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A New Observer for Nonlinear Fractional Order Systems

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Abstract— **In this work an observer structure for a certain class of nonlinear fractional order systems is proposed. For solving this task we introduce a Fractional Algebraic Observability (FAO) property which is used as a main tool in the design of the observer system. We apply our proposals in the master-slave synchronization problem, where the coupling signal is viewed as output and the slave system is regarded as observer (the slave is requested to recover the unknown state trajectories of the master). Finally, as numerical example we consider a fractional order Rössler hyperchaotic system and by means of some simulations we show the effectiveness of the suggested approach.**

I. INTRODUCTION

Fractional calculus is as old as conventional calculus, but is not as popular in science and engineering as conventional calculus. In the last three centuries this subject was studied only in mathematics, but in recent years it has been used in many fields of engineering and science [1]. "It might be that this mathematical tool help us modelling the reality in a better way and also, might be, that this is the calculus of the XXI century" [2].

Among the publications dedicated to fractional order systems, some subjects have been studied, e.g., linear systems [3], chaotic dynamics and its synchronization [4]–[7], stability [8], [9], delayed systems [10], systems identification [2], control systems [11], optimal control [12], quantitative finances [13], quantic evolution of complex systems [14], digital image processing, variational principles and its applications, Euler-Lagrange equations, applications in finances and economy, bioengineering applications, fractional Fourier transform, sliding modes, robotics, among others [15].

The synchronization problem is an interesting topic in fractional chaotic systems [16]. The synchronization of integer order chaotic systems has been extensively investigated since its introduction by Pecora and Carroll [17]. On the other hand, [18] is the first work concerning on synchronization of fractional systems, the authors showed by means of a control law that fractional order chaotic systems can be synchronized by using the similar scheme as that of their integer counterparts.

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Some techniques related to chaos synchronization in fractional systems have been proposed. For instance, we mention the works [6], [19], [20] in which the authors propose the employment of feedback controllers, which allows to achieve the synchronization between two identical fractional-order chaotic systems, the theoretical analysis is derived by utilizing the stability criterion of linear fractional systems; in [21] is studied the synchronization of fractional order chaotic systems with unidirectional linear error feedback coupling; in [22] is presented a classical Luenberger observer design for the synchronization of fractional-order chaotic systems, i.e., the observer structure needs a copy of the system and a linear output error feedback, the application is restricted to scalar coupling signals; in [23], [24] are given sufficient conditions for the synchronization between two identical fractional systems by using the Laplace transform theory.

The main contribution in this work is to show a novel technique for the synchronization problem in nonlinear fractional-order systems via observer design. Here arises a basic practical question: would it be possible to reconstruct the unknown signals? We give an answer to this question by introducing a basic definition (similar to the differential and algebraic approach used in nonlinear integer order systems [25]) related with the estimation (reconstruction) of the unknown variables, so-called Fractional Algebraic Observability (FAO) property[1]. As far as we know in the literature this class of observer structure has not been used in fractional order systems.

The rest of this paper is organized as follows. In Section II is given a brief note about fractional derivatives and Mittag-Leffler type function. Section III presents the problem statement and its solution, based on FAO and the master-slave synchronization scheme. In Section IV we apply the methodology presented in Section III to the fractional order Rössler hyperchaotic system, also some numerical results are shown. The intention of choosing this system is to clarify the proposed methodology and to highlight the simplicity and flexibility of the suggested approach. Finally, we conclude with some remarks in Section V.

II. ON FRACTIONAL DERIVATIVES

There are several definitions of a fractional derivative of order α [11], [26], [27], we will use the Caputo fractional operator in the definition of fractional order systems, because

[1] An observable system in this sense can be regarded as a system in which the unknown variables can be expressed in terms of the output signal and a finite number of its fractional derivatives.

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the meaning of the initial conditions for systems described using this operator is the same as for integer order systems.

Definition 1 (Caputo Fractional derivative): The Caputo fractional derivative of order $\alpha \in \mathbb{R}^+$ of a function x is defined as: see [11]

$$x^{(\alpha)} = {}_{t_0}D_t^\alpha x(t) = \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^{t} \frac{d^m x(\tau)}{d\tau^m}(t-\tau)^{m-\alpha-1}\, d\tau, \tag{1}$$

where: $m-1 \leq \alpha < m$, $\frac{d^m x(\tau)}{d\tau^m}$ is the m-th derivative of x in the usual sense, $m \in \mathbb{N}$ and Γ is the gamma function[2].□

Now we define the following notation

$$\mathcal{D}^{(r\alpha)}x(t) = \underbrace{{}_{t_0}D_t^\alpha\, {}_{t_0}D_t^\alpha \ldots {}_{t_0}D_t^\alpha\, {}_{t_0}D_t^\alpha}_{\text{r-times}}\, x(t) \tag{2}$$

i. e., it is the Caputo fractional derivative of order α applied $r \in \mathbb{N}$ times sequentially, with $\mathcal{D}^{(0)}x(t) = x(t)$, we can note that if $r = 1$ then $\mathcal{D}^{(\alpha)}x(t) = x^{(\alpha)}$.

A. Mittag-Leffler type function

The Mittag-Leffler function with two parameters is defined as [28]:

$$E_{\alpha,\beta}(z) = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(\alpha i + \beta)}, \quad z, \beta \in \mathbb{C},\ \mathbb{R}e(\alpha) > 0 \tag{3}$$

this function is used to solve fractional differential equations as the exponential function in integer order systems. In the particular case when $\alpha = \beta = 1$, we have that $E_{1,1}(z) = e^z$. Now if we have particular values of α, the function (3) has asymptotic behavior at infinity.

Theorem 1 ([11]): If $\alpha \in (0, 2)$, β is an arbitrary complex number and μ is an arbitrary real number such that

$$\frac{\pi\alpha}{2} < \mu < \min\{\pi,\, \pi\alpha\} \tag{4}$$

then for an arbitrary integer $\kappa \geq 1$ the following expansion holds:

$$E_{\alpha,\beta}(z) = -\sum_{i=1}^{\kappa} \frac{1}{\Gamma(\beta - \alpha i)z^i} + O\left(\frac{1}{|z|^{\kappa+1}}\right) \tag{5}$$

with $|z| \to \infty$, $\mu \leq |\arg(z)| \leq \pi$. ■

The Mittag-Leffler function has the following properties:

Property 1 [11].

$$\int_0^t \tau^{\beta-1} E_{\alpha,\beta}(-k\tau^\alpha)\, d\tau = t^\beta E_{\alpha,\beta+1}(-kt^\alpha)\ ,$$

with $\beta > 0$.

Property 2 [29]. $E_{\alpha,\beta}(-x)$ is completely monotonic, i. e., $(-1)^n E_{\alpha,\beta}^{(n)}(-x) \geq 0$ for $0 < \alpha \leq 1$ and $\beta \geq \alpha$, for all $x \in (0, \infty)$ and $n \in \mathbb{N} \cup \{0\}$.

We will use these facts in the following problem.

[2]To simplify the notation we omitted the time dependence in $x^{(\alpha)}$, in what follows we take $t_0 = 0$

III. Problem Statement and Main Result

We take the initial condition problem for an autonomous fractional order nonlinear system, with $0 < \alpha < 1$,

$$\begin{aligned} x^{(\alpha)} = &\ f(x),\ \ x(0) = x_0 \\ y = &\ h(\bar{x}) \end{aligned} \tag{6}$$

where $x \in \Omega \subset \mathbb{R}^n$, $f : \Omega \to \mathbb{R}^n$ is a Lipschitz continuous function[3], with $x_0 \in \Omega \subset \mathbb{R}^n$, in this case y denotes the output of the system (the measure that we can obtain), $\bar{x} \in \mathbb{R}^p$ represents the states that we can observe (known states), $h : \mathbb{R}^p \to \mathbb{R}^q$ is a continuous function and $1 \leq p < n$.

Consider the system given by (6), we will separate in two dynamical systems with states $\bar{x} \in \mathbb{R}^p$ and $\eta \in \mathbb{R}^{n-p}$ respectively with $x^T = (\bar{x}^T, \eta^T)$, the first system will describe the known states and the second represents unknown states, then the system (6) can be written as:

$$\begin{aligned} \bar{x}^{(\alpha)} &= \bar{f}(\bar{x}, \eta) \\ \eta^{(\alpha)} &= \Delta(\bar{x}, \eta) \\ y_{\bar{x}} &= y = h(\bar{x}) \end{aligned} \tag{7}$$

where $f^{\,T}(x) = \left(\bar{f}^{\,T}(\bar{x},\eta), \Delta^T(\bar{x},\eta)\right)$, $\bar{f} \in \mathbb{R}^p$ and $\Delta \in \mathbb{R}^{n-p}$. Now the problem is: How can we estimate the η's states? this question arises because if we know the η's states we can use these signals to generate measuring depending on them. In order to solve this observation problem let us introduce the following observability property.

Definition 2 (FAO): A state variable $\eta_i \in \mathbb{R}$ satisfies the Fractional Algebraic Observability (FAO) if it is a function of the first $r \in \mathbb{N}$ sequential derivatives (in the sense of the equation (2)) of the available output $y_{\bar{x}}$, i.e.,

$$\eta_i = \phi_i\left(y_{\bar{x}},\, y_{\bar{x}}^{(\alpha)}, \mathcal{D}^{(2\alpha)}y_{\bar{x}}, \ldots,\, \mathcal{D}^{(r\alpha)}y_{\bar{x}}\right) \tag{8}$$

where $\phi_i : \mathbb{R}^{(r+1)q} \to \mathbb{R}$. □

If we assume that the components of unknown state vector η are FAO, then we can describe our problem in terms of the master-slave synchronization scheme, which is defined in the following way.

Let us consider the master system:

$$\eta_i^{(\alpha)} = \Delta_i(\bar{x}, \eta) \tag{9}$$

$$y_{\eta_i} = \phi_i\left(y_{\bar{x}},\, y_{\bar{x}}^{(\alpha)}, \mathcal{D}^{(2\alpha)}y_{\bar{x}}, \ldots,\, \mathcal{D}^{(r\alpha)}y_{\bar{x}}\right) \tag{10}$$

for $i \in \{p+1, \ldots, n\}$, where η_i is a component of the state vector η and y_{η_i} denotes the output of the i-th master system.

Now let us propose a fractional dynamical system with the same order α, which will be the slave system (observer):

$$\hat{\eta}_i^{(\alpha)} = k_{\hat{\eta}_i}(y_{\eta_i} - \hat{\eta}_i), \tag{11}$$

$$y_{\hat{\eta}_i} = \hat{\eta}_i, \tag{12}$$

for $i \in \{p+1, \ldots, n\}$, where $\hat{\eta}_i$ is the state, and $y_{\hat{\eta}_i}$ denotes the output of the slave system and $k_{\hat{\eta}_i}$ is a positive constant.

In the master-slave synchronization scheme, the output of the master system (10) describes the target signal, while (12)

[3]This assures the unique solution [28]

represents the response signal. Therefore the synchronization problem can be established as:
Given the master system (9) and our slave system (11), it should be determined some conditions, such that the output of the slave system (12) synchronizes with the output of the master system (10).

Let us define the synchronization error as:

$$e_i = y_{\eta_i} - y_{\hat{\eta}_i} = \eta_i - \hat{\eta}_i. \tag{13}$$

Now we establish a convergence analysis of the synchronization error.

Proposition 1: Let the system (6) which can be expressed in the form (7), where the following conditions are fulfilled:

H1: η_i satisfies the FAO property for $i \in \{p+1, \ldots, n\}$.
H2: Δ_i is bounded, i.e., $\exists\, M \in \mathbb{R}^+$ such that $\|\Delta(x)\| \leq M$, $\forall\, x \in \Omega$.
H3: $k_{\hat{\eta}_i} \in \mathbb{R}^+$.

Then, the synchronization of the master output (10) with the slave output (12) is achieved, for global initial condition of the states.

Proof. From **H1** we can write equations (9)-(13). Taking the fractional derivative of the equation (13), we have

$$e_i^{(\alpha)} = \eta_i^{(\alpha)} - \hat{\eta}_i^{(\alpha)} \tag{14}$$

Substituting the fractional dynamics (9) and (11) into (14), we obtain

$$e_i^{(\alpha)} + k_{\hat{\eta}_i} e_i = \Delta_i(x) \tag{15}$$

There exists a unique solution for the system (15), due to $\Delta_i(x(t)) - k_{\hat{\eta}_i} e_i(t)$ is a Lipschitz continuous function on e.[4]

Solving the above equation [28], we have

$$\begin{aligned} e_i(t) = \;& e_{i0} E_{\alpha,1}(-k_{\hat{\eta}_i} t^\alpha) \\ & + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(k_{\hat{\eta}_i}(t-\tau)^\alpha)\Delta_i(x(\tau))\mathrm{d}\tau \end{aligned} \tag{16}$$

where $e_i(0) = e_{i0}$.

Using Triangle and Cauchy-Schwarz inequalities and **H2**

$$\begin{aligned} |e_i(t)| \leq \;& |e_{i0} E_{\alpha,1}(-k_{\hat{\eta}_i} t^\alpha)| \\ & + M \int_0^t |(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-k_{\hat{\eta}_i}(t-\tau)^\alpha)|\mathrm{d}\tau \end{aligned}$$

The functions $(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-k_{\hat{\eta}_i}(t-\tau)^\alpha)$ and $E_\alpha(-k_{\hat{\eta}_i} t^\alpha)$ are not negative due to **Property 2** of Mittag-Leffler function and **H3**

$$\begin{aligned} |e_i(t)| \leq \;& |e_{i0}| E_{\alpha,1}(-k_{\hat{\eta}_i} t^\alpha) \\ & + M \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-k_{\hat{\eta}_i}(t-\tau)^\alpha)\mathrm{d}\tau \end{aligned}$$

Using **Property 1** of Mittag-Leffler function

$$|e_i(t)| \leq \; |e_{i0}| E_{\alpha,1}(-k_{\hat{\eta}_i} t^\alpha) + M t^\alpha E_{\alpha,\alpha+1}(-k_{\hat{\eta}_i} t^\alpha)$$

[4] Equation (15) is non-autonomous, but the Lipschitz condition assures a unique solution [28].

If $t \to \infty$, we use the equation (5) with $\mu = 3\pi\alpha/4$ due to **H3**.

$$\begin{aligned} \lim_{t\to\infty} |e_i(t)| \leq \;& |e_{i0}| \lim_{t\to\infty} E_{\alpha,1}(-k_{\hat{\eta}_i} t^\alpha) \\ & + M \lim_{t\to\infty} t^\alpha E_{\alpha,\alpha+1}(-k_{\hat{\eta}_i} t^\alpha) \\ = \;& \frac{M}{k_{\hat{\eta}_i}} \end{aligned}$$

■

Remark 1: If the FAO of a state variable is expressed in terms of the fractional sequential derivatives of the output y, which are unknown, then is necessary to introduce an artificial variable (if it is possible) in order to avoid the use of these unknown derivatives.

IV. Numerical Example

In this section, the synchronization of the fractional order Rössler hyperchaotic system is treated.

Remark 2: Chaotic systems are characterized by global boundedness of the trajectories [30]. By this fact, H2 is always satisfied.

First, consider the fractional order Rössler hyperchaotic system [31]

$$x^{(\alpha)} = \begin{pmatrix} x_3 + a x_1 + x_2 \\ -c x_4 + d x_2 \\ -x_1 - x_4 \\ b + x_3 x_4 \end{pmatrix} \tag{17}$$

$$y = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

where $x = (x_1, x_2, x_3, x_4)^T$ is the state vector, y_1 and y_2 are the considered outputs. When $a = 0.32$, $b = 3$, $c = -0.5$, $d = 0.05$, and $\alpha = 0.95$, the Rössler equations (17) has a hyperchaotic attractor (see Fig. 1).

Now, we rewrite system (17) in the form (7) as follows

$$\begin{aligned} \bar{x}^{(\alpha)} &= \begin{pmatrix} \eta_3 + \mathrm{a}\bar{x}_1 + \bar{x}_2 \\ -\mathrm{c}\eta_4 + \mathrm{d}\bar{x}_2 \end{pmatrix} \\ \eta^{(\alpha)} &= \begin{pmatrix} -\bar{x}_1 - \eta_4 \\ \mathrm{b} + \eta_3\eta_4 \end{pmatrix} \end{aligned} \tag{18}$$

where $x_1 = \bar{x}_1$, $x_2 = \bar{x}_2$, $\eta_3 = x_3$, $\eta_4 = x_4$, $y_{\bar{x}_1} = \bar{x}_1$ and $y_{\bar{x}_2} = \bar{x}_2$. From (18), it is not difficult to find the following relations

$$\eta_3 = \phi_3\left(y_{\bar{x}}, y_{\bar{x}}^{(\alpha)}\right) = y_{\bar{x}_1}^{(\alpha)} - \mathrm{a} y_{\bar{x}_1} - y_{\bar{x}_2} \tag{19}$$

$$\eta_4 = \phi_4\left(y_{\bar{x}}, y_{\bar{x}}^{(\alpha)}\right) = -\frac{1}{\mathrm{c}} y_{\bar{x}_2}^{(\alpha)} + \frac{\mathrm{d}}{\mathrm{c}} y_{\bar{x}_2} \tag{20}$$

then we say that $\eta_3 = x_3$ and $\eta_4 = x_4$ are FAO and therefore H1 is fulfilled.

From above, the master systems are given by

$$\begin{cases} \eta_3^{(\alpha)} = & -\bar{x}_1 - \eta_4 \\ y_{\eta_3} = & \eta_3 = y_{\bar{x}_1}^{(\alpha)} - \mathrm{a} y_{\bar{x}_1} - y_{\bar{x}_2} \end{cases} \tag{21}$$

$$\begin{cases} \eta_4^{(\alpha)} = & \mathrm{b} + \eta_3\eta_4 \\ y_{\eta_4} = & \eta_4 = -\frac{1}{\mathrm{c}} y_{\bar{x}_2}^{(\alpha)} + \frac{\mathrm{d}}{\mathrm{c}} y_{\bar{x}_2} \end{cases} \tag{22}$$

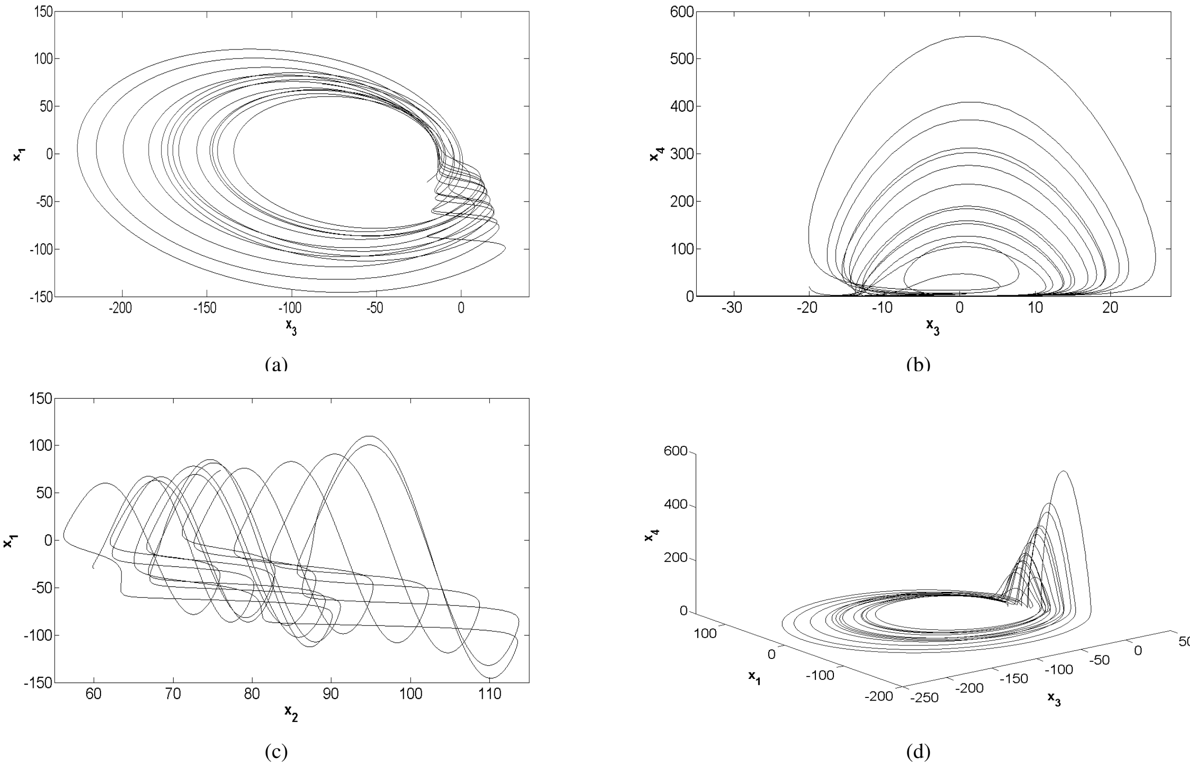


Fig. 1. Phase plot of the fractional-order Rössler hyperchaotic system with $a = 0.32$, $b = 3$, $c = -0.5$, $d = 0.05$, $\alpha = 0.95$, and initial conditions $x_1(0) = -30$, $x_2 = 60$, $x_3(0) = -20$, and $x_4(0) = 20$: (a) x_3-x_1 plane, (b) x_3-x_4 plane, (c) x_2-x_1 plane, and (d) x_3-x_1-x_4 space.

Now we design the corresponding slave systems for (21) and (22). By using (11), we have

$$\hat{\eta}_3^{(\alpha)} = k_{\hat{\eta}_3}(y_{\eta_3} - \hat{\eta}_3) \tag{23}$$

with $k_{\hat{\eta}_3} \in \mathbb{R}^+$ (condition H3).

Replacing (19) into (23) leads to

$$\hat{\eta}_3^{(\alpha)} = k_{\hat{\eta}_3}\left(y_{\bar{x}_1}^{(\alpha)} - \mathrm{a}y_{\bar{x}_1} - y_{\bar{x}_2}\right) - k_{\hat{\eta}_3}\hat{\eta}_3 \tag{24}$$

In order to avoid the use of the fractional derivative $y_{\bar{x}}^{(\alpha)}$, we introduce an auxiliary variable $\gamma_{\hat{\eta}_3}$:

$$\gamma_{\hat{\eta}_3} = -k_{\hat{\eta}_3}y_{\bar{x}_1} + \hat{\eta}_3 \tag{25}$$

then

$$\hat{\eta}_3 = \gamma_{\hat{\eta}_3} + k_{\hat{\eta}_3}y_{\bar{x}_1} \tag{26}$$

Substituting (26) and its fractional derivative of order α into (24), we obtain

$$\gamma_{\hat{\eta}_3}^{(\alpha)} = -k_{\hat{\eta}_3}\gamma_{\hat{\eta}_3} - k_{\hat{\eta}_3}\left(\mathrm{a}y_{\bar{x}_1} + y_{\bar{x}_2}\right) - k_{\hat{\eta}_3}^2 y_{\bar{x}_1} \quad , \tag{27}$$

with $\gamma_{\hat{\eta}_3}(0) = \gamma_{\hat{\eta}_{3_0}}$.

Then, the corresponding slave system of (21) is given by

$$\begin{cases} \hat{\eta}_3 = \gamma_{\hat{\eta}_3} + k_{\hat{\eta}_3}y_{\bar{x}_1} \\ y_{\hat{\eta}_3} = \hat{\eta}_3 \end{cases} \tag{28}$$

By means of the same procedure we have obtained the following slave system for (22)

$$\begin{cases} \hat{\eta}_4 = \gamma_{\hat{\eta}_4} - \dfrac{k_{\hat{\eta}_4}}{\mathrm{c}}y_{\bar{x}_2} \\ y_{\hat{\eta}_4} = \hat{\eta}_4 \end{cases} \tag{29}$$

where the dynamics of the auxiliary variable $\gamma_{\hat{\eta}_4}$ is given by

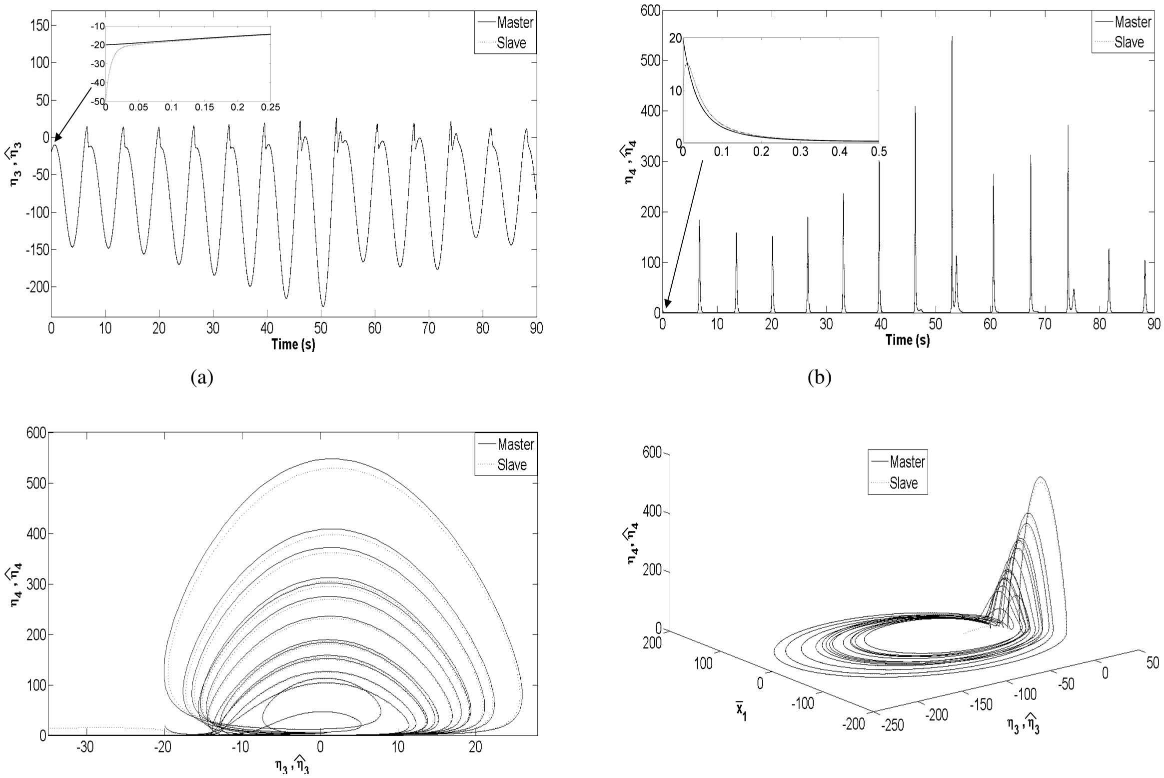
$$\gamma_{\hat{\eta}_4}^{(\alpha)} = -k_{\hat{\eta}_4}\gamma_{\hat{\eta}_4} + \frac{\mathrm{d}}{\mathrm{c}}k_{\hat{\eta}_4}y_{\bar{x}_2} + \frac{k_{\hat{\eta}_4}^2}{\mathrm{c}}y_{\bar{x}_2} \quad , \tag{30}$$

with $\gamma_{\hat{\eta}_4}(0) = \gamma_{\hat{\eta}_{4_0}}$ and $k_{\hat{\eta}_4} \in \mathbb{R}^+$ (condition H3).

Numerical simulations are performed for $a = 0.32$, $b = 3$, $c = -0.5$, $d = 0.05$, and $\alpha = 0.95$. We consider the following initial conditions to the master system $\bar{x}_1(0) = -30$, $\bar{x}_2(0) = 60$, $\eta_3(0) = -20$, $\eta_4(0) = 20$, the initial conditions to the slave system $\hat{\eta}_3(0) = -50$, $\hat{\eta}_4(0) = 10$, and the gain parameters are taken as $k_{\hat{\eta}_3} = k_{\hat{\eta}_4} = 100$. The synchronization between masters (21)-(22) and slaves (28)-(29) is shown in Fig. 2.

V. Conclusions

It was introduced a new concept so-called Fractional Algebraic Observability (FAO) which is a fundamental issue to determinate the unknown variables of nonlinear fractional order systems by means of the master-slave synchronization scheme, in particular we applied the results to a hyperchaotic fractional order system with success, however this technique can be applied to other class of systems which satisfy the properties of Proposition 1. Some numerical simulations have illustrated the effectiveness of the suggested approach.



(a) (b) (c) (d)

Fig. 2. Synchronization of the fractional-order Rössler hyperchaotic system.

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