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A Geometric Perspective on H_2-Optimal Rejection by Measurement Feedback in Strictly-Proper Systems: The Continuous-Time Case

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Abstract— **In this work, we develop a geometric method for solving the problem of H_2-optimal rejection of disturbance inputs in continuous-time linear systems without feedthrough terms from the control input and the disturbance input to the controlled output and the measured output. A necessary and sufficient condition for the solvability of the problem is expressed in terms of a pair of subspaces, a controlled-invariant subspace and a conditioned-invariant subspace, derived from the Hamiltonian systems associated with the problem. The if-part of the proof shows how to synthesize the feedback regulator, which is non-strictly-proper in general.**

I. Introduction

The continuous-time H_2-optimal control problem by measurement feedback is completely solved and well-settled in the regular case: i.e., when the linear map from the control input to the controlled output is injective, the linear map from the disturbance input to the measured output is surjective, and the subsystems involved do not have invariant zeros on the imaginary axis (see, e.g., [1], [2], [3] as recent references). The treatment becomes much more intricate when the singular problem is tackled: i.e., when the abovementioned assumptions of injectivity and surjectivity are removed. As was pointed out in [4], the separation principle, that, in the regular case, allows us to reduce the original problem to an optimal control problem by state feedback and an optimal filtering problem, does not hold anymore, in general, and the infimum of the performance index is not always attainable.

The singular H_2-optimal control problem was often approached by considering two different solutions separately: the solution where the dynamic feedback regulator is assumed to be strictly-proper and that where the dynamic feedback regulator is assumed to be non-strictly-proper (see, e.g., [1], [4], [5], [6]). The methodologies developed in the abovementioned works in order to prove necessary and sufficient conditions for solvability of the H_2-optimal control problem by dynamic measurement feedback and the corresponding procedures for synthesizing the dynamic feedback controllers are based on the use of tools like linear matrix inequalities and special coordinate basis (see, e.g., [1], but also the more recent [7] and [8]).

In this work, we concentrate our attention on the H_2-optimal rejection problem in the special case where there are no feedthrough terms from the control input and the disturbance input to the controlled output and the measured output and we propose a procedure that, by means of the sole geometric tools (see, e.g., [9], [10]), provides the feedback regulator on the more general assumption that this be non-strictly-proper. Besides, the procedure retrieves the strictly-proper feedback regulator when the separation principle holds. The methodological approach is inspired by that of [11] and [12], where the singular H_2-optimal control problem by state feedback and the finite-horizon linear quadratic optimal control problem, respectively, were solved by referring to the associated Hamiltonian systems. In this work, the study of the geometric properties of the pair of the Hamiltonian systems related to the original problem, one connected with the optimal control problem by state feedback and the other with the optimal filtering problem, yields a pair of resolving subspaces for the original problem. Then, the synthesis of the feedback regulator is based on the computation of linear maps which are friends of the resolving subspaces and on the application of suitable projections.

Notation: $\mathbb{R}$ stands for the set of real numbers and $\mathbb{R}^+$ for the set of the nonnegative real numbers. $\mathbb{C}$, $\mathbb{C}^-$, and $\mathbb{C}^0$ respectively stand for the complex plane, the open left-half complex plane, and the imaginary axis. Matrices and linear maps are denoted by capital letters, like A. The spectrum, the image, and the kernel of A are denoted by $\sigma(A)$, $\operatorname{im} A$, and $\ker A$, respectively. The trace, the transpose, the inverse, and the Moore-Penrose inverse of A are denoted by $\operatorname{tr}(A)$, $A^\top$, A^{-1}, and A^+, respectively. The restriction of a linear map A to an A-invariant subspace $\mathcal{J}$ is denoted by $A|_{\mathcal{J}}$. The quotient space of a vector space $\mathcal{X}$ over a subspace $\mathcal{V} \subseteq \mathcal{X}$ is denoted by $\mathcal{X}/\mathcal{V}$. The orthogonal complement of $\mathcal{V}$ is denoted by $\mathcal{V}^\perp$. The direct sum of two subspaces $\mathcal{V}$ and $\mathcal{W}$ is denoted by $\mathcal{V} \oplus \mathcal{W}$. The dimension of $\mathcal{V}$ is denoted by $\dim \mathcal{V}$. The symbol $\uplus$ denotes union with multiplicity count. The symbols $I?_n$ and $O_{m\times n}$ are respectively used for the identity matrix of dimension n and the $m \times n$ zero matrix (subscripts are omitted when the dimensions are clear from the context). The symbol $\|x\|$ denotes the Euclidean norm of the vector $x \in \mathbb{R}^n$. The symbol $\mathbf{E}[\cdot]$ stands for the expectation operation. The symbol $\|v(t)\|_{\ell_2}$ denotes the ℓ_2 norm of the signal $v(t)$. The symbol $\|w(t)\|_{\mathrm{rms}}$ denotes the root mean square norm of the signal $w(t)$. Moreover, specific geometric notions and properties extensively used in this work are collected in Appendix.

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II. Problem Statement

Consider the continuous-time linear time-invariant system

$$\dot{x}(t) \;=\; A\,x(t) + B\,u(t) + H\,h(t), \tag{1}$$
$$y(t) \;=\; C\,x(t), \tag{2}$$
$$e(t) \;=\; E\,x(t), \tag{3}$$

where $x \in \mathcal{X} = \mathbb{R}^n$ is the state, $u \in \mathbb{R}^p$ the control input, $h \in \mathbb{R}^s$ the disturbance input, $y \in \mathbb{R}^q$ the measured output, $e \in \mathbb{R}^r$ the to-be-controlled output (with $p, s, q, r \leq n$). The set of the admissible control input signals is assumed to be the set $\mathcal{U}_f$ of all piecewise-continuous functions defined in $\mathbb{R}^+$ and with finite values in $\mathbb{R}^p$. A, B, H, C, E are constant real matrices of appropriate dimensions. Without loss of generality, B and H are assumed to be of full column rank, C and E of full row rank.

Moreover, let the following assumptions hold:

$\mathcal{A}\,1$. the pair (A, B) be stabilizable, or, equivalently, $\mathcal{R} = \min \mathcal{J}(A, \mathcal{B})$ be externally stable;

$\mathcal{A}\,2$. the pair (A, C) be detectable, or, equivalently, $\mathcal{Q} = \max \mathcal{J}(A, \mathcal{C})$ be internally stable.

In system (1)–(3), there are no feedthrough terms from the inputs u and h to the outputs y and e. With a slight abuse of terminology, we will characterize a system with this property as a strictly-proper system.

Consider the dynamic feedback controller

$$\dot{z}(t) \;=\; N\,z(t) + M\,y(t), \tag{4}$$
$$u(t) \;=\; L\,z(t) + K\,y(t), \tag{5}$$

where $z \in \mathcal{X} = \mathbb{R}^n$ is the state and N, M, L, K are constant real matrices of appropriate dimensions.

Then, the closed-loop system is described by the state and output equations

$$\dot{x}_c(t) \;=\; A_c\,x_c(t) + H_c\,h(t), \tag{6}$$
$$e(t) \;=\; E_c\,x_c(t), \tag{7}$$

where

$$A_c = \begin{bmatrix} A + BKC & BL \\ MC & N \end{bmatrix}, \quad H_c = \begin{bmatrix} H \\ O \end{bmatrix}, \tag{8}$$
$$E_c = \begin{bmatrix} E & O \end{bmatrix}, \tag{9}$$

or by the transfer matrix

$$G(s) = E_c(sI - A_c)^{-1}H_c. \tag{10}$$

The H_2-optimal rejection problem by measurement feedback consists in finding a dynamic feedback controller like (4), (5) such that the closed-loop system (6), (7), with (8), (9), be asymptotically stable and the H_2-norm of the transfer matrix $G(s)$, defined by (10), namely

$$\|G(s)\|_2 = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{tr}[G^*(j\omega)G(j\omega)]\mathrm{d}\omega \right)^{1/2}, \tag{11}$$

where $G^*(j\omega)$ denotes the complex conjugate transpose of $G(j\omega)$, be minimal.

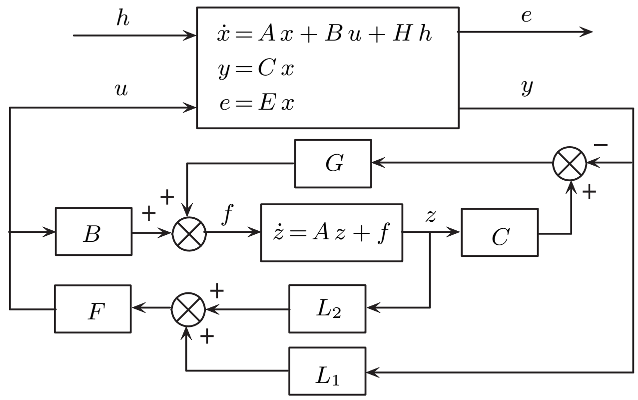

Fig. 1. Block diagram for H_2-optimal rejection by measurement dynamic feedback in strictly-proper systems

III. Scheme of the Feedback Controller

In this section, we will go into the details of the design of the dynamic feedback regulator. The peculiarity of the solution that will be illustrated is that the information on the state of the to-be-controlled system is derived as a linear combination of the measured output of the system and the state of the observer.

A similar scheme was formerly adopted to achieve exact disturbance decoupling by dynamic measurement feedback in [10]. However, in this work, the computation of the specific linear combination will be made in order to guarantee the minimal H_2-norm of the closed-loop system. Hence, it will be based on the use of subspaces which are functional to this objective and that will be determined in Section IV. Moreover, as will be shown in Section V, our procedure directly provides the strictly-proper feedback regulator when this is able to attain the optimal solution.

A detailed block diagram of the dynamic feedback regulator and its connections with the to-be-controlled system is shown in Fig. 1. Consequently, the matrices in (4), (5) are

$$N \;=\; A + BFL_2 + GC, \quad M = BFL_1 - G, \tag{12}$$
$$L \;=\; FL_2, \quad K = FL_1. \tag{13}$$

Then, from now on, we will concentrate our attention on the computation of the matrices F, G, L_1, L_2.

IV. Hamiltonian Systems and Resolving Subspaces

In this section, the H_2-optimal control problem by state feedback and the H_2-optimal filtering problem respectively associated with the H_2-optimal rejection problem by measurement feedback stated in Section II will be reconsidered from a geometric approach perspective. The aim of this discussion is to derive the subspaces needed to express the main condition for solvability of the H_2-optimal rejection problem by measurement feedback and to synthesize the feedback regulator.

A. The H_2-Optimal Control Problem by State Feedback

First, we will focus on the H_2-optimal control problem by state feedback and we will review the basics of the geometric approach to its solution. This approach was first presented for discrete-time systems in [11] and later transferred to continuous-time systems (see, e.g., [13]).

As mentioned in Section II, the set of the admissibile control input signals is the set $\mathcal{U}_f$ of all piecewise continuous functions defined in $\mathbb{R}^+$ and with finite values in $\mathbb{R}^p$. In particular, this means that distributions are not to be regarded as admissible control inputs. In the light of this restraint, it is convenient to replace the usual statement of the H_2-optimal control problem in the frequency domain with a modified statement in the time domain, the solution of which consists of a subspace of the state space and a state feedback.

Problem 1: Consider the system

$$\dot{x}(t) = A\,x(t) + B\,u(t), \tag{14}$$

$$e(t) = E\,x(t). \tag{15}$$

Find the maximal subspace $\mathcal{V}^*_{\mathcal{H}_2} \subseteq \mathcal{X}$ and a state feedback $F_{\mathcal{H}_2}$ such that any state trajectory $x(t)$, with $t \geq 0$, of

$$\dot{x}(t) = (A + BF_{\mathcal{H}_2})\,x(t),$$

starting from $x(0) = x_0 \in \mathcal{V}^*_{\mathcal{H}_2}$, satisfies

(i) $\lim_{t\to\infty} \|x(t)\| = 0$,

(ii) $\int_0^\infty x^\top(t)\,E^\top E\,x(t)\mathrm{d}t = \min_{F\in\mathcal{F}} \|e(t)\|^2_{\ell_2}$,

where $\mathcal{F}$ is the set of all F such that (i) holds.

The Hamiltonian function associated with Problem 1 is

$$\boldsymbol{H}(x(t),\lambda(t),u(t)) = x^\top(t)\,E^\top E\,x(t) + \lambda^\top(t)\,[A\,x(t) + B\,u(t)], \tag{16}$$

where $\lambda(t)$ is an undetermined multiplier also called the costate. The state and costate equations and the stationarity condition are

$$\dot{x}(t) = \left(\frac{\partial \boldsymbol{H}}{\partial \lambda}\right)^\top = A\,x(t) + B\,u(t), \tag{17}$$

$$\dot{\lambda}(t) = -\left(\frac{\partial \boldsymbol{H}}{\partial x}\right)^\top = -2\,E^\top E\,x(t) - A^\top\,\lambda(t), \tag{18}$$

$$0 = \left(\frac{\partial \boldsymbol{H}}{\partial u}\right)^\top = B^\top\,\lambda(t), \tag{19}$$

respectively. Let $p(t) = 2\lambda(t)$. Then, the differential equations (17), (18), and the algebraic equation (19) can be written as the system

$$\begin{bmatrix} \dot{x}(t) \\ \dot{p}(t) \end{bmatrix} = \begin{bmatrix} A & O \\ -E^\top E & -A^\top \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} + \begin{bmatrix} B \\ O \end{bmatrix} u(t), \tag{20}$$

$$\eta(t) = \begin{bmatrix} O & B^\top \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix}, \tag{21}$$

where the output $\eta(t)$ is required to be zero for all $t \geq 0$. The system (20), (21) will henceforth be referred to as the continuous-time Hamiltonian system associated with Problem 1. It can also be written in compact form as

$$\dot{\tilde{x}}(t) = \tilde{A}\,\tilde{x}(t) + \tilde{B}\,u(t), \tag{22}$$

$$\eta(t) = \tilde{E}\,\tilde{x}(t), \tag{23}$$

where

$$\tilde{A} = \begin{bmatrix} A & O \\ -E^\top E & -A^\top \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B \\ O \end{bmatrix}, \tag{24}$$

$$\tilde{E} = \begin{bmatrix} O & B^\top \end{bmatrix}. \tag{25}$$

Problem 2: Consider system (22), (23), with (24), (25), and let $\tilde{\mathcal{X}}$ denote the corresponding state space. Find the maximal subspace $\tilde{\mathcal{V}}^*_g \subseteq \tilde{\mathcal{X}}$ and a linear map $\tilde{F}$ such that any state trajectory $\tilde{x}(t)$, with $t \geq 0$, of the autonomous system

$$\dot{\tilde{x}}(t) = (\tilde{A} + \tilde{B}\tilde{F})\,\tilde{x}(t),$$

with initial state $\tilde{x}(0) = \tilde{x}_0 \in \tilde{\mathcal{V}}^*_g$, satisfies

(i) $\lim_{t\to\infty} \|\tilde{x}(t)\| = 0$,

(ii) $\eta(t) = 0$ for all $t \geq 0$.

Problem 2 is a variant of the classical disturbance decoupling problem with stability (see, e.g., [3], [9], [10]). As can be proved by adapting the results of the geometric approach to this slightly-different formulation, a solution of Problem 2 is a pair $(\tilde{\mathcal{V}}^*_g, \tilde{F})$, where $\tilde{\mathcal{V}}^*_g$ is the maximal internally stabilizable $(\tilde{A}, \tilde{\mathcal{B}})$-controlled invariant subspace contained in $\tilde{\mathcal{E}}$ ($\tilde{\mathcal{B}}$ stands for $\operatorname{im}\tilde{B}$ and $\tilde{\mathcal{E}}$ for $\ker\tilde{E}$) and $\tilde{F}$ is a linear map such that

$$(\tilde{A} + \tilde{B}\tilde{F})\tilde{\mathcal{V}}^*_g \subseteq \tilde{\mathcal{V}}^*_g, \tag{26}$$

$$\sigma((\tilde{A} + \tilde{B}\tilde{F})|_{\tilde{\mathcal{V}}^*_g}) \subset \mathbb{C}^-. \tag{27}$$

Note that $\tilde{F}$ is nonunique, in general, due to assignability of the spectra $\sigma((\tilde{A}+\tilde{B}\tilde{F})|_{\mathcal{R}_{\tilde{\mathcal{V}}^*}})$ and $\sigma((\tilde{A}+\tilde{B}\tilde{F})|_{(\tilde{\mathcal{V}}^*+\tilde{\mathcal{R}})/\tilde{\mathcal{V}}^*})$, where $\tilde{\mathcal{V}}^* = \max\mathcal{V}(\tilde{A},\tilde{\mathcal{B}},\tilde{\mathcal{E}})$, $\mathcal{R}_{\tilde{\mathcal{V}}^*} = \tilde{\mathcal{V}}^* \cap \min\mathcal{S}(\tilde{A},\tilde{\mathcal{E}},\tilde{\mathcal{B}})$, and $\tilde{\mathcal{R}} = \min\mathcal{J}(\tilde{A},\tilde{\mathcal{B}})$ (see Appendix for references).

A pair $(\tilde{\mathcal{V}}^*_g, \tilde{F})$ that solves Problem 2 is related to a pair $(\mathcal{V}^*_{\mathcal{H}_2}, F_{\mathcal{H}_2})$ that solves Problem 1 as follows. Let

$$\tilde{\mathcal{V}}^*_g = \operatorname{im}\tilde{V}^*_g = \operatorname{im}\begin{bmatrix} V_X \\ V_P \end{bmatrix}, \tag{28}$$

where the matrix on the right-hand side of (28) is a basis matrix of $\tilde{\mathcal{V}}^*_g$ partitioned according to $\tilde{A}$ in (24). Let

$$\tilde{F} = \begin{bmatrix} F_X & F_P \end{bmatrix}, \tag{29}$$

be the matrix, partitioned according to $\tilde{A}$ and $\tilde{B}$ in (24), that represents the linear map $\tilde{F}$ with respect to the same coordinates. Then, a pair $(\mathcal{V}^*_{\mathcal{H}_2}, F_{\mathcal{H}_2})$ consists of the subspace

$$\mathcal{V}^*_{\mathcal{H}_2} = \operatorname{im}V_X \tag{30}$$

and the linear map represented by

$$F_{\mathcal{H}_2} = F_X + F_P V_P V_X^+, \tag{31}$$

with respect to the same coordinates.

Note that, while in the discrete-time case the matrix V_X is an $n \times n$ invertible matrix since the subspace of the admissible initial states is the whole state space of the original system, in the continuous-time case the subspace of the initial states that can be driven asymptotically to the origin with an admissible state feedback control law does not match the whole state space in general. Hence, in formula (31), which is the continuous-time counterpart of that in [11, Section IV], the Moore-Penrose inverse of V_X replaces the inverse. The use of the Moore-Penrose inverse is correct since V_X is a full column rank matrix.

In view of the approach to the solution of the H_2-optimal rejection problem by measurement feedback that will be developed in Section V, it is worth pointing out the following property of the pair $(\mathcal{V}^*_{\mathcal{H}_2}, F_{\mathcal{H}_2})$.

Property 1: The subspace $\mathcal{V}^*_{\mathcal{H}_2}$, defined by (30), is an internally stable $(A+BF_{\mathcal{H}_2})$-invariant subspace, where $F_{\mathcal{H}_2}$ is defined by (31).

Proof: Since the subspace $\tilde{\mathcal{V}}^*_g$ is an internally stable $(\tilde{A}+\tilde{B}\tilde{F})$-invariant subspace with basis matrix $\tilde{V}^*_g$, a matrix X exists, such that

$$(\tilde{A}+\tilde{B}\tilde{F})\,\tilde{V}^*_g = \tilde{V}^*_g X, \tag{32}$$

$$\sigma(X) = \sigma((\tilde{A}+\tilde{B}\tilde{F})|_{\tilde{\mathcal{V}}^*_g}) \subset \mathbb{C}^-. \tag{33}$$

Equation (32) can also be written as

$$\begin{bmatrix} A+BF_X & BF_P \\ E^\top E & -A^\top \end{bmatrix} \begin{bmatrix} V_X \\ V_P \end{bmatrix} = \begin{bmatrix} V_X \\ V_P \end{bmatrix} X, \tag{34}$$

where (24), (28) and (29) were taken into account. From the first block of rows of (34) one gets

$$AV_X + B(F_X V_X + F_P V_P) = V_X X. \tag{35}$$

Since V_X a full column rank matrix, $V_X^+ V_X = I$. Hence,

$$F_X V_X + F_P V_P = F_{\mathcal{H}_2} V_X, \tag{36}$$

follows from (31). Finally, (35) and (36) imply

$$(A+BF_{\mathcal{H}_2})V_X = V_X X, \tag{37}$$

which, in light of (30) and (33), proves the thesis. ■

B. The H_2-Optimal Filtering Problem

This section will be focused on the H_2-optimal filtering problem associated with the H_2-optimal rejection problem by measurement feedback. Similarly to the H_2-optimal control problem by state feedback, also the H_2-optimal filtering problem will have a slightly modified formulation with respect to the usual one in the frequency domain. This is exactly the dual counterpart of Problem 1.

Problem 3: Consider the system

$$\dot{x}(t) = A\,x(t) + H\,h(t), \tag{38}$$

$$y(t) = C\,x(t), \tag{39}$$

where $h(t)$, with $t \geq 0$, is a zero-mean wide-sense-stationary white-noise stochastic process with unit intensity. Find the minimal subspace $\mathcal{S}^*_{\mathcal{H}_2} \subseteq \mathcal{X}$ and an output injection $G_{\mathcal{H}_2}$ such that any state trajectory $\hat{e}(t)$, with $t \geq 0$, of

$$\dot{\hat{e}}(t) = (A + G_{\mathcal{H}_2} C)\,\hat{e}(t) + H\,h(t),$$

with initial state $\hat{e}(0) = x_0 \notin \mathcal{S}^*_{\mathcal{H}_2}$, satisfies

(i) $\lim_{t\to\infty} \|\hat{e}(t)\| = 0$, with $h(t) = 0$ for all $t \geq 0$,

(ii) $\mathbf{E}[\hat{e}^\top(t)\,\hat{e}(t)] = \min_{G\in\mathcal{G}} \|\hat{e}(t)\|^2_{\mathrm{rms}}$,
where $\mathcal{G}$ is the set of all G such that (i) holds.

The variable $\hat{e}(t)$ is the error in the estimate of the state $x(t)$ of system (38), (39), with initial state $x(0) = x_0$, obtained with a full-order observer

$$\dot{z}(t) = (A + G_{\mathcal{H}_2} C)\,z(t) - G_{\mathcal{H}_2}\,y(t),$$

with initial state $z(0) = 0$.

The solution of Problem 3 will be derived from that of Problem 1, where the triple (A, B, E) is replaced by the triple $(A^\top,\, C^\top,\, H^\top)$, by means of simple duality arguments. Let $(\mathcal{V}^*_{\mathcal{H}_2,(A^\top,\,C^\top,\,H^\top)}, F_{\mathcal{H}_2,(A^\top,\,C^\top,\,H^\top)})$ be a solution of Problem 1 stated for the triple $(A^\top,\, C^\top,\, H^\top)$. Then, a pair $(\mathcal{S}^*_{\mathcal{H}_2}, G_{\mathcal{H}_2})$ that solves Problem 3 is defined by

$$\mathcal{S}^*_{\mathcal{H}_2} = \left(\mathcal{V}^*_{\mathcal{H}_2,(A^\top,\,C^\top,\,H^\top)}\right)^\perp, \tag{40}$$

$$G_{\mathcal{H}_2} = F^\top_{\mathcal{H}_2,(A^\top,\,C^\top,\,H^\top)}. \tag{41}$$

Property 2: The subspace $\mathcal{S}^*_{\mathcal{H}_2}$, defined by (40), is an externally stable $(A+G_{\mathcal{H}_2}C)$-invariant subspace, where $G_{\mathcal{H}_2}$ is defined by (41).

Proof: It follows from (40), (41), and Property 1 by duality arguments. ■

As mentioned at the beginning of Section IV, the subspaces $\mathcal{V}^*_{\mathcal{H}_2}$ and $\mathcal{S}^*_{\mathcal{H}_2}$, respectively defined by (30) and (40), and the linear maps $F_{\mathcal{H}_2}$ and $G_{\mathcal{H}_2}$, respectively defined by (31) and (41), will play a crucial role in the synthesis of the feedback regulator (4), (5).

V. Main Results

In this section, we will present the main results on solvability of the H_2-optimal rejection problem by measurement feedback, by exploiting the properties of the subspaces introduced in Section IV. The proof of the main result is constructive. Hence, it will directly show how to synthesize the dynamic feedback regulator according to the scheme described in Section III. The proofs of the lemmas and properties, which are of a strictly technical nature, will be omitted for the sake of brevity.

Lemma 1: The subspace $\mathcal{V}^*_{\mathcal{H}_2}$, defined by (30), is an externally stabilizable $(A, \mathcal{B})$-controlled invariant subspace.

Property 3: Let $(\mathcal{V}^*_{\mathcal{H}_2}, F_{\mathcal{H}_2})$ be a pair that solves Problem 1. Then, a linear map F exists, such that

$$(A+BF)\mathcal{V}^*_{\mathcal{H}_2} \subseteq \mathcal{V}^*_{\mathcal{H}_2}, \tag{42}$$

$$\sigma((A+BF)|_{\mathcal{V}^*_{\mathcal{H}_2}}) = \sigma((A+BF_{\mathcal{H}_2})|_{\mathcal{V}^*_{\mathcal{H}_2}}), \tag{43}$$

$$\sigma((A+BF)|_{\mathcal{X}/\mathcal{V}^*_{\mathcal{H}_2}}) \subset \mathbb{C}^-. \tag{44}$$

Property 3 states that, on the assumption that the pair (A, B) be stabilizable, the subspace $\mathcal{V}^*_{\mathcal{H}_2}$ can be externally stabilized without affecting its internal eigenvalues, which are those determined by solving the H_2-optimal control problem. For reasons that will be clarified in the following, any linear map F that satisfies conditions (42)–(44) of

Property 3 can be assumed as the matrix F in the design of the feedback regulator described in Section III.

Lemma 2: The subspace $\mathcal{S}^*_{\mathcal{H}_2}$, defined by (40), is an internally stabilizable $(A,\mathcal{C})$-conditioned invariant subspace.

Property 4: Let $(\mathcal{S}^*_{\mathcal{H}_2}, G_{\mathcal{H}_2})$ be a pair that solves Problem 3. Then, a linear map G exists, such that

$$(A+GC)\mathcal{S}^*_{\mathcal{H}_2} \subseteq \mathcal{S}^*_{\mathcal{H}_2}, \quad (45)$$
$$\sigma((A+GC)|_{\mathcal{X}/\mathcal{S}^*_{\mathcal{H}_2}}) = \sigma((A+G_{\mathcal{H}_2}C)|_{\mathcal{X}/\mathcal{S}^*_{\mathcal{H}_2}}), \quad (46)$$
$$\sigma((A+GC)|_{\mathcal{S}^*_{\mathcal{H}_2}}) \subset \mathbb{C}^-. \quad (47)$$

Property 4 states that, on the assumption that the pair (A,C) be detectable, the subspace $\mathcal{S}^*_{\mathcal{H}_2}$ can be internally stabilized without affecting its external eigenvalues, which are those determined by solving the H_2-optimal filtering problem. For reasons that will be discussed later, any linear map G that satisfies conditions (45)–(47) of Property 4 can be assumed as the matrix G in the design of the feedback regulator described in Section III.

According to the scheme depicted in Fig. 1, the information on the state of the to-be-controlled system is obtained as a linear combination of the measured output of the system and the state of the observer. The matrices L_1 and L_2 that define the specific linear combination will be derived through the following property, by exploiting some features of the subspace $\mathcal{S}^*_{\mathcal{H}_2}$.

Property 5: Consider system (38), (39) and the subspace $\mathcal{S}^*_{\mathcal{H}_2}$ defined by (40). Let $\mathcal{L} \subseteq \mathcal{X}$ be a subspace such that

$$\mathcal{L} \oplus (\mathcal{S}^*_{\mathcal{H}_2} \cap \mathcal{C}) = \mathcal{S}^*_{\mathcal{H}_2}. \quad (48)$$

Then, linear maps L_1 and L_2 exist, such that

$$L_1 C + L_2 = I_n, \quad (49)$$
$$\ker L_2 = \mathcal{L}. \quad (50)$$

The following theorem states a necessary and sufficient conditions for solvability of the H_2-optimal rejection problem by measurement feedback.

Theorem 1: Consider system (1)–(3). Let assumptions $\mathcal{A}\,1$ and $\mathcal{A}\,2$ hold. The problem of finding a dynamic feedback controller like (4), (5), with (12) and (13), such that the closed-loop system (6), (7), with (8) and (9), be asymptotically stable and the H_2-norm of the transfer matrix $G(s)$, defined by (10), be minimal, is solvable if and only if

$$\mathcal{S}^*_{\mathcal{H}_2} \subseteq \mathcal{V}^*_{\mathcal{H}_2}, \quad (51)$$

where $\mathcal{V}^*_{\mathcal{H}_2}$ and $\mathcal{S}^*_{\mathcal{H}_2}$ are the resolving subspaces respectively defined by (30) and (40).

Proof: If. Let the matrices F, G, L_1, L_2 in the regulator equations (4), (5), with (12) and (13), be respectively defined as in Property 3, Property 4, and Property 5. Consider the closed-loop system (6), (7), with (8) and (9). The matrix A_c can be explicited with respect to F, G, L_1, L_2 as

$$A_c = \begin{bmatrix} A+BFL_1C & BFL_2 \\ BFL_1C-GC & A+BFL_2+GC \end{bmatrix}. \quad (52)$$

Perform the similarity transformation

$$T_c = \begin{bmatrix} I & O \\ I & -I \end{bmatrix},$$

so that the overall system can be written as

$$\dot{x}'_c(t) = A'_c\, x_c(t) + H'_c\, h(t), \quad (53)$$
$$e(t) = E'_c\, x_c(t), \quad (54)$$

where, also in the light of (49),

$$A'_c = \begin{bmatrix} A+BF & -BFL_2 \\ O & A+GC \end{bmatrix}, \quad H'_c = \begin{bmatrix} H \\ H \end{bmatrix}, \quad (55)$$
$$E'_c = E_c. \quad (56)$$

Consider the subspace $\mathcal{W}$, defined by

$$\mathcal{W} = \left\{ \begin{bmatrix} x \\ \hat{e} \end{bmatrix} : x \in \mathcal{V}^*_{\mathcal{H}_2},\ \hat{e} \in \mathcal{S}^*_{\mathcal{H}_2} \right\}, \quad (57)$$

with respect to the new coordinates. We will show that $\mathcal{W}$ is an internally and externally stable A_c-invariant subspace. With F determined according to Property 3, the subspace $\mathcal{V}^*_{\mathcal{H}_2}$ is an internally and externally stable $(A+BF)$-invariant subspace. With G determined according to Property 4, the subspace $\mathcal{S}^*_{\mathcal{H}_2}$ is an internally and externally stable $(A+GC)$-invariant subspace. Hence, showing that the subspace $\mathcal{W}$ is an A_c-invariant subspace reduces to showing that

$$BFL_2\, \mathcal{S}^*_{\mathcal{H}_2} \subseteq \mathcal{V}^*_{\mathcal{H}_2}. \quad (58)$$

Equations (51) and (42) imply

$$(A+BF)\, \mathcal{S}^*_{\mathcal{H}_2} \subseteq \mathcal{V}^*_{\mathcal{H}_2}.$$

Hence, all the more reasons for

$$(A+BF)\, (\mathcal{S}^*_{\mathcal{H}_2} \cap \mathcal{C}) \subseteq \mathcal{V}^*_{\mathcal{H}_2}. \quad (59)$$

By virtue of (49), (59) can also be written as

$$(A + BFL_1C + BFL_2)\, (\mathcal{S}^*_{\mathcal{H}_2} \cap \mathcal{C}) \subseteq \mathcal{V}^*_{\mathcal{H}_2}. \quad (60)$$

Since $\mathcal{S}^*_{\mathcal{H}_2}$ is an $(A,\mathcal{C})$-conditioned invariant subspace,

$$A\, (\mathcal{S}^*_{\mathcal{H}_2} \cap \mathcal{C}) \subseteq \mathcal{S}^*_{\mathcal{H}_2}. \quad (61)$$

Since $(\mathcal{S}^*_{\mathcal{H}_2} \cap \mathcal{C}) \subseteq \mathcal{C}$,

$$BFL_1C\, (\mathcal{S}^*_{\mathcal{H}_2} \cap \mathcal{C}) = \{0\}. \quad (62)$$

Therefore, in the light of (61) and (62), (60) implies

$$BFL_2\, (\mathcal{S}^*_{\mathcal{H}_2} \cap \mathcal{C}) \subseteq \mathcal{V}^*_{\mathcal{H}_2}, \quad (63)$$

Finally, (63), (48), and (50) imply (58). Furthermore, $\mathcal{W}$ is an internally and externally stable A_c-invariant subspace since $\sigma(A_c) \subset \mathbb{C}^-$. In fact, the upper block triangular structure of A'_c shows that, with F and G respectively determined according to Property 3 and Property 4, the overall system is asymptotically stable, since

$$\begin{aligned} \sigma(A'_c) &= \sigma(A+BF) \uplus \sigma(A+GC) = \\ &\sigma((A+BF)|_{\mathcal{V}^*_{\mathcal{H}_2}}) \uplus \sigma((A+BF)|_{\mathcal{X}/\mathcal{V}^*_{\mathcal{H}_2}}) \uplus \\ &\sigma((A+GC)|_{\mathcal{S}^*_{\mathcal{H}_2}}) \uplus \sigma((A+GC)|_{\mathcal{X}/\mathcal{S}^*_{\mathcal{H}_2}}), \end{aligned}$$

and, with those particular F and G, the invariant subspaces $\mathcal{V}^*_{\mathcal{H}_2}$ and $\mathcal{S}^*_{\mathcal{H}_2}$ are both internally and externally stable.

Only if. It is direct consequence of minimality of $\mathcal{S}^*_{\mathcal{H}_2}$ and maximality of $\mathcal{V}^*_{\mathcal{H}_2}$ as resolving subspaces of the associated optimal filtering problem and optimal control problem, respectively. ■

As was mentioned in Section III, the procedure described provides a strictly-proper feedback controller when specific conditions are satisfied. More precisely, if $\mathcal{S}^*_{\mathcal{H}_2} \subseteq \mathcal{C}$, then $\mathcal{S}^*_{\mathcal{H}_2} \cap \mathcal{C} = \mathcal{S}^*_{\mathcal{H}_2}$. Consequently, according to (48), $\mathcal{L} = \{0\}$. Furthermore, according to (49) and (50), $L_2 = I$ and $L_1 = 0$. This means that, in this case, the direct feedthrough from the measured output to the control input disappears and the static feedback directly involves the whole state estimate. In other words, this shows that, on the assumption $\mathcal{S}^*_{\mathcal{H}_2} \subseteq \mathcal{C}$, the separation principle holds.

VI. Conclusion

The H_2-optimal rejection problem by measurement feedback in strictly-proper systems was solved by means of pure geometric approach arguments applied to the Hamiltonian systems associated with the original problem. In particular, a necessary and sufficient condition for solvability of the problem was stated in terms of a pair of resolving subspaces directly derived from the Hamiltonian systems. The the proof of the sufficiency of the condition, which is constructive, has shown the procedure for the synthesis of the feedback regulator.

Appendix
Geometric Approach Notation and Background

The aim of this section is to review and collect some notions of the geometric approach extensively used in this paper. The reader is referred to [10] for more details.

Consider the continuous-time linear time-invariant system

$$\dot{x}(t) = A\,x(t) + B\,u(t), \tag{64}$$

$$y(t) = C\,x(t), \tag{65}$$

where A, B, C are constant real matrices of appropriate dimensions. Let B be of full column rank and C of full row rank.

Geometric objects widely employed in this work are the following: $\mathcal{X}$, the state space of (64), (65); $\mathcal{B}$, the image of B; $\mathcal{C}$, the kernel of C; $\mathcal{R} = \min \mathcal{J}(A, \mathcal{B})$, the minimal A-invariant subspace containing $\mathcal{B}$ or, equivalently, the reachable subspace of (A, B); $\mathcal{Q} = \max \mathcal{J}(A, \mathcal{C})$, the maximal A-invariant subspace contained in $\mathcal{C}$ or, equivalently, the unobservable subspace of (A, C); $\mathcal{V}^* = \max \mathcal{V}(A, \mathcal{B}, \mathcal{C})$, the maximal $(A, \mathcal{B})$-controlled invariant subspace contained in $\mathcal{C}$; $\mathcal{S}^* = \min \mathcal{S}(A, \mathcal{C}, \mathcal{B})$, the minimal $(A, \mathcal{C})$-conditioned invariant subspace containing $\mathcal{B}$; $\mathcal{R}_{\mathcal{V}^*} = \mathcal{V}^* \cap \mathcal{S}^*$, the controllability subspace on $\mathcal{V}^*$.

Some basic geometric definitions and properties exploited in the work are reviewed below. A subspace $\mathcal{J} \subseteq \mathcal{X}$, with basis matrix J, is an A-invariant subspace if and only if a matrix X exists such that $A\,J = J\,X$. The matrix X represents the linear map $A|_{\mathcal{J}}$ with respect to the same coordinates. An A-invariant subspace $\mathcal{J} \subseteq \mathcal{X}$ is said to be internally stable if $\sigma(A|_{\mathcal{J}}) \subset \mathbb{C}^-$. An A-invariant subspace $\mathcal{J}$ is said to be externally stable if $\sigma(A|_{\mathcal{X}/\mathcal{J}}) \subset \mathbb{C}^-$. A subspace $\mathcal{V} \subseteq \mathcal{X}$ is an $(A, \mathcal{B})$-controlled invariant subspace if and only if a linear map F exists, such that $(A + BF)\mathcal{V} \subseteq \mathcal{V}$. An $(A, \mathcal{B})$-controlled invariant subspace $\mathcal{V} \subseteq \mathcal{X}$ is said to be internally stabilizable if a linear map F exists, such that $(A + BF)\mathcal{V} \subseteq \mathcal{V}$ and $\sigma((A + BF)|_{\mathcal{V}}) \subset \mathbb{C}^-$. An $(A, \mathcal{B})$-controlled invariant subspace $\mathcal{V} \subseteq \mathcal{X}$ is said to be externally stabilizable if a linear map F exists, such that $(A + BF)\mathcal{V} \subseteq \mathcal{V}$ and $\sigma((A + BF)|_{\mathcal{X}/\mathcal{V}}) \subset \mathbb{C}^-$. A subspace $\mathcal{S} \subseteq \mathcal{X}$ is an $(A, \mathcal{C})$-conditioned invariant subspace if and only if a linear map G exists, such that $(A + GC)\,\mathcal{S} \subseteq \mathcal{S}$. The relation $\max \mathcal{V}(A, \mathcal{B}, \mathcal{C}) = \left(\min \mathcal{S}(A^\top, \mathcal{B}^\perp, \mathcal{C}^\perp\right)^\perp$ can be proven by means of duality arguments. Let a linear map F be such that $(A + BF)\mathcal{V}^* \subseteq \mathcal{V}^*$, then $(A + BF)\mathcal{R}_{\mathcal{V}^*} \subseteq \mathcal{R}_{\mathcal{V}^*}$ holds with the same F. The spectrum $\sigma((A + BF)|_{\mathcal{R}_{\mathcal{V}^*}})$ is assignable. The spectrum $\sigma((A + BF)|_{\mathcal{V}^*/\mathcal{R}_{\mathcal{V}^*}})$ is fixed and is also known as the set of the internal unassignable eigenvalues of $\mathcal{V}^*$ or, equivalently, as the set $\mathcal{Z}(A, B, C)$ of the invariant zeros of system (64), (65). Moreover, the spectrum $\sigma((A + BF)|_{(\mathcal{V}^* + \mathcal{R})/\mathcal{V}^*})$ is assignable and the spectrum $\sigma((A + BF)|_{\mathcal{X}/(\mathcal{V}^* + \mathcal{R})})$ is fixed.

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