact that the function $e^x + e^{-x} - 2$ is positive for $x \neq 0$. For the case of the variance, using $Var[g(t)] = \mathbb{E}[g^2(t)] - \mathbb{E}^2[g(t)]$, we again obtain a linear combination of the moments of X, and, a straightforward substitution for μ_Y, σ_Y, a and c leads to the result. $\square$

Note the remarkable property of the SLS feedback controller. The expected value of the resulting trading gain is positive. That is, for $\mu \neq 0$, one expects a positive trading gain by following the SLS feedback controller, regardless of the realized drift and volatility in the idealized GBM market.

C. Quantifying the Win/Loss Boundary

Suppose we wish to estimate the ranges of price at time t for which the trade is winning or losing. Noting that $g(t) < 0$ when $p(t) = p(0)$ and $t > 0$, it is straightforward to show that there is an open interval of prices $(p_-(\sigma, K, t), p_+(\sigma, K, t))$, about $p(0)$, in which the SLS feedback controller results in a losing trade. The following lemma quantifies this interval exactly and provides an explicit formula for the probability of a losing trade.

Lemma 3.1: The SLS feedback controller results in a losing trade $g(t) < 0$ if and only if

$$p(t) \in (p_-(\sigma, K, t), p_+(\sigma, K, t))$$

where

$$p_\pm(\sigma, K, t) \doteq \left[e^{-\frac{1}{2}\sigma^2(K-K^2)t}\left(1 - \sqrt{1 \pm e^{-\sigma^2 K^2 t}}\right)\right]^{1/K} p(0).$$

Additionally, the probability of a loss is

$$P(g(t) < 0) = \Phi\left(\frac{y_+ - \mu_Y}{\sigma_Y}\right) - \Phi\left(\frac{y_- - \mu_Y}{\sigma_Y}\right)$$

where

$$y_\pm = \log\left[\frac{1}{a}\left(1 \pm \sqrt{1 - ac}\right)\right]^{1/K},$$

and $\Phi(\cdot)$ is the cumulative distribution function of the standard normal.

Proof: Substituting $X = \left(\frac{p(t)}{p(0)}\right)^K$ into the gain equation and setting it equal to zero gives $aX^2 - 2X + c = 0$. Then, solving using this quadratic equation and substituting back in for a, c and X leads to lower and upper loss boundary limits

$$p_\pm(\sigma, K, t) = \left[e^{-\frac{1}{2}\sigma^2(K-K^2)t}\left(1 - \sqrt{1 \pm e^{-\sigma^2 K^2 t}}\right)\right]^{1/K} p(0).$$

Using the simplifying notation a and c introduced above, the probability of a loss becomes

$$P(g(t) < 0) = P\left(\left[\frac{1}{a}\left(1 - \sqrt{1 - ac}\right)\right]^{1/K} < \frac{p(t)}{p(0)} < \left[\frac{1}{a}\left(1 + \sqrt{1 - ac}\right)\right]^{1/K}\right).$$

Recalling that $p(t)/p(0)$ is generated via Geometric Brownian Motion, this random variable is log normally distributed. That is, the random variable $Y = \log \frac{p(t)}{p(0)}$ is normally distributed with mean $\mu_Y = (\mu - \frac{1}{2}\sigma^2)t$ and standard deviation given by $\sigma_Y = \sigma\sqrt{t}$. Now introducing the classical normalization, it follows that the random variable $Z \doteq \frac{Y - \mu_Y}{\sigma_Y}$ is standard normal; i.e., $\mathcal{N}(0, 1)$. Thus, defining

$$y_\pm \doteq \log\left[\frac{1}{a}\left(1 \pm \sqrt{1 - ac}\right)\right]^{1/K}$$

the probability of loss can be written as

$$P(g(t) < 0) = \Phi\left(\frac{y_+ - \mu_Y}{\sigma_Y}\right) - \Phi\left(\frac{y_- - \mu_Y}{\sigma_Y}\right). \square$$

Figure 3 shows the probability of loss as a function of the volatility σ and the gain K.

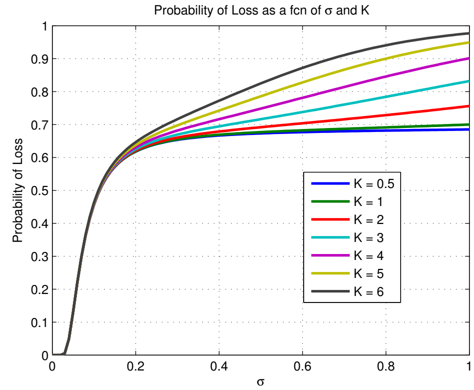

Fig. 3. Probability of Loss for $\mu = 0.15$, $t = 0.5$, $I_0 = 1$.

D. PDF for the SLS trading gain

The main result in this section is a closed form expression for the probability density function for the trading gain $g(t)$. The following notation facilitates presentation of results in the theorem to follow.

$$X_\pm(x, t) \doteq \frac{1}{2}e^{\frac{1}{2}\sigma^2(K-K^2)t}\left[\left(\frac{K}{I_0}x + 2\right) \pm \sqrt{\left(\frac{K}{I_0}x + 2\right)^2 - 4e^{-\sigma^2 K^2 t}}\right];$$

$$\nu \doteq \mu - \frac{\sigma^2}{2}; \quad Z_+(x, t) \doteq \left(\frac{\log X_+^{\frac{1}{K}}(x, t) - \nu t}{\sigma\sqrt{t}}\right)^2;$$

$$Z_-(x, t) \doteq \left(\frac{\log X_-^{\frac{1}{K}}(x, t) - \nu t}{\sigma\sqrt{t}}\right)^2;$$

$$A(x, t) \doteq \frac{1}{\sigma I_0\sqrt{2\pi t}\sqrt{\left(\frac{K}{I_0}x + 2\right)^2 - 4e^{-\sigma^2 K^2 t}}}.$$

Theorem 3.3: For $t > 0$, the probability density function $f(x, t)$ for the trading gain $g(t)$ is as follows: For

$$x \le \frac{2I_0}{K}\left[e^{-\frac{1}{2}\sigma^2 K^2 t} - 1\right] \doteq g^*(t),$$

$f(x, t) \equiv 0$, and, for $x > g^*(t)$,

$$f(x, t) = A(x, t)\left(e^{-\frac{1}{2}Z_+^2(x,t)} + e^{-\frac{1}{2}Z_-^2(x,t)}\right).$$

Proof: Beginning with the formula

$$g(t) = \frac{I_0}{K}\left[\left(\frac{p(t)}{p(0)}\right)^K e^{\frac{1}{2}\sigma^2(K-K^2)t} + \left(\frac{p(t)}{p(0)}\right)^{-K} e^{-\frac{1}{2}\sigma^2(K+K^2)t} - 2\right]$$

for the trading gain, to simplify calculations, we introduce the temporary shorthand notation $X \doteq \left(\frac{p(t)}{p(0)}\right)^K$. Noting that the pdf $f(x, t)$ is zero when x is below the minimum of $g(t)$, we break the proof into two cases:

The Zero pdf Case: To minimize $g(t)$ with respect to X, we set the derivative of the convex function

$$G(X) \doteq aX + cX^{-1} - 2$$

to zero and obtain minimizer $X^* = \sqrt{\frac{c}{a}} = e^{-\frac{1}{2}\sigma^2 Kt}$ and associated price ratio $\frac{p^*(t)}{p(0)} = e^{-\frac{1}{2}\sigma^2 t}$. Hence, the trading gain corresponds to a loss given by $g^*(t)$. Now it follows that for $x \leq g^*(t)$, the cdf is given by

$$F(x,t) \doteq P\left(g(t) \leq x\right) = 0$$

and the corresponding pdf is $f(x,t) = \frac{\partial F(x,t)}{\partial x} = 0$.

The Positive pdf Case: We first calculate the cdf

$$F(x,t) = P\left(g(t) \leq x\right) = P\left(\frac{I_0}{K}(aX + cX^{-1} - 2) \leq x\right).$$

Letting $b \doteq \frac{K}{I_0}x + 2$, the cdf above reduces to

$$\begin{aligned} F(x,t) &= P(aX^2 - bx + c \leq 0) \\ &= P(X_-(x,t) \leq X \leq X_+(x,t)) \end{aligned}$$

where $X_\pm \doteq \frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$. Now expressing the cdf in terms of the underlying prices and noting that the random variable

$$Y \doteq \log \frac{p(t)}{p(0)} = \log X^{\frac{1}{K}}$$

is $\mathcal{N}(\nu t, \sigma^2 t)$, using the classical normalization, it follows that $Z \doteq \frac{\log X^{\frac{1}{K}} - \nu t}{\sigma \sqrt{t}}$ is $\mathcal{N}(0,1)$ and

$$F(x,t) = P(Z_- \leq Z \leq Z_+) = \frac{1}{\sqrt{2\pi}} \int_{Z_-}^{Z_+} e^{-\frac{\zeta^2}{2}} d\zeta.$$

Now, to obtain the pdf, we use Leibnitz rule to differentiate the integral above. That is, $f(x,t) = f_+(x,t) - f_-(x,t)$ where

$$f_\pm(x,t) \doteq \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}Z_\pm^2} \frac{\partial Z_+}{\partial x}.$$

Now calculating the partial derivative

$$\frac{\partial Z_+}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\log X_+^{\frac{1}{K}} - \nu t}{\sigma\sqrt{t}}\right) = \frac{1}{K X_+ \sigma \sqrt{t}} \frac{\partial X_+}{\partial x}$$

and substituting into f_+ (and similarly for f_-), we obtain

$$f_\pm(x,t) = \frac{1}{K X_\pm \sigma \sqrt{2\pi t}} e^{-\frac{1}{2}Z_\pm^2} \frac{\partial X_\pm}{\partial x}.$$

To complete the calculation of f, we note that it is straightforward to verify that

$$\begin{aligned} \frac{1}{X_+}\frac{\partial X_+}{\partial x} &= \frac{K}{I_0\sqrt{b^2 - 4ac}} \\ &= \frac{K}{I_0\sqrt{\left(\frac{K}{I_0}x + 2\right)^2 - 4e^{-\sigma^2 K^2 t}}} \\ &= -\frac{1}{X_-}\frac{\partial X_-}{\partial x}. \end{aligned}$$

Now substituting into f_+, f_- and f and recalling the definition of $A(x,t)$, obtain the formula for $f(x,t)$. □

Figure 4 includes illustrative plots of the probability density function $f(x,t)$ for the trading gain $g(t)$ which were obtained via the theorem with $K = 4$, $t = 0.5$ and $\sigma = 0.2$. One salient feature of these plots, consistent with common sense, is that the larger the realized value of the ratio $\gamma \doteq \frac{|\mu|}{\sigma}$, the more attractive the trade becomes. For example, in the figure, for those plots when $\gamma > 2$, it is obvious by inspection that the probability of a significant rate of return is quite high; e.g., when $|\mu| = 1$, by performing a straightforward integration, and recalling that $I_0 = 1$, we conclude that the probability that the raw rate of return is 100% is about 0.55. For this same scenario, the probability of a 25% return is about 0.94.

On the other hand, when the realized value of the γ ratio is small, the likelihood of a loss can be quite high. For example, for the "driftless" case corresponding to $\gamma = 0$, a straightforward integration of the density function results in a probability $p \approx 0.62$ that $g^*(0.5) < 0$ and a probability $p \approx 0.2$ that a raw return of 10% or more will result. Finally, to provide a balanced picture of this low γ trade, it is important to recall one important aspect of the theorem: That is, for the simultaneous long-short feedback controller, the loss is no more than $g^*(t)$ with probability one. For this $\gamma = 0$ scenario under consideration, using the formula in the theorem, we obtain $g^*(0.5) \approx -0.07$. That is, a loss of about 7% is the maximum that can occur.

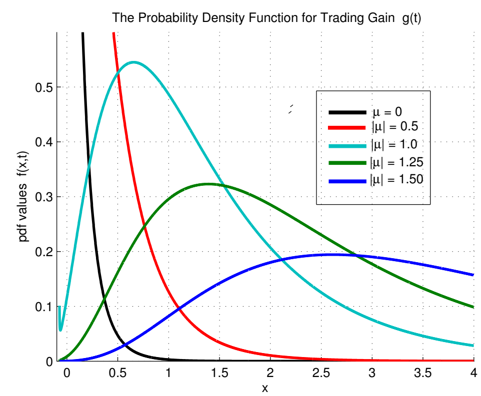

Fig. 4. Probability Density Function for $g(t)$ for Various Values of μ with $\sigma = 0.2$, $K = 4$, $t = 0.5$ and $I_0 = 1$.

IV. Numerical Example

We performed a massive numerical test of the SLS feedback controller using $2,400,000$ GBM price paths. Each path was obtained by making a selection of μ and σ and then obtaining 252 discrete-time prices corresponding to daily closing for one year. Subsequently, we used an initial investment of $I_0 = 1$ and a feedback gain of $K = 4$. To illustrate the robustness of the SLS feedback controller, for each value of μ satisfying $-0.3 \leq \mu \leq 0.3$, we generated $N = 40,000$ GBM price paths obtained by randomly selecting σ using a uniform distribution over the interval $[0, 2\mu]$.

Evaluation of Performance: Corresponding to each value of μ above, a "raw return" was calculated as follows:

Letting $G_i(\mu)$ and $I_i(\mu)$ denote the trading gain and mean absolute investment respectively, via simulation, we obtained a raw annualized return $R_i(\mu) \doteq \frac{G_i(\mu)}{I_i(\mu)}$. Then, to obtain the overall return corresponding to drift μ, the raw return $R_i(\mu)$ for the i-th path was assigned weight w_i in accordance with the amount invested; i.e.,$w_i \doteq \frac{I_i(\mu)}{\sum_{i=1}^{N} I_i(\mu)}$. Hence, for drift μ, the overall return is calculated to be

$$R(\mu) \doteq \sum_{i=1}^{N} w_i R_i(\mu) = \frac{\sum_{i=1}^{N} G_i(\mu)}{\sum_{i=1}^{N} I_i(\mu)}.$$

In Figure 5, a plot of $R(\mu)$ versus μ is given. Consistent

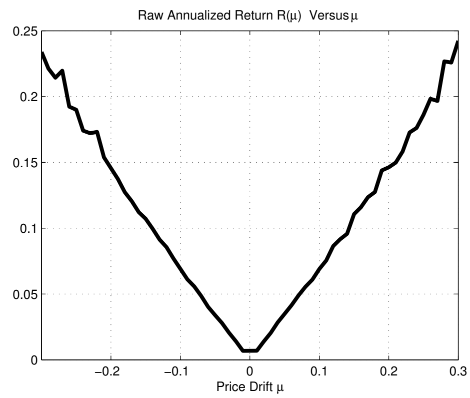

Fig. 5. Plot of the Raw Annualized Return for the SLS Feedback Control.

with the results of Theorem 3.2, in every case, the expected trading gain of the SLS feedback controller is positive. In generating the $R(\mu)$ plot, we also kept track of the convergence of partial sums over the course of the simulation. Our simulations indicate that the distribution of trading gains is heavily positively skewed. For example, it is apparent that a small number of the GBM paths result in very high trading gains $G_k(\mu)$. There are also a large number of paths resulting in a loss. However, in view of Theorem 3.3, these losses are very small compared to the large profits which result from the exceptional trades; see the conclusion for further discussion.

V. Conclusions and Further Research

In this paper, we provided a detailed analysis of the performance of the Simultaneous Long-Short (SLS) feedback controller in an idealized Geometric Brownian Motion (GBM) market. In addition to deriving closed form expressions for the trading gain and its density function, we also proved that the SLS controller possesses the remarkable property of a positive expected trading gain $\mathbb{E}[g(t)] > 0$ for all non-zero drifts μ and all $t > 0$. The simulations in Section IV indicate that profits often resulted from "exceptional trades." The attainment of such profits typically involved a large investment $I(t)$ which was permitted by the adequate resource assumption. In view of these considerations, it would be of interest to conduct future research which includes an upper bound $|I(t)| \leq I_{max}$ on the investment. Another natural continuation of this work involves the development of an "adaptive SLS" feedback controller which essentially adjusts the gain K entering into the investment $I(t)$ based on considerations such as the "realized" drift and volatility or other factors such as the need to regulate the amount at risk.

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